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Extension of Euler's beta function

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Abstract

An extension of Euler's beta function, analogous to the recent generalization of Euler's gamma function and Riemann's zeta function, for which the usual properties and representation are naturally and simply extended, is introduced. It is proved that the extension is connected to the Macdonald, error and Whittaker functions. In addition, the extended beta distribution is introduced.

Keywords: Generalized gamma function; Euler's gamma function; Euler's beta function; Beta distribution

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1. Introduction

A generalization of a well-known special function, which extends the domain of that function, can be expected to be useful provided that the important properties of the special function are carried over to the generalization in a natural and simple manner. Of course, the original special function and its properties must be recoverable as a particular case of the generalization. Thus Euler (1707–1983) generalized the factorial function from the domain of natural numbers to the gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\,dt, \quad \mathrm{Re}(\alpha) > 0, \tag{1.1}$$

defined over the right half of the complex plane. This led Legendre (in 1811) to decompose the gamma function into the incomplete gamma functions, $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$, which are obtained from

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(1.1) by replacing the upper and lower limits by x, respectively. These functions develop singularities at the negative integers.

Two of us (MAC and SMZ) extended the domain of these functions to the entire complex plane [5] by inserting a regularization factor $e^{-b/t}$ in the integrand of (1.1). For $\mathrm{Re}(b) > 0$, this factor clearly removes the singularity coming from the $t = 0$ limit and for $b = 0$ reduces to the original gamma function. It turns out that this generalized gamma function, $\Gamma_b(\alpha)$, is related to the Macdonald function, $K_\alpha(2\sqrt{b})$, by

$$\Gamma_b(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t-b/t}\, dt = 2b^{\alpha/2} K_\alpha(2\sqrt{b}), \quad \mathrm{Re}(b) > 0, \tag{1.2}$$

and satisfies the recursion relation and reflection formula

$$\Gamma_b(\alpha + 1) = \alpha\Gamma_b(\alpha) + b\Gamma_b(\alpha - 1), \tag{1.3}$$

$$\Gamma_b(-\alpha) = b^{-\alpha}\Gamma_b(\alpha). \tag{1.4}$$

Note that the relationships between the generalized gamma and Macdonald functions could not have been apparent in the original gamma function. These generalized gamma functions proved very useful in diverse engineering and physical problems [4-7, 18].

The regularizer $e^{-b/t}$ also proved very useful in extending the domain of Riemann's zeta function [3], thereby providing relationships that could not have been obtained with the original zeta function.

In view of the effectiveness of the above regularizer for gamma and zeta functions, it seems worthwhile to look into the possibility that the domains of other special functions could be usefully extended in a similar manner. In particular, Euler's beta function, $B(x, y)$, has a close relationship to his gamma function,

$$B(x, y) = B(y, x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \tag{1.5}$$

and could be expected to be usefully extendable in a similar manner. It has the integral representation

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt, \quad \mathrm{Re}(x) > 0, \ \mathrm{Re}(y) > 0. \tag{1.6}$$

It is clear that the simple regularizer given above would destroy the symmetry property so essential for the beta function. Since for the x-y symmetry to be preserved there must be symmetry of the integrand in t and $1 - t$, we try the extension

$$B(x, y; b) = \int_0^1 t^{x-1}(1 - t)^{y-1} e^{-b/[t(1-t)]}\, dt, \quad \mathrm{Re}(b) > 0. \tag{1.7}$$

This extension will be seen to be extremely useful, in that most properties of the beta function carry over naturally and simply for it and it provides connections with the error and Whittaker functions, and even new representations for special case of these functions. Clearly, when $b = 0$ it reduces to the original beta function.

This paper is divided into seven sections. Different integral representations of the extended beta function are given in Section 2. Some properties of the function are proved in Section 3. Section 4 deals with the relation of some special cases of the function with some known special functions. The Mellin transform representation of our extended beta function is given in Section 5. We introduce the extended beta distribution in Section 6. Some concluding remarks about the function are given in Section 7.

2. Integral representations of the extended beta function

It is important and useful to look for other integral representations of the extended beta function, both as a check that the extension is natural and simple and for use later. It is also useful to look for relationships between the original beta function and its extension. In this connection, we first provide a relationship between them.

Theorem 2.1.

$$\int_0^\infty b^{s-1}B(x,y;b)\,\mathrm{d}b = \Gamma(s)B(x+s,y+s), \quad \mathrm{Re}(s)>0,\ \mathrm{Re}(x+s)>0,\ \mathrm{Re}(y+s)>0. \tag{2.1}$$

Proof. Multiplying (1.7) by b^{s-1} and integrating with respect to b from $b=0$ to $b=\infty$, we get

$$\int_0^\infty b^{s-1}B(x,y;b)\,\mathrm{d}b = \int_0^\infty b^{s-1}\left(\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{e}^{-b/[t(1-t)]}\,\mathrm{d}t\right)\mathrm{d}b. \tag{2.2}$$

The order of integration in (2.2) can be interchanged because of the uniform convergence of the integral. Therefore, we have

$$\int_0^\infty b^{s-1}B(x,y;b)\,\mathrm{d}b = \int_0^1 t^{x-1}(1-t)^{y-1}\left(\int_0^\infty b^{s-1}\,\mathrm{e}^{-b/[t(1-t)]}\,\mathrm{d}b\right)\mathrm{d}t. \tag{2.3}$$

However, the integral in (2.3) can be simplified in terms of the gamma function to give

$$\int_0^\infty b^{s-1}\mathrm{e}^{-b/[t(1-t)]}\,\mathrm{d}b = t^s(1-t)^s\Gamma(s), \quad \mathrm{Re}(s)>0,\ 0<t<1. \tag{2.4}$$

From (2.3) and (2.4), our proof follows. □

By putting $s=1$, in (2.1), we get the interesting relation

$$\int_0^\infty B(x,y;b)\,\mathrm{d}b = B(x+1,y+1), \quad \mathrm{Re}(x)>-1,\ \mathrm{Re}(y)>-1, \tag{2.5}$$

between the original and the extended beta functions.

Remark 2.2. It is interesting to note that all the derivatives of the extended beta function with respect to the parameter b can be expressed in terms of the function to give

$$\frac{\partial^n}{\partial b^n} B(x, y; b) = (-1)^n B(x-n, y-n; b), \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

Relations (2.1) and (2.6) may be helpful to derive a partial differential equation for the function.

Remark 2.3. The usual integral representations of the beta function carry over naturally to our extension and can be recovered from them by taking $b = 0$.

Theorem 2.4. (Integral representations).

$$B(x, y; b) = 2\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1} \mathrm{e}^{-b\sec^2\theta\csc^2\theta}\,\mathrm{d}\theta, \tag{2.7}$$

$$B(x, y; b) = \mathrm{e}^{-2b}\int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \mathrm{e}^{-b(u+u^{-1})}\,\mathrm{d}u, \tag{2.8}$$

$$B(x, y; b) = 2^{1-x-y}\int_{-1}^{1} (1+t)^{x-1}(1-t)^{y-1} \mathrm{e}^{-4b/(1-t^2)}\,\mathrm{d}t, \tag{2.9}$$

$$B(x, y; b) = \tfrac{1}{2}\mathrm{e}^{-2b}\int_0^{\infty} \frac{u^{x-1}+u^{y-1}}{(1+u)^{x+y}} e^{-b(u+u^{-1})}\,\mathrm{d}u, \tag{2.10}$$

$$B(x, y; b) = (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1} \exp\left[-b\frac{(c-a)^2}{(u-a)(c-u)}\right]\mathrm{d}u, \tag{2.11}$$

$\mathrm{Re}(b) > 0$. For $b = 0$, $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$,

$$B(\alpha, \beta; b) = 2^{1-\alpha-\beta}\int_{-\infty}^{\infty} \exp((\alpha-\beta)x - 4b\cosh^2 x)\frac{\mathrm{d}x}{(\cosh x)^{\alpha+\beta}}, \tag{2.12}$$

$$B(\alpha, \beta; b) = 2^{2-\alpha-\beta}\int_0^{\infty} \frac{\cosh((\alpha-\beta)x)}{(\cosh x)^{\alpha+\beta}} \exp(-4b\cosh^2 x)\,\mathrm{d}x, \tag{2.13}$$

$$B(\alpha, \beta; b) = 2^{1-\alpha-\beta}\int_{-\infty}^{\infty} \exp\left(\tfrac{1}{2}(\alpha-\beta)x - 2b\cosh x\right)\frac{\mathrm{d}x}{(\cosh\frac{1}{2}x)^{\alpha+\beta}}, \tag{2.14}$$

$$B(\alpha, \beta; b) = 2^{2-\alpha-\beta}\int_0^{\infty} \frac{\cosh(\frac{1}{2}(\alpha-\beta)x)}{(\cosh\frac{1}{2}x)^{\alpha+\beta}} \exp(-2b\cosh x)\,\mathrm{d}x, \tag{2.15}$$

$\mathrm{Re}(b) > 0$. For $b = 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Proof. The proofs of (2.7)–(2.15) are straightforward. In particular, (2.7) follows when we use the transformation $t = \cos^2\theta$ in (1.7) and (2.8) follows from (1.7) when we use the transformation $t = u/(1+u)$. Similarly, (2.11) follows from (1.7) when we use the transformation $t = (u-a)/(c-a)$. □

A useful inequality

$$|B(x, y; b)| \leqslant e^{-4b} B(x, y), \quad x > 0,\ y > 0,\ b \geqslant 0, \tag{2.16}$$

follows from the integral representation (2.8). Since the function $e^{-b(u+u^{-1})}$ attains its maximum at $u = 1$,

$$|B(x, y; b)| \leqslant e^{-4b} \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du. \tag{2.17}$$

Hence the inequality.

It is worth remarking that this property already foreshadows the regularization of the beta function for negative x and/or y seen in Section 7, in that it shows how the extended beta function has an exponentially damped magnitude compared with the original beta function.

3. Properties of the extended beta function

In this section, we consider the analogue of (1.3) for the extended beta function and the extension of (1.5). While the former comes very naturally, it must be admitted that the extension of (1.5) is not so simple.

Theorem 3.1 (Functional relation).

$$B(x, y+1; b) + B(x+1, y; b) = B(x, y; b). \tag{3.1}$$

Proof. The left-hand side of (3.1) equals

$$\int_0^1 \{t^{x-1}(1-t)^y + t^x(1-t)^{y-1}\} e^{-b/[t(1-t)]}\, dt, \tag{3.2}$$

which, after simple algebraic manipulation, yields

$$\int_0^1 t^{x-1}(1-t)^{y-1} e^{-b/[t(1-t)]}\, dt,$$

which is equal to the right hand side of (3.1). □

As a direct corollary, on putting $b = 0$ in (3.1), we get the usual relation for the beta function.

Since the generalized gamma function (1.2) extends the Euler gamma function in a natural way, it seems worthwhile to extend the classical relationship (1.5) proved by Euler.

Theorem 3.2 (Product formula). *Let $\Gamma_b(x)$ be the generalized gamma function as defined in* (1.2). *Then*

$$\Gamma_b(x)\Gamma_b(y) = 2\int_0^\infty r^{2(x+y)-1} e^{-r^2} B\left(x, y; \frac{b}{r^2}\right) dr, \quad \mathrm{Re}(b) > 0. \text{ For } b = 0,\ \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0. \tag{3.3}$$

Proof. Substituting $t = \eta^2$ in (1.2), we get

$$\Gamma_b(x) = 2\int_0^\infty \eta^{2x-1} e^{-\eta^2 - b\eta^{-2}}\, d\eta. \tag{3.4}$$

Therefore,

$$\Gamma_b(x)\Gamma_b(y) = 4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1} e^{-(\eta^2+\xi^2)} e^{-b(\eta^{-2}+\xi^{-2})}\, d\eta\, d\xi. \tag{3.5}$$

The substitutions $\eta = r\cos\theta$, $\xi = r\sin\theta$ in (3.5) yield

$$\Gamma_b(x)\Gamma_b(y) = 4\int_0^{\pi/2}\int_0^\infty r^{2(x+y)-1} e^{-r^2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\exp\left[-\frac{b}{r^2\sin^2\theta\cos^2\theta}\right] dr\, d\theta. \tag{3.6}$$

Interchanging the order of integration on the left-hand side in (3.6), we get

$$\Gamma_b(x)\Gamma_b(y) = 2\int_0^\infty r^{2(x+y)-1} e^{-r^2}\left(2\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\exp\left[-\frac{b/r^2}{\sin^2\theta\cos^2\theta}\right] d\theta\right) dr. \tag{3.7}$$

From (2.7) and (3.7), the proof of the theorem is complete. □

Again we see that putting $b = 0$ in (3.3) gives the classical relation (1.5).

4. Connection with other special functions

As mentioned in the Introduction, this extension of the beta function is justified not only by the fact that most properties of the beta function are carried over simply, but also by the fact that this function is related to other special functions for particular values of the variables. In this section, we demonstrate this fact for the cases $y = -x$ and $y = x$. There may well be other such relations to be discovered for other special cases.

Theorem 4.1.

$$B(\alpha, -\alpha; b) = 2e^{-2b}K_\alpha(2b), \quad \mathrm{Re}(b) > 0. \tag{4.1}$$

Proof. The substitutions $x = \alpha$ and $y = -\alpha$ in (2.8) yield

$$B(\alpha, -\alpha; b) = \int_0^\infty u^{\alpha-1} e^{-b(u+u^{-1})}\, du. \tag{4.2}$$

The integral on the right-hand side of (4.2) is the Mellin transform of $e^{-b(u+u^{-1})}$ in α that can be solved in terms of Macdonald and exponential functions [11, p. 384] to give (4.1). □

According to [13, p. 978 (8.468)]

$$K_{n-1/2}(z)=\sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\sum_{k=0}^{n}\frac{(n+k)!(2z)^{-k}}{k!(n-k)!},\quad n=0,1,2,\ldots,\ \mathrm{Re}(z)>0. \tag{4.3}$$

Therefore, the extended beta function $B(n+\frac{1}{2}, -n-\frac{1}{2}; b)$ can be written as the function

$$B(n+\tfrac{1}{2}, -n-\tfrac{1}{2}; b)=\sqrt{\frac{\pi}{b}}\,\mathrm{e}^{-4b}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(4b)^k},\quad \mathrm{Re}(b)>0. \tag{4.4}$$

In particular, for $n=0$, this yields

$$B(\tfrac{1}{2}, -\tfrac{1}{2}; b)=\sqrt{\frac{\pi}{b}}\,\mathrm{e}^{-4b},\quad \mathrm{Re}(b)>0. \tag{4.5}$$

Theorem 4.2. *The extended beta function is related to the Whittaker function by*

$$B(\alpha,\alpha;b)=\sqrt{\pi}\,2^{-\alpha}b^{(\alpha-1)/2}\mathrm{e}^{-2b}W_{-\alpha/2,\alpha/2}(4b),\quad \mathrm{Re}(b)>0. \tag{4.6}$$

Proof. The substitution $y=x$ in (2.9) yields

$$B(x,x;b)=2^{1-2x}\int_{-1}^{1}(1-t^2)^{x-1}\mathrm{e}^{-4b/(1-t^2)}\,\mathrm{d}t. \tag{4.7}$$

Since the integrand on the right-hand side is even, it follows that

$$B(x,x;b)=2^{2-2x}\int_{0}^{1}(1-t^2)^{x-1}\mathrm{e}^{-4b/(1-t^2)}\,\mathrm{d}t. \tag{4.8}$$

The substitution $\xi=1-t^2$ in (4.8) yields

$$B(x,x;b)=2^{1-2x}\int_{0}^{1}\xi^{x-1}(1-\xi)^{1/2-1}\mathrm{e}^{-4b/\xi}\,\mathrm{d}\xi. \tag{4.9}$$

The integral on the right-hand side of (4.9) is a special case of the result [13, p. 384 (3.471) (2)] (see [12, p. 187 (18)])

$$\int_0^u x^{v-1}(u-x)^{\mu-1}\mathrm{e}^{-\beta/x}\,\mathrm{d}x=\beta^{(v-1)/2}u^{(2\mu+v-1)/2}\exp\left(-\frac{\beta}{2u}\right)\Gamma(\mu)W_{(1-2\mu-v)/2,v/2}\left(\frac{\beta}{u}\right)$$

$((\mathrm{Re})\mu>0,\ \mathrm{Re}(\beta)>0,\ u>0)$.

With $\beta=4b$, $u=1$, $v=x$ and $\mu=\frac{1}{2}$, this gives

$$B(x,x;b)=\sqrt{\pi}\,2^{-x}b^{(x-1)/2}\mathrm{e}^{-2b}W_{-x/2,x/2}(4b),\quad \mathrm{Re}(b)>0. \tag{4.10}$$

Replacing x by α in (4.10) completes the proof of (4.6). □

In particular, taking $\alpha=\frac{1}{2}$ and using [10, p. 432] we obtain

$$B(\tfrac{1}{2},\tfrac{1}{2};b)=\pi\,\mathrm{Erfc}(2\sqrt{b}),\quad \mathrm{Re}(b)>0, \tag{4.11}$$

and taking $\alpha = 0$, we obtain

$$B(0, 0; b) = 2e^{-2b}K_0(2b), \quad \mathrm{Re}(b) > 0. \tag{4.12}$$

Remark. The relationship [1, p. 317] may be exploited to express (4.6) in terms of the confluent hypergeometric function $\psi(a, c; z)$,

$$B(\alpha, \alpha; b) = \sqrt{\pi}\,2^{1-2\alpha}e^{-4b}\psi(\tfrac{1}{2}, 1-\alpha; 4b), \quad \mathrm{Re}(b) > 0, \tag{4.13}$$

which incidently again gives (4.11).

We can use the above theorems to express other extended beta functions as series of Whittaker or Macdonald functions provided $x - y$ or $x + y$ are integers. This fact is shown in the following two theorems.

Theorem 4.3.

$$B(\alpha, -\alpha - n; b) = 2e^{-2b}\sum_{k=0}^{n}\binom{n}{k}K_{\alpha+k}(2b). \tag{4.14}$$

Proof. On setting $x = \alpha$ and $y = -\alpha - n$ in (3.1), we get

$$B(\alpha, -\alpha - n; b) = B(\alpha, -\alpha - n + 1; b) + B(\alpha + 1, -\alpha - n; b). \tag{4.15}$$

We can write this formula recursively, starting with $n = 1$, to obtain

$$\begin{aligned} B(\alpha, -\alpha - 1; b) &= B(\alpha, -\alpha; b) + B(\alpha + 1, -\alpha - 1; b),\\ B(\alpha, -\alpha - 2; b) &= B(\alpha, -\alpha; b) + 2B(\alpha + 1, -\alpha - 1; b) + B(\alpha + 2, -\alpha - 2; b),\\ B(\alpha, -\alpha - 3; b) &= B(\alpha, -\alpha; b) + 3B(\alpha + 1, -\alpha - 1; b) + 3B(\alpha + 2, -\alpha - 2; b)\\ &\quad + B(\alpha + 3, -\alpha - 3; b), \end{aligned} \tag{4.16}$$

and so on. The series arises exactly as the binomial series does and so we can guess that

$$B(\alpha, -\alpha - n; b) = \sum_{k=0}^{n}\binom{n}{k}B(\alpha + k, -\alpha - k; b). \tag{4.17}$$

This result can be simply proved by induction, assuming it true for some n and writing $B(\alpha, -\alpha - n - 1; b)$ by using (4.15). It immediately follows that (4.17) holds for $(n + 1)$. Now (4.1) and (4.17) directly yield (4.14). □

Theorem 4.4.

$$B(\alpha, \alpha + n; b) = (\sqrt{\pi}e^{-2b})\tfrac{1}{2}n\sum_{k=0}^{[n/2]}2^{-\alpha-k}b^{(\alpha+k-1)/2}\frac{(-1)^k}{n-k}\binom{n-k}{k}W_{-(\alpha+k)/2,(\alpha+k)/2}(4b). \tag{4.18}$$

Proof. Setting $x = \alpha$ and $y = \alpha + n - 1$ in (3.1), we obtain

$$B(\alpha, \alpha + n; b) = B(\alpha, \alpha + n - 1; b) - B(\alpha + 1, \alpha + n - 1; b). \tag{4.19}$$

Also, taking $x = y = \alpha$ in (3.1) and using the symmetry property of the extended beta function gives

$$B(\alpha, \alpha + 1; b) = \tfrac{1}{2} B(\alpha, \alpha; b). \tag{4.20}$$

Using (4.19) recursively with $n = 2, 3, \ldots$ yields

$$\begin{aligned} B(\alpha, \alpha + 2; b) &= \tfrac{1}{2} B(\alpha, \alpha; b) - \tfrac{2}{2} B(\alpha + 1, \alpha + 1; b), \\ B(\alpha, \alpha + 3; b) &= \tfrac{1}{2} B(\alpha, \alpha; b) - \tfrac{3}{2} B(\alpha + 1, \alpha + 1; b), \\ B(\alpha, \alpha + 4; b) &= \tfrac{1}{2} B(\alpha, \alpha; b) - \tfrac{4}{2} B(\alpha + 1, \alpha + 1; b) + \tfrac{2}{2} B(\alpha + 2, \alpha + 2; b), \\ B(\alpha, \alpha + 5; b) &= \tfrac{1}{2} B(\alpha, \alpha; b) - \tfrac{5}{2} B(\alpha + 1, \alpha + 1; b) + \tfrac{5}{2} B(\alpha + 2, \alpha + 2; b), \end{aligned} \tag{4.21}$$

and so on. Constructing the sequence of coefficients of $B(\alpha + k, \alpha + k; b)$ for different values of k, on the right-hand side of (4.21), we can fit constant, linear, quadratic, cubic, quartic, etc. polynomials for $k = 0, 1, 2, 3, 4, \ldots$, respectively. We find that they satisfy the formula

$$B(\alpha, \alpha + n; b) = \tfrac{1}{2} n \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n - k} \binom{n - k}{k} B(\alpha + k, \alpha + k; b). \tag{4.22}$$

We prove this formula by induction. Let us assume that it holds for some n and use (4.19) to write the coefficient of a typical term in the right hand side of the resulting equation. $B(\alpha + p, \alpha + p; b)$ has the coefficient

$$\frac{n}{2} \frac{(-1)^p}{n - p} \binom{n - p}{p} - \frac{n - 1}{2} \frac{(-1)^p}{n - 1 - p + 1} \binom{n - 1 - p + 1}{p - 1}.$$

Using the usual formula for addition of binomial coefficients and simplifying, this coefficient reduces to

$$\frac{(-1)^p}{2(n - p)} \frac{(n - p)!}{(n - 2p - 1)!\,p!} (n^2 - np + n - p),$$

which further reduces to

$$\frac{n + 1}{2} (-1)^p \frac{1}{n + 1 - p} \frac{(n - p + 1)!}{(n - 2p + 1)!\,p!}.$$

This is the coefficient of the corresponding term in (4.22) with n replaced by $(n + 1)$. Hence (4.22) holds for all n. Now, using (4.6) and (4.22), we obtain (4.18). □

Not only can we express the extended beta functions as finite series of Macdonald and Whittaker functions (when the sum or difference of the arguments is an integer) we can express the Whittaker, Macdonald, error and exponential functions as an infinite series of extended beta functions. This fact follows as a direct consequence of the following theorem.

Theorem 4.5.

$$B(x, 1 - y; b) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B(x + n, 1; b), \quad \mathrm{Re}(b) > 0. \tag{4.23}$$

Proof. Here $(y)_n$, the factorial function, is the coefficient arising in the binomial series for arbitrary powers

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!}. \tag{4.24}$$

Using (4.24) in (1.6), we obtain

$$B(x, 1-y; b) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-1} e^{-b/[t(1-t)]} \, dt. \tag{4.25}$$

Now the summation could be divergent at the end points of integration for particular values of x and y. However, for $\mathrm{Re}(b) > 0$, the regularizing factor damps the singularity arising there. Thus we can interchange the order of integration and summation to obtain

$$B(x, 1-y; b) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} \int_0^1 t^{x+n-1} e^{-b/[t(1-t)]} \, dt. \tag{4.26}$$

Using (1.6), we obtain (4.23). □

Corollary.

$$K_\alpha(2b) = \tfrac{1}{2} e^{2b} \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} B(\alpha+n, 1; b), \tag{4.27}$$

$$W_{-\alpha/2, \alpha/2}(4b) = \frac{1}{\sqrt{\pi}} 2^\alpha b^{(1-\alpha)/2} e^{2b} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} B(\alpha+n, 1; b). \tag{4.28}$$

Proof. These results follow directly by taking $y - x = 1$ and $y + x = 1$, respectively, with $x = \alpha$. Notice that both reduce to the Macdonald function, K_0, in the case, $\alpha = 0$. Further, with $\alpha = \frac{1}{2}$ in (4.28), the Whittaker function reduces to $\mathrm{Erfc}(2\sqrt{b})$. □

Remark. Notice that, with a negative integer α in (4.27) or a positive integer α in (4.28), the series terminates and thus reduces to a finite series instead of an infinite series. With a negative integer y in (4.23), the series will, of course, again terminate (as would have been anticipated by the symmetry of the extended beta function in its arguments).

5. Mellin transform representation of the extended beta function

Theorem 5.1.

$$B(x, y; b) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma(x+y+2s)} b^{-s} \, ds, \quad \mathrm{Re}(b) > 0. \tag{5.1}$$

Proof. Let $\mathcal{M}$ be the Mellin transform operator as defined by [11, p.305]. Then, we can write (2.1) in operational form to give

$$\mathcal{M}\{B(x, y; b); b \to s\} = \Gamma(s)B(x+s, y+s), \quad \mathrm{Re}(s) > 0, \ \mathrm{Re}(x+s) > 0, \ \mathrm{Re}(y+s) > 0. \tag{5.2}$$

Taking the inverse Mellin transform of both sides of (5.2), we get

$$B(x, y; b) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s)B(x+s, y+s)b^{-s}\,\mathrm{d}s, \quad \mathrm{Re}(b) > 0. \tag{5.3}$$

The substitution for $B(x+s, y+s)$ in (5.3) from (1.5) completes the proof of (5.1). □

Note that we cannot recover a corresponding formula for the original beta function by taking the limit as $b \to 0$.

The substitutions $x = v$ and $y = -v$ in (5.1), using the Legendre duplication formula and replacing b by $\frac{1}{4}b$ yields (see [11, p. 350, Eq. (25)])

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(s+v)\Gamma(s-v)}{\Gamma(s+\frac{1}{2})} b^{-s}\,\mathrm{d}s = \pi^{-1/2} e^{-b/2} K_v(\tfrac{1}{2}b), \quad \mathrm{Re}(b) > 0. \tag{5.4}$$

Further, the substitutions $x = y = \alpha$ and $b = \frac{1}{4}\xi$ in (5.1) yield

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(s)\Gamma(s+\alpha)}{\Gamma(s+\alpha+\frac{1}{2})} \xi^{-s}\,\mathrm{d}s = \xi^{(\alpha-1)/2} \mathrm{e}^{-\xi/2} W_{-\alpha/2, \alpha/2}(\xi), \quad \mathrm{Re}(b) > 0. \tag{5.5}$$

Remark. It is important to note that the representation (5.1) proves closed form evaluation of the inverse Mellin transform of a class of products of gamma functions in terms of the extended beta function. Only one of its special cases, namely (5.4) seems to be known in the literature. The special case (5.5) appears to be new. This is clear evidence of the usefulness of the extended beta function.

6. The extended beta distribution

It is expected that there will be many applications of the extended beta function, like there were of the generalized gamma function. One application that springs to mind is to Statistics. For example, the conventional beta distribution can be extended, by using our extended beta function, to variables p and q with an infinite range. It appears that such an extension may be desirable for the project evaluation and review technique used in some special cases.

We define the extended beta distribution by

$$f(t) = \begin{cases} \dfrac{1}{B(p, q; b)} t^{p-1}(1-t)^{q-1} \mathrm{e}^{-b/[t(1-t)]}, & 0 < t < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{6.1}$$

A random variable X with probability density function (pdf) given by (6 .1) will be said to have the extended beta distribution with parameters p and q, $-\infty < p < \infty$, $-\infty < q < \infty$ and $b > 0$. If v is any real number then [17]

$$E(X^v) = \frac{B(p+v, q; b)}{B(p, q; b)}. \tag{6.2}$$

In particular, for $v = 1$,

$$\mu = E(X) = \frac{B(p+1, q; b)}{B(p, q; b)} \tag{6.3}$$

represents the mean of the distribution and

$$\begin{aligned}\sigma^2 &= E(X^2) - (E(X))^2 \\ &= \frac{B(p, q; b)B(p+2, q; b) - B^2(p+1, q; b)}{B^2(p, q; b)}\end{aligned} \tag{6.4}$$

is the variance of the distribution.

The moment generating function of the distribution is

$$\begin{aligned}M(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) \\ &= \frac{1}{B(p, q; b)} \sum_{n=0}^{\infty} B(p+n, q; b) \frac{t^n}{n!}.\end{aligned} \tag{6.5}$$

The cumulative distribution of (6.1) can be written as

$$F(x) = \frac{B_x(p, q; b)}{B(p, q; b)}, \tag{6.6}$$

where

$$B_x(p, q; b) = \int_0^x t^{p-1}(1-t)^{q-1} e^{-b/[t(1-t)]} \, dt, \quad b > 0, \quad -\infty < p < \infty, \quad -\infty < q < \infty, \tag{6.7}$$

is the extended incomplete beta function. For $b = 0$, we must have $p > 0$ and $q > 0$ in (6.7) for convergence, and then, $B_x(p, q; 0) = B_x(p, q)$, where $B_x(p, q)$ is the incomplete beta function [13, p. 960] given by

$$B_x(p, q) = \frac{x^p}{p} \, {}_2F_1(p, 1-q; p+1; x). \tag{6.8}$$

It is to be noted that the problem of expressing $B_x(p, q; b)$ in terms of other special functions remains open.

Presumably, this distribution should be useful in extending the statistical results for strictly positive variables to deal with variables that can take arbitrarily large negative values as well.

7. Discussion and conclusion

The classical beta function $B(x, y)$ is defined in the first quadrant of the x–y plane. The extra exponential factor in the extended beta function plays the role of a regularizer and allows us to state the Euler beta function as the limit of a function defined in the whole plane. The extension also meets the requirement that the previous results for the beta function are naturally and simply extended. Of course, some results analogous to previous ones would hold for any extension of the function. What we have required and obtained is that the results for the extension should be no less elegant, or more cumbersome, than those for the original function.

The numerical values of $B(x, y; b)$ can easily be obtained by most mathematical software packages. Using QDAGI [14, 15] we have provided a graph of $B(x, y; b)$ against x for different values of b, see Fig. 1. By the symmetry between x and y the general behaviour can be easily visualized. Note how b "pulls the function down".

A more exciting feature that emphasizes the importance of the extension is its relationship with the Macdonald, error and Whittaker functions. This leads us to believe in the wide applications of the extended beta function in different areas of statistics, engineering and applied mathematics. One of us (MAC) intends to pursue these relationships and extensions further [2].

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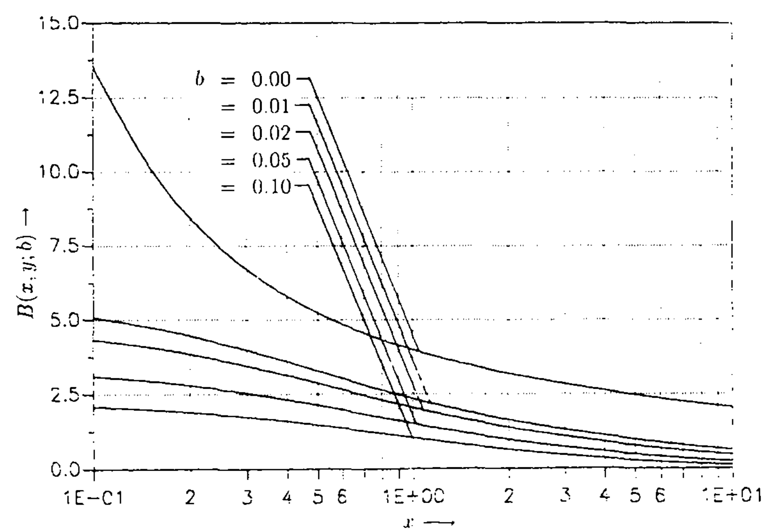

Fig. 1. Graphical representation of $B(x, y; b)$ for $y = 0.25$ and $b = 0, 0.01, 0.02, 0.05, 0.1$. Note that $b = 0$ simply gives the usual beta function and we can see, here, how increasing b "pulls the graph down".

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