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Itô's theory of excursion point processes and its developments

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Abstract

Itô's theory of excursion point processes is reviewed and the following topics are discussed: Application of the theory to one-dimensional diffusion processes on half-intervals satisfying Feller's boundary conditions, and its multi-dimensional extension, i.e., the application of the theory to multi-dimensional diffusion processes satisfying Wentzell's boundary conditions.

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1. Introduction

In this paper, dedicated to the memory of Professor Kiyosi Itô, we shall review *Itô's theory of excursion point processes* and some of its further developments. As Itô himself remembered in the Foreword of [13], this theory is a natural outgrowth of his joint work with Henry McKean [14,15] on one-dimensional diffusion processes: in order to understand better the class of diffusion processes on a linear interval satisfying Feller's boundary conditions, Itô introduced in [11,12] the notion of excursion point processes, which are point processes with values in a function space (i.e. path space).

On the other hand, we know well that the notion of Poisson point processes has been introduced and it has played a key role in Lévy–Itô theory on the structure of sample paths of Lévy processes [8,10]. This class of Poisson point processes has been defined by the size of

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jumps of path functions, and so point processes take their values in a finite dimensional Euclidean space. However, the notion of point processes is defined generally with state spaces which can be rather arbitrary. Itô's idea in [11,12] is that, for a strong Markov process, the totality of excursions away from a recurrent state can be formulated as a stationary Poisson point process with values in a function space. This is indeed a breakthrough in the study of boundary problems for diffusion processes. We quote here a recollection of Itô's given in the Foreword of [13] in his study of the description of all possible extensions of a minimal diffusion up to the hitting time of the boundary point: *After several years it became my habit to observe even finite dimensional facts from the infinite dimensional viewpoint. This habit led me to reduce the problem above to a Poisson point process with values in the space of excursions.*

A most typical and important example is the case of the *Poisson point process of Brownian excursions* which is treated, e.g., in Chap. III, Subsection 4.3 of [7], Chap. 6 of [17], Chap. VI, Section 8 of [21], Chap. XII of [20]. The Poisson point process of Brownian excursions is a correct and precise mathematical realization of the decomposition of Brownian sample functions into pieces called excursions. Once this is correctly formulated, not only Brownian sample functions but also their functionals, like local times, the maximum process, down-crossing numbers, occupation times, etc., can be recovered and represented in terms of the point process of Brownian excursions. Many facts concerning these functionals and their limit theorems can be deduced from such representations. Even if the facts considered thereby may not be new, this approach can often provide us with a deeper and better understanding of such facts and can produce sometimes a remarkable progress of the theory. Some such examples are the Williams formula for the occupation times on the positive half-line of Brownian paths which proves the arcsine law without appealing to the Feynman–Kac theorem, Lévy's down-crossing theorem, asymptotic laws of planar Brownian motion due to Pitman and Yor, among many others.

As was stated above, the motivation of Itô in the study of point processes with values in a function space originated from the joint work with McKean on analyzing and constructing path functions of linear diffusions satisfying the most general Feller boundary conditions. I could never forget those hot days in the summer of 1969, when Professor Itô returned to his home in Kyoto, on vacation from his regular work at Aarhus, and gave us a series of lectures on excursion point processes (which was a source of [11]). I remember that he told us then that his theory was motivated by a question raised by Lamperti, "What are all possible boundary conditions at an exit boundary of a linear diffusion, typically the case of the boundary 0 of Feller's diffusion on $[0, \infty)$ with the generator $x \frac{d^2}{dx^2}$, which is a typical scaling limit of critical Galton–Watson branching processes?" We shall review this topic in Section 3.

A main objective of the present paper is to extend Itô's idea to the same problem in *multi-dimensional cases*, in particular, to the problem of constructing and analyzing *multi-dimensional diffusion processes in a domain with the boundary satisfying the most general Wentzell boundary conditions*. This was a problem which, after the success of the Itô–McKean theory, was attacked by many people by various approaches. In Section 4, we shall show that Itô's approach by Poisson point processes of excursions can still be applied well to this problem.

We shall discuss the problem in Section 4, going from a simpler case to more general cases. As we see there, the problem can be solved by determining and constructing two stochastic processes. The first one, denoted by $(\xi(t))$, is a process moving on the boundary. The second one, denoted by $(A(t))$, is an increasing process.

A simple case of the problem is when the generator, which describes the behavior of the diffusion inside the domain before hitting the boundary, and the boundary condition, which describes the behavior of the diffusion on the boundary, are both *of constant coefficients*. In this case, both processes $(\xi(t))$ and $(A(t))$ are Lévy processes and they can be given directly by non-canonical representations of Lévy–Itô type, as in Example 2.1 of Section 2, from two Poisson point processes with values in certain function spaces. So, in Section 4, we first set up these two Poisson point processes from the given data for the generator and the boundary condition.

In more complicated cases of the data given by variable coefficients, we start again with the same two Poisson point processes with values in function spaces. In this case, however, the processes $(\xi(t))$ and $(A(t))$ are no longer Lévy processes, but are semimartingales and should be determined as unique solutions of stochastic differential equations (SDE's) of the jump type based on these Poisson point processes. So far, a SDE of the jump type has usually been considered only in the case when the SDE is based on a point process with values in a Euclidean space, as discussed, e.g., by Itô [9] and Gihman and Skorohod [5]. In the present problem, we need and use essentially SDE's based on Poisson point processes with values in function spaces.

2. Point processes and Poisson point processes

Let $(U, \mathcal{B}_U)$ be a measurable space. In this paper, we always assume that U is a standard Borel space in the sense of [19] and $\mathcal{B}_U$ is the topological σ-field on U (equivalently, U is a Lusin space in the sense of [2] and $\mathcal{B}_U$ is the totality of Borel subsets of U). By a *point function* on U, we mean a mapping

$$p : (0,\infty) \supset \mathbf{D}_p \ni t \mapsto p(t) \in U$$

where the domain $\mathbf{D}_p$ of the mapping p is a countable subset of $(0,\infty)$. p defines a counting measure $N_p(\mathrm{d}s, \mathrm{d}u)$ on $(0,\infty) \times U$ with σ-field $\mathcal{B}(0,\infty) \otimes \mathcal{B}_U$ via

$$N_p((0,t] \times A) = \sum_{s \in \mathbf{D}_p, s \le t} 1_A(p(s)), \quad t > 0,\ A \in \mathcal{B}_U. \tag{2.1}$$

Let Π_U be the totality of point functions on U and $\mathcal{B}(\Pi_U)$ be the smallest σ-field on Π_U with respect to which the mappings $\Pi_U \ni p \mapsto N_p((0,t] \times A) \in \{0, 1, \dots, \infty\}$, for each $t > 0$ and $A \in \mathcal{B}_U$, are all measurable. By a *point process* $\mathbf{p}$, we mean a $(\Pi_U, \mathcal{B}(\Pi_U))$-valued random variable, i.e., a map $\mathbf{p} : \Omega \ni \omega \to p(\omega) \in \Pi_U$, which is $\mathcal{F}/\mathcal{B}(\Pi_U)$-measurable, defined on a probability space $(\Omega, \mathcal{F}, P)$. A point process $\mathbf{p}$ is called *σ-finite* if there exist $A_n \in \mathcal{B}_U$, $n = 1, 2, \dots$, such that $A_n \subset A_{n+1}$, $\bigcup_n A_n = U$ and $N_p((0,t] \times A_n) < \infty$ a.s. for all $t > 0$ and n. A general theory of point processes is developed in the framework of semimartingale theory (cf., e.g., [7,16,18]). In this paper, we treat mainly the notion of *Poisson point processes*.

Definition 2.1. A point process $\mathbf{p}$ is called a *stationary Poisson point process* if the following holds:

$$E\left(\exp\left\{-\sum_{k=1}^{l} \lambda_k N_p((s,t] \times A_k)\right\} | \mathcal{F}_s^p\right) = \exp\left\{-(t-s)\sum_{k=1}^{l} n(A_k)(1 - \mathrm{e}^{-\lambda_k})\right\} \tag{2.2}$$

for every $t > s \ge 0$, $\lambda_k > 0$ and $A_k \in \mathcal{B}_U$ such that $\{A_k\}$ are mutually disjoint, where n is a positive measure on $(U, \mathcal{B}_U)$ defined by

$$n(A) = E\{N_p((0,1] \times A)\}, \quad A \in \mathcal{B}_U,$$

and $\mathcal{F}_s^p$ is the sub-σ-field of $\mathcal{F}$ generated by the family $\{(r, \mathbf{p}(r)); r \in \mathbf{D_p}, r \le s\}$ of random points in $(0, s] \times U$.

Thus, the law of a stationary Poisson point process $\mathbf{p}$ is uniquely determined by the measure n. We call n the *characteristic measure* of $\mathbf{p}$.

When the underlying probability space is endowed with a filtration $(\mathcal{F}_t)$, a point process $\mathbf{p}$ is called an $(\mathcal{F}_t)$-*stationary Poisson point process* if $N_p((0, t] \times A)$ is $\mathcal{F}_t$-measurable for every $t > 0$ and $A \in \mathcal{B}_U$, and (2.2) holds with $\mathcal{F}_s^p$ replaced by $\mathcal{F}_s$.

Itô [12] characterized a stationary Poisson point process as a point process having the following two properties:

(i) It is *stationary* in the sense that, for every $t > 0$, $\theta_t(\mathbf{p}) \stackrel{d}{=} \mathbf{p}$, where $\theta_t(\mathbf{p})$ is the point process defined by

$$\mathbf{D}_{\theta_t(\mathbf{p})} = \{s; t + s \in \mathbf{D_p}\} \quad \text{and} \quad \theta_t(\mathbf{p})(s) = \mathbf{p}(t + s), \quad s \in \mathbf{D}_{\theta_t(\mathbf{p})}.$$

(ii) It is a *renewal* in the sense that, for every $t > 0$, the point process $\theta_t(\mathbf{p})$ is independent of the family $\{(s, \mathbf{p}(s)); s \in \mathbf{D_p}, s \le t\}$ of random points in $(0, t] \times U$.

A characterization of an $(\mathcal{F}_t)$-stationary Poisson point process in the semimartingale framework is as follows (cf. e.g. [7,16]): An $(\mathcal{F}_t)$-adapted point process is an $(\mathcal{F}_t)$-stationary Poisson point process if and only if its *compensator* $\widehat{N}_p((0, t] \times A)$, $A \in \mathcal{B}_U$, is given by the deterministic measure $t \cdot n(A)$.

A stationary Poisson point process is σ-finite if and only if its characteristic measure n is a σ-finite measure on $(U, \mathcal{B}_U)$. In the following, we treat only σ-finite stationary Poisson point processes.

The most basic existence theorem is stated as follows: *Given a σ-finite measure n on U, there exists a (law unique) stationary Poisson point process* $\mathbf{p}$ *with characteristic measure n*. A standard construction of $\mathbf{p}$ from families of i.i.d. exponential times and i.i.d. U-valued random variables is explained in, e.g., [12,11,7].

Consider a point process $\mathbf{p}$ on a standard Borel space U. Let V be another standard Borel space and suppose we are given a Borel map $T : U \to V$. Then a point process $\mathbf{q}$ on V is defined by setting $\mathbf{D_q} = \mathbf{D_p}$ and $\mathbf{q}(t) = T(\mathbf{p}(t))$ for $t \in \mathbf{D_p}$. We define $\mathbf{q} = T(\mathbf{p})$ and call it the *image point process* of $\mathbf{p}$ under the map T.

When $\mathbf{p}$ is a stationary Poisson point process with characteristic measure n and if we suppose that the image measure $T(n) := n \circ T^{-1}$ of n under the map T is a σ-finite measure on $(V, \mathcal{B}_V)$, then it is obvious that $T(\mathbf{p})$ is a stationary Poisson point process on V with characteristic measure $T(n)$.

Now we review the important notion of *stochastic integrals* based on Poisson point processes. This is usually treated in the framework of semimartingale theory, and so we assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is endowed with a filtration $(\mathcal{F}_t)$.

Let $\mathbf{p}$ be an $(\mathcal{F}_t)$-stationary Poisson point process with values in a standard Borel space U and with characteristic measure n. A (real or $\mathbf{R}^n$-valued) function $f(t, u, \omega)$ defined on $[0, \infty) \times U \times \Omega$ is called *predictable* if the map $f : ([0, \infty) \times \Omega) \times U \ni ((t, \omega), u) \mapsto f(t, u, \omega)$ is $\mathcal{P} \otimes \mathcal{B}_U$-measurable, where $\mathcal{P}$ is the $(\mathcal{F}_t)$-predictable σ-field on $[0, \infty) \times \Omega$.

First, if a predictable function $f(s, u, \omega)$ satisfies the condition

$$\int_{(0,t]\times U} |f(s, u, \cdot)| N_p(\mathrm{d}s, \mathrm{d}u) < \infty \quad \text{a.s., for every } t > 0, \tag{2.3}$$

then we have, obviously,

$$\int_{(0,t]\times U} f(s,u,\cdot)N_p(\mathrm{d}s,\mathrm{d}u) = \sum_{s\in\mathbf{D_p},s\le t} f(s,\mathbf{p}(s),\cdot), \tag{2.4}$$

the sum in the right-hand side (RHS) being absolutely convergent almost surely. If $f(s,u,\omega)$ satisfies the condition

$$\int_0^t \mathrm{d}s \int_U E\left(|f(s,u,\cdot)|\right) n(\mathrm{d}u) < \infty, \quad \text{for every } t>0, \tag{2.5}$$

then (2.3) holds and we have the identity

$$E\left(\int_{(0,t]\times U} f(s,u,\cdot)N_p(\mathrm{d}s,\mathrm{d}u)\right) = \int_0^t \mathrm{d}s \int_U E(f(s,u,\cdot))n(\mathrm{d}u). \tag{2.6}$$

Next, if a predictable function $f(s,u,\omega)$ satisfies the condition

$$\int_0^t \mathrm{d}s \int_U E\left(|f(s,u,\cdot)|^2\right) n(\mathrm{d}u) < \infty, \quad \text{for every } t>0, \tag{2.7}$$

then the stochastic integral $\int_{(0,t]\times U} f(s,u,\cdot)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u)$, which is also often denoted by $\int_0^{t+}\int_U f(s,u,\cdot)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u)$, is defined as a square-integrable $(\mathcal{F}_t)$-martingale: here, we define, formally, $\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u) = N_p(\mathrm{d}s,\mathrm{d}u) - \mathrm{d}s\, n(\mathrm{d}u)$ and the previous stochastic integral is defined as a limit of compensated sums (cf. e.g. [7] for details). If $g(s,u,\omega)$ has the same property as $f(s,u,\omega)$, then we have the identity

$$\begin{aligned} &E\left(\int_{(0,t]\times U} f(s,u,\cdot)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u)\cdot\int_{(0,t]\times U} g(s,u,\cdot)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u)\right)\\ &\quad = \int_0^t \mathrm{d}s \int_U E[f(s,u,\cdot)\cdot g(s,u,\cdot)]n(\mathrm{d}u). \end{aligned} \tag{2.8}$$

(In the case of $\mathbf{R}^n$-valued integrands, $\cdot$ is understood as the inner product.)

Let $U_0 \in \mathcal{B}_U$ be such that

$$n(U\setminus U_0) < \infty. \tag{2.9}$$

Then, for an $\mathbf{R}^n$-valued predictable function $f(s,u,\omega)$, the function $f(s,u,\omega)\cdot\mathbf{1}_{U\setminus U_0}$ satisfies condition (2.3) because $N_p((0,t]\times(U\setminus U_0))$ is finite a.s. for every $t>0$. Suppose further that $f(s,u,\omega)$ satisfies the condition

$$\int_0^t \mathrm{d}s \int_{U_0} E\left(|f(s,u,\cdot)|^2\right) n(\mathrm{d}u) < \infty, \quad \text{for every } t>0. \tag{2.10}$$

Under these conditions, both

$$\int_{(0,t]\times U} f(s,u,\cdot)\mathbf{1}_{U\setminus U_0}(u)N_p(\mathrm{d}s,\mathrm{d}u)$$

and

$$\int_{(0,t]\times U} f(s,u,\cdot)\mathbf{1}_{U_0}(u)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u)$$

are well-defined, and so an n-dimensional $(\mathcal{F}_t)$-semimartingale $\xi(t)$ is defined by

$$\xi(t) = \int_{(0,t]\times U} f(s,u,\cdot)\mathbf{1}_{U\setminus U_0}(u)N_p(\mathrm{d}s,\mathrm{d}u) + \int_{(0,t]\times U} f(s,u,\cdot)\mathbf{1}_{U_0}(u)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u). \tag{2.11}$$

We shall meet many examples of semimartingales represented in this form in Section 4.

Example 2.1 (*Canonical and Non-canonical Lévy–Itô Representations of Lévy Processes*). In the case when $f(s,u,\cdot)$ is given by $f(s,u,\cdot) = \phi(u)$, where ϕ is a Borel map $\phi : U \ni u \to \phi(u) \in \mathbf{R}^n$ satisfying

$$\int_{U_0} |\phi(u)|^2 n(\mathrm{d}u) < \infty,$$

the above $(\mathcal{F}_t)$-semimartingale $\xi(t)$ given by (2.11) is a stationary $(\mathcal{F}_t)$-Lévy process in the sense that, for every $t > s \geq 0$, the increment $\xi(t) - \xi(s)$ is independent of $\mathcal{F}_s$ and its law depends only on $t - s$. Thus, the very definition

$$\xi(t) = \int_{(0,t]\times U} \phi(u)\mathbf{1}_{U\setminus U_0}(u)N_p(\mathrm{d}s,\mathrm{d}u) + \int_{(0,t]\times U} \phi(u)\mathbf{1}_{U_0}(u)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}u) \tag{2.12}$$

is a representation of the Lévy–Itô type for $\xi(t)$. Let $\widetilde{U} = \{u \in U; |\phi(u)| > 0\}$. If $\mathbf{q}$ is the image point process, under the map ϕ, of the point process $\mathbf{p}$ restricted to $\widetilde{U}$, then $\mathbf{q}$ is an $(\mathcal{F}_t)$-stationary Poisson point process with values in $\mathbf{R}^n \setminus \{0\}$ and with characteristic measure ν, which is the image measure, under the map ϕ, of the measure n restricted to $\widetilde{U}$. It is easy to verify that $\int_{\mathbf{R}^n\setminus\{0\}}(1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$. It is also easy to deduce the following formula:

$$\xi(t) = \int_{(0,t]\times(\mathbf{R}^n\setminus\{0\})} x\mathbf{1}_{\{|x|>1\}}N_q(\mathrm{d}s,\mathrm{d}x) + \int_{(0,t]\times(\mathbf{R}^n\setminus\{0\})} x\mathbf{1}_{\{|x|\leq 1\}}\widetilde{N}_q(\mathrm{d}s,\mathrm{d}x) + bt \tag{2.13}$$

where $b \in \mathbf{R}^n$ is given by

$$b = \int_{(U\setminus U_0)\cap\{|\phi(u)|\leq 1\}} \phi(u)n(\mathrm{d}u) - \int_{U_0\cap\{|\phi(u)|>1\}} \phi(u)n(\mathrm{d}u). \tag{2.14}$$

This is the canonical Lévy–Itô representation of the Lévy process $\xi(t)$ and the Lévy measure in the Lévy–Khinchin canonical form of the law of $\xi(1)$ coincides with the characteristic measure ν of the point process $\mathbf{q}$.

When $f(s,u,\cdot)$ is not a function of u only, the semimartingale $\xi(t)$ given by (2.11) is no longer a Lévy process. An important special case is when (2.11) defines a stochastic differential equation. An important result along these lines is summarized in the following theorem which will play a crucial role in Section 4. We set up, as above, a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$ and suppose an $(\mathcal{F}_t)$-stationary Poisson point process $\mathbf{p}$ with values in a standard Borel space U and with characteristic measure n to be given. Let $U_0 \in \mathcal{B}_U$ be given as above such that

$$n(U \setminus U_0) < \infty. \tag{2.15}$$

Suppose we are given the following:

(i) A Borel map $\sigma(x) = (\sigma_k^i(x))$: $\mathbf{R}^n \ni x \mapsto \sigma(x) \in \mathbf{R}^n \otimes \mathbf{R}^r$.
(ii) A Borel map $b(x) = (b^i(x))$: $\mathbf{R}^n \ni x \mapsto b(x) \in \mathbf{R}^n$.
(iii) A Borel map $f(x,u) = (f^i(x,u))$: $\mathbf{R}^n \times U \ni (x,u) \mapsto f(x,u) \in \mathbf{R}^n$.

We set $\sigma_k(x) = (\sigma_k^i(x))_{i=1}^n \in \mathbf{R}^n$ and assume that these functions satisfy (for some constant $K > 0$)

$$\sum_{k=1}^r |\sigma_k(x)|^2 + |b(x)|^2 + \int_{U_0} |f(x,u)|^2 n(\mathrm{d}u) \le K(1+|x|^2) \tag{2.16}$$

and

$$\sum_{k=1}^r |\sigma_k(x) - \sigma_k(y)|^2 + |b(x) - b(y)|^2 + \int_{U_0} |f(x,u) - f(y,u)|^2 n(\mathrm{d}u) \le K|x-y|^2. \tag{2.17}$$

Theorem 2.1 (*Cf.* [7]). *Given an r-dimensional $(\mathcal{F}_t)$-Brownian motion $(B^k(t))$ and an $\mathcal{F}_0$-measurable and $\mathbf{R}^n$-valued random variable ξ, the following SDE for an $\mathbf{R}^n$-valued $(\mathcal{F}_t)$-semimartingale $(\xi(t))$ such that $\xi(0) = \xi$ has a pathwise unique solution:*

$$\begin{aligned}\xi(t) = \xi(0) &+ \sum_{k=1}^r \int_0^t \sigma_k(\xi(s))\mathrm{d}B^k(s) + \int_0^t b(\xi(s))\mathrm{d}s \\ &+ \int_0^{t+}\int_U f(\xi(s-),u)\cdot \mathbf{1}_{U\setminus U_0}(u)\, N_p(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^{t+}\int_U f(\xi(s-),u)\cdot \mathbf{1}_{U_0}(u)\, \widetilde{N}_p(\mathrm{d}s,\mathrm{d}u).\end{aligned} \tag{2.18}$$

Finally, we give an example of Poisson point processes with values in a function space. As was explained in the Introduction, the study of such point processes has been a main motivation of Itô's works in [11,12]. Also, this example plays a fundamental role in the following sections.

Example 2.2 (*Excursion Point Process of Brownian Excursions*). Let $B = (B(t))$ be a reflecting Brownian motion on the half-line $[0,\infty)$, with $B(0) = 0$, and let $l(t)$ be the local time at 0 of B, that is,

$$l(t) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[0,\varepsilon)}(B(s))\mathrm{d}s.$$

We introduce the following notation for path spaces:

$$\mathcal{W}_0 = \Big\{\omega : [0,\infty) \ni t \mapsto \omega(t) \in [0,\infty);\ \text{continuous}, \omega(0) = 0,\ 0 < \exists\sigma(\omega) < \infty \\ \text{such that } \omega(t) > 0 \text{ if } t \in (0,\sigma(\omega)) \text{ and } \omega(t) = 0 \text{ if } t \ge \sigma(\omega)\Big\}, \tag{2.19}$$

and, more generally, for $x \in [0,\infty)$,

$$\mathcal{W}_x = \Big\{\omega : [0,\infty) \ni t \mapsto \omega(t) \in [0,\infty);\ \text{continuous}, \omega(0) = x,\ 0 < \exists\sigma(\omega) < \infty \\ \text{such that } \omega(t) > 0 \text{ if } t \in (0,\sigma(\omega)) \text{ and } \omega(t) = 0 \text{ if } t \ge \sigma(\omega)\Big\}. \tag{2.20}$$

These spaces are all standard Borel spaces (cf. [21], p. 413).

Let $l^{-1}(t) := A(t)$ be the right-continuous inverse of $t \mapsto l(t)$. Then $t \mapsto A(t)$ is strictly increasing and $\lim_{t\to\infty} A(t) = \infty$ almost surely. Then a random set in $(0, \infty)$ is defined by $\mathbf{D} = \{s > 0; A(s) > A(s-)\}$. For $s \in \mathbf{D}$, we define a path $p_s \in \mathcal{W}_0$ by setting

$$p_s(t) = \begin{cases} B(A(s-)+t) - B(A(s-)), & \text{if } 0 \le t < A(s) - A(s-), \\ 0, & \text{if } t \ge A(s) - A(s-), \end{cases} \tag{2.21}$$

so $\sigma(p_s) = A(s) - A(s-)$.

By setting $\mathbf{D_p} = \mathbf{D}$ and $\mathbf{p}(s) = p_s$ for $s \in \mathbf{D_p}$, we have a point process $\mathbf{p}$ with values in the path space $\mathcal{W}_0$ and Itô proved that this is a stationary Poisson point process. It is called the *excursion point process of Brownian positive excursions*. The excursion point process of Brownian negative excursions is defined similarly, from a reflecting Brownian motion on the negative half-line $(-\infty, 0]$.

For the excursion point process of Brownian positive excursions $\mathbf{p}$, its characteristic measure n_+ is a σ-finite measure on $\mathcal{W}_0$, which we call the *Brownian (positive) excursion measure*. This can be described in several different ways: a standard one is to define it as a Markovian measure associated with the law of absorbing Brownian motion on $(0, \infty)$ having an entrance law from the origin. (An entrance law is determined up to a multiplicative constant, which corresponds to a normalization of the local time.)

We give here a description in terms of BES(3)-diffusion process: BES(3) (denote it as $\mathrm{BES}^a(3)$ when it starts at a) is a conservative diffusion on the half-line $[0, \infty)$ with the generator $\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)$ and the boundary 0 as an immediate entrance state. The transition probability density $p(t, x, y)$, $t > 0$, $x, y \in [0, \infty)$, with respect to the measure $m(\mathrm{d}y) = y^2\mathrm{d}y$ is given by

$$p(t,x,y) = p(t,y,x) = \begin{cases} \dfrac{1}{xy}(g(t, x-y) - g(t, x+y)), & \text{if } x, y > 0 \\ \dfrac{2}{t} g(t,x), & \text{if } x \ge 0 \text{ and } y = 0, \end{cases}$$

where $g(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$.

Since $p(t, x, y)$ is strictly positive and continuous on $(0, \infty) \times [0, \infty) \times [0, \infty)$, we can precisely define, for each fixed $T > 0$, the pinned or tied down $\mathrm{BES}^0(3)$-diffusion $X_0^T = (X(t))$, $0 \le t \le T$, with the condition $X(0) = 0$ and $X(T) = 0$ almost surely. It is obtained from $\mathrm{BES}^0(3)$, $(X(t))$, by a Girsanov–Maruyama transformation (a change of the drift) defined by the following positive martingale $(M(t))$ with expectation 1:

$$M(t) = \frac{p(T-t, X(t), 0)}{p(T, 0, 0)}, \quad 0 \le t < T.$$

The law of X_0^T defines a probability measure on $\mathcal{W}_0 \cap \{\sigma(\omega) = T\}$, which we denote by P_0^T. Then, we have the following representation of the positive Brownian excursion measure (cf. e.g. [7], p. 125): For $A \in \mathcal{B}(\mathcal{W}_0)$,

$$n_+(A) = \int_0^\infty P_0^T(A \cap \{\omega; \sigma(\omega) = T\}) \frac{\mathrm{d}T}{\sqrt{2\pi T^3}}.$$

There is another beautiful description of n_+ due to Williams using two independent $\mathrm{BES}^0(3)$ (cf. e.g. [7], p. 144): Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be mutually independent $\mathrm{BES}^0(3)$'s and, for $a > 0$, let $\sigma_i^a = \inf\{t; X_i(t) = a\}$, for $i = 1, 2$. Set $\sigma^a = \sigma_1^a + \sigma_2^a$ and define $\{Y(t)\}$ by

$$Y(t) = \begin{cases} X_1(t), & \text{if } 0 \le t < \sigma_1^a, \\ X_2(\sigma^a - t), & \text{if } \sigma_1^a \le t < \sigma^a, \\ 0, & \text{if } t \ge \sigma^a. \end{cases}$$

Then, the path $t \mapsto Y(t)$ is in $\mathcal{W}_0$. We denote its law on $\mathcal{W}_0$ by R_a. Then it holds that, for $A \in \mathcal{B}(\mathcal{W}_0)$,

$$n_+(A) = \int_0^\infty R_a(A) \frac{\mathrm{d}a}{a^2}.$$

Starting from the Brownian excursion measure n_+, which is an infinite but σ-finite measure on $\mathcal{W}_0$, we have a stationary Poisson point process $\mathbf{p}$ on $\mathcal{W}_0$. Then, from $\mathbf{p}$, we can recover the path function of the reflecting Brownian motion $t \mapsto B(t)$ and its local time $l(t)$ at 0 in the following way (cf. e.g., [7], Chap. III, Subsection 4.3).

First, set

$$A(t) = \int_{(0,t]\times\mathcal{W}_0} \sigma(\omega) N_p(\mathrm{d}s, \mathrm{d}\omega) = \sum_{s \in \mathbf{D_p}, s \le t} \sigma(\mathbf{p}(s)).$$

As we saw in Example 2.1 above, this is a stationary Lévy process which is obviously increasing, so it is a subordinator. By rewriting it in canonical Lévy–Itô form as we did above, we can identify it as a stable subordinator of exponent 1/2. $t \mapsto A(t)$ is strictly increasing and $\lim_{t\to\infty} A(t) = \infty$ a.s.. Then, setting $A(0-) = 0$, we see that, for every $t \in [0, \infty)$, there is a unique $s \in [0, \infty)$, denoted by $s = l(t)$, such that $A(s-) \le t \le A(s)$. If $A(s-) < A(s)$, this implies that $s \in \mathbf{D_p}$ and we set $B(t) = [\mathbf{p}(s)](t - A((s-)))$. If $A(s-) = A(s)$, we set $B(t) = 0$. We can identify $t \mapsto B(t)$ and $t \mapsto l(t)$ so defined with the reflecting Brownian sample function and its local time at 0.

From a Poisson point process of Brownian positive excursions $\mathbf{p}^+$ and a Poisson point process of Brownian negative excursions $\mathbf{p}^-$ which are mutually independent, their sum $\mathbf{p} = \mathbf{p}^+ + \mathbf{p}^-$, defined in an obvious way, is a Poisson point process with values in the disjoint union $\mathcal{W}_0 \cup \{-\mathcal{W}_0\}$ and with the domain given by the disjoint union of $\mathbf{D}_{\mathbf{p}^+}$ and $\mathbf{D}_{\mathbf{p}^-}$. Then a sample function of Brownian motion on the real line starting at the origin and its local time at the origin can be defined from the point process $\mathbf{p}$ by the same procedure as above. If, in this construction, the point processes $\mathbf{p}^+$ and $\mathbf{p}^-$ are modified so that, for a given constant p with $0 < p < 1$, their characteristic measures are replaced by $p \cdot n_+$ and $(1 - p) \cdot n_-$, respectively, then we obtain a sample function of the *skew Brownian motion* with the skew parameter p.

3. Excursion point processes associated with diffusion processes on the half-line satisfying Feller's boundary conditions

3.1

Here we give a typical example of Itô's excursion point process and thereby the construction of sample functions in the case of diffusion processes on the half-line $[0, \infty)$ satisfying Feller's boundary conditions. First, we consider the case when the diffusions are Brownian motions.

Let $\mathbf{X} = (X_t, P_x)$, $x \in [0, \infty)$, be a diffusion process (i.e., a strong Markov process with continuous paths) on the half-interval $[0, \infty)$ which behaves as a Brownian motion inside $(0, \infty)$; however, we allow a discontinuity of the sample path when it is on the boundary $x = 0$, and so a jump may take place from the boundary to inside. Also, we attach an extra state point Δ as the terminal point and allow some extinction (i.e., the jump to the terminal point Δ) when the process is on the boundary. As is shown in p. 194 of [14], such a diffusion is described by the following parameters: *Three nonnegative numbers* p_1, p_2, p_3 *and a nonnegative Borel measure* $p_4(\mathrm{d}x)$ *on* $(0,\infty)$ subject to the normalization

$$p_1 + p_2 + p_3 + \int_{(0,\infty)} (1 \wedge x) p_4(\mathrm{d}x) = 1 \tag{3.1}$$

and also having the property that

$$\text{if } p_1 < 1 \quad \text{and} \quad p_2 + p_3 = 0, \quad \text{then } p_4((0,1]) = \infty. \tag{3.2}$$

So the local generator $\mathcal{G}$ inside $(0,\infty)$ is given by $\mathcal{G}u(x) = \frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x)$ and Feller's boundary condition is given by

$$p_1 u(0) + p_3(\mathcal{G}u)(0) = p_2 \frac{\mathrm{d}u}{\mathrm{d}x}(0) + \int_{(0,\infty)} (u(x) - u(0)) p_4(\mathrm{d}x) \tag{3.3}$$

for $\mathcal{C}^2$ functions $u(x)$ on $[0,\infty)$ which are bounded together with the derivatives. We always extend u to $[0,\infty) \cup \{\Delta\}$ so that $u(\Delta) = 0$. We denote by $\mathcal{C}_b^2([0,\infty))$ the class of all such functions. Set, for $u \in \mathcal{C}_b^2([0,\infty))$,

$$(Lu)(0) = -p_1 u(0) + p_2 \frac{\mathrm{d}u}{\mathrm{d}x}(0) - p_3(\mathcal{G}u)(0) + \int_{(0,\infty)} (u(x) - u(0)) p_4(\mathrm{d}x). \tag{3.4}$$

One way of stating that the local generator $\mathcal{G}$ and Feller's boundary condition (3.3) determine the diffusion $\mathbf{X}$ is the following: $\mathbf{X}$ is a stochastic process on $[0,\infty) \cup \{\Delta\}$ with Δ as a trap uniquely characterized by the following property: for any $u \in \mathcal{C}_b^2([0,\infty))$, the process $t \mapsto [u(X(t)) - u(X(0))]$ is a semimartingale with the semimartingale decomposition given by

$$u(X(t)) - u(X(0)) = \text{a martingale} + \int_0^t (\mathcal{G}u)(X(s))\mathrm{d}s + (Lu)(0)\phi(t),$$

where $\phi(t)$ is a continuous increasing process such that

$$\phi(t) = \int_0^t \mathbf{1}_{\{0\}}(X(s))\mathrm{d}\phi(s) \quad \text{and} \quad \int_0^t \mathbf{1}_{\{0\}}(X(s))\mathrm{d}s = p_3\phi(t). \tag{3.5}$$

The constant p_1 and the measure $p_4(\mathrm{d}x)$ can be put together to define a nonnegative Borel measure $p_5(\mathrm{d}x)$ on the extended half-line $(0,\infty) \cup \{\Delta\}$ so that

$$\text{the restriction } p_5(\mathrm{d}x)|_{(0,\infty)} = p_4(\mathrm{d}x) \quad \text{and} \quad \mathbf{1}_{\{\Delta\}}(x) p_5(\mathrm{d}x) = p_1 \delta_{\{\Delta\}}(\mathrm{d}x), \tag{3.6}$$

where $\delta_{\{\Delta\}}(\mathrm{d}x)$ is the unit measure at the point Δ. Obviously, conversely p_1 and p_4 can be recovered from p_5. We can also rewrite (3.3) in the form

$$p_3(\mathcal{G}u)(0) = p_2 \frac{\mathrm{d}u}{\mathrm{d}x}(0) + \int_{(0,\infty)\cup\{\Delta\}} (u(x) - u(0)) p_5(\mathrm{d}x), \quad x \in [0,\infty) \tag{3.7}$$

for any function $u(x) \in \mathcal{C}_b^2([0,\infty))$. The rewriting (3.7) of (3.3) corresponds, in a probabilistic language, to the identification of the absorption (i.e., the extinction or killing) with the jump to the terminal state Δ. Note also that, when $p_1 = 1$, the point 0 and Δ must be identified so that our diffusion $\mathbf{X}$ is strong Markov and right-continuous.

We are now going to construct the path functions of $\mathbf{X}$ by Itô's method; for a given starting point $a \in [0,\infty)$, we construct the path function $X^a(t)$ of $\mathbf{X}$ such that $X^a(0) = a$. *We disregard the trivial case* $p_1 = 1$; in this case, $\mathbf{X}$ starting from inside is a Brownian motion before it hits the boundary 0 and is killed at once on hitting 0, so we need to identify 0 and Δ. Thus, we assume that $p_1 < 1$ from now on.

First of all, we take a sufficiently large probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$ on which we can realize the following objects:

(i) A filtration $(\mathcal{G}_t)$ on Ω such that $\mathcal{G}_t \subset \mathcal{F}_0$ for every $t > 0$ and a one-dimensional $(\mathcal{G}_t)$-Brownian motion $\widehat{\mathbf{B}}^a = (\widehat{B}^a(t))$ such that $\widehat{B}^a(0) = a$.
(ii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_1$ on $\mathcal{W}_0$ with characteristic measure n_+, i.e. a Poisson point process of Brownian positive excursions (cf. Example 2.2).
(iii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_2$ with values in the product space $[0,\infty) \times W_1$ with characteristic measure given by the product measure $p_4(\mathrm{d}x) \times P_1$, P_1 being the Wiener measure on $W_1 := \{w \in \mathcal{C}([0,\infty) \to \mathbf{R}); w(0) = 1\}$.
(iv) An exponential time $\mathbf{e}$ such that $P(\mathbf{e} > t) = \mathrm{e}^{-p_1 t}$, $t \geq 0$, which is independent of $\widehat{\mathbf{B}}^a$, $\mathbf{p}_1$ and $\mathbf{p}_2$.

Remark 3.1. In the same spirit as combining p_1 and p_4 to obtain p_5, the above point process $\mathbf{p}_2$ and the exponential time $\mathbf{e}$ can be combined together to obtain a point process $\mathbf{p}_3$ with values in the product space $([0,\infty) \cup \{\Delta\}) \times W_0$. Though such an approach may be more elegant mathematically, we shall follow a more elementary and standard route here.

We recall the notation for the spaces of paths of excursions. For $x \in [0,\infty)$,

$$\begin{aligned}\mathcal{W}_x = \{&\omega : [0,\infty) \ni t \mapsto \omega(t) \in [0,\infty); \text{continuous}, \ \omega(0) = x, 0 < \exists\sigma(\omega) < \infty \\ &\text{such that } \omega(t) > 0 \text{ if } t \in (0,\sigma(\omega)) \text{ and } \omega(t) = 0 \text{ if } t \geq \sigma(\omega)\}.\end{aligned} \tag{3.8}$$

Note that, if $x > 0$, then for $\omega \in \mathcal{W}_x$, $\sigma(\omega) = \min\{s \geq 0 | \omega(s) = 0\}$; while, for $\omega \in \mathcal{W}_0$, $\omega(0) = 0$ and $\sigma(\omega) = \min\{s > 0 | \omega(s) = 0\} > 0$. Also, we introduce the following notation:

$$\mathcal{W}_{++} = \bigcup_{x>0} \mathcal{W}_x.$$

We introduce the following two operations on path spaces; here, $\mathbf{0}$ denotes the constant zero path: $\mathbf{0}(t) \equiv 0$.

(i) For each $c \geq 0$, a map $T_1^{(c)} : \mathcal{W}_0 \ni \omega \mapsto T_1^{(c)}(\omega) \in \mathcal{W}_0 \cup \{\mathbf{0}\}$ is defined by

$$T_1^{(c)}(\omega)(t) = \begin{cases} c\omega\left(\dfrac{t}{c^2}\right), & \text{if } c > 0 \\ 0, & \text{if } c = 0. \end{cases} \tag{3.9}$$

(ii) A map $T_2 : (0,\infty) \times W_1 \ni (x,w) \mapsto T_2((x,w)) \in \mathcal{W}_{++}$ is defined by

$$T_2((x,w))(t) = \begin{cases} x \cdot w\left(\dfrac{t}{x^2}\right), & \text{if } 0 \leq t < x^2 m_0(w) \\ 0, & \text{if } t \geq x^2 m_0(w) \end{cases} \tag{3.10}$$

where $m_0(w) = \inf\{s > 0 | w(s) = 0\}$ for $w \in W_1$. (Note that $w(0) = 1$ if $w \in W_1$.)

Let $\mathbf{q}_1$ be the image under the map $T_1^{(p_2)}$ of the point process $\mathbf{p}_1$ and $\mathbf{q}_2$ be the image under the map T_2 of the point process $\mathbf{p}_2$, so that $\mathbf{q}_1$ and $\mathbf{q}_2$ are stationary $(\mathcal{F}_t)$-Poisson point processes with values in $\mathcal{W}_0$ and in $\mathcal{W}_{++}$, respectively, and with characteristic measures $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively, given by

$$\mathbf{m}_1(A) = p_2 n_+(A), \quad A \in \mathcal{B}(\mathcal{W}_0)$$

and

$$\mathbf{m}_2(A) = \int_{(0,\infty)} P_1\left(\left\{w; \left[s \mapsto x \cdot w\left(\frac{s}{x^2} \wedge m_0(w)\right)\right] \in A\right\}\right) p_4(\mathrm{d}x), \quad A \in \mathcal{B}(\mathcal{W}_{++}).$$

Then the images $\sigma(\mathbf{q}_1)$ and $\sigma(\mathbf{q}_2)$ of $\mathbf{q}_1$ and $\mathbf{q}_2$, respectively, under the map $\sigma : \mathcal{W}_0 \cup \mathcal{W}_{++} \ni \omega \mapsto \sigma(\omega) \in (0,\infty)$ are stationary $(\mathcal{F}_t)$-Poisson point processes with values in $(0,\infty)$, which we denote by π_1 and π_2, respectively. Their characteristic measures, denoted by $\nu_1(\mathrm{d}x)$ and $\nu_2(\mathrm{d}x)$, respectively, are given by

$$\nu_1(\mathrm{d}x) = p_2 \frac{\mathrm{d}x}{\sqrt{2\pi x^3}}, \qquad \nu_2(\mathrm{d}x) = \left(\int_{(0,\infty)} \frac{y}{\sqrt{2\pi x^3}} \mathrm{e}^{-y^2/(2x)} p_4(\mathrm{d}y)\right) \mathrm{d}x.$$

Let $m_0^{\widehat{B}^a} = \min\{s \geq 0 | \widehat{B}^a(s) = 0\}$. We set

$$\begin{aligned} A(t) &= m_0^{\widehat{B}^a} + \sum_{s \in \mathbf{D}_{\mathbf{q}_1}; s \leq t} \sigma(\mathbf{q}_1(s)) + \sum_{s \in \mathbf{D}_{\mathbf{q}_2}; s \leq t} \sigma(\mathbf{q}_2(s)) + p_3 t \\ &= m_0^{\widehat{B}^a} + \int_0^{t+} \int_{\mathcal{W}_0} \sigma(\omega) N_{\mathbf{q}_1}(\mathrm{d}s, \mathrm{d}\omega) + \int_0^{t+} \int_{\mathcal{W}_{++}} \sigma(\omega) N_{\mathbf{q}_2}(\mathrm{d}s, \mathrm{d}\omega) + p_3 t \\ &= m_0^{\widehat{B}^a} + \int_0^{t+} \int_{(0,\infty)} x N_{\pi_1}(\mathrm{d}s, \mathrm{d}x) + \int_0^{t+} \int_{(0,\infty)} x N_{\pi_2}(\mathrm{d}s, \mathrm{d}x) + p_3 t. \end{aligned} \tag{3.11}$$

It is easy to see that $\int_{(0,\infty)} (1 \wedge x)\nu_1(\mathrm{d}x) + \int_{(0,\infty)} (1 \wedge x)\nu_2(\mathrm{d}x) < \infty$ and hence (3.11) defines an $(\mathcal{F}_t)$-subordinator (i.e., a right-continuous, increasing $(\mathcal{F}_t)$-adapted process $\{A(t)\}$ such that $A(t_2) - A(t_1)$ is independent of $\mathcal{F}_{t_1}$ for every $0 \leq t_1 < t_2$). By our assumptions (3.1) and (3.2), we easily see that, with probability 1, $t \mapsto A(t)$ is strictly increasing and $\lim_{t\to\infty} A(t) = \infty$.

Now we define the path $t \in [0,\infty) \mapsto X^a(t) \in [0,\infty) \cup \{\Delta\}$ as follows:
If $0 \leq t \leq m_0^{\widehat{B}^a}$, we set $X^a(t) = \widehat{B}^a(t)$.
If $A(\mathbf{e}) > t > m_0^{\widehat{B}^a}$, then there exists a unique $\mathbf{e} \geq s > 0$ such that $A(s-) = A(s) = t$ or $A(s-) \leq t < A(s)$.

In the first case, we set $X^a(t) = 0$.

In the second case, it must happen that either $s \in \mathbf{D}_{\mathbf{q}_1}$ or $s \in \mathbf{D}_{\mathbf{q}_2}$, and exactly one case can occur with probability 1 because $\mathbf{D}_{\mathbf{q}_1} \cap \mathbf{D}_{\mathbf{q}_2} = \emptyset$ with probability 1. If $s \in \mathbf{D}_{\mathbf{q}_1}$, we set $X^a(t) = [\mathbf{q}_1(s)](t - A(s-))$, while if $s \in \mathbf{D}_{\mathbf{q}_2}$, we set $X^a(t) = [\mathbf{q}_2(s)](t - A(s-))$.
Finally, if $t \geq A(\mathbf{e})$, we set $X^a(t) = \Delta$.

In this way, $\{X^a(t)\}$ is completely defined and we can identify it as a Brownian motion on the half-line satisfying Feller's boundary condition (3.3).

Remark 3.2. In the above construction, we have used the scaling property of the Wiener process in (3.9) and (3.10). This seemingly troublesome procedure, which we can spare in the one-dimensional case, will play a crucial role in its multi-dimensional extension, as we discuss in the next section.

3.2

We consider the general case in which the minimal diffusion $\widehat{B}$ is replaced by a more general Feller diffusion process $\widehat{X}$. Itô's method, based on the Poisson point process of excursions, as we have discussed in the case of the process $\widehat{B}$, still works. We shall only remark upon some necessary modifications. For simplicity, we assume that $\widehat{X}$ is given on the half-interval $(0, \infty)$ as a Feller diffusion with generator $\mathcal{G} = \frac{\mathrm{d}}{m(\mathrm{d}x)} \frac{\mathrm{d}}{\mathrm{d}x}$, so the canonical scale is the Euclidean scale: $s(x) = x$, and the speed measure is an everywhere positive Radon measure on $(0, \infty)$. The boundary ∞ cannot be reached from inside and so we consider only the problem concerning the boundary 0.

If $\int_{(0,1)} m\{(x, 1]\}\mathrm{d}x = \infty$, then the boundary 0 is natural and the process $\widehat{X}$, starting from inside, cannot reach 0 in a finite time. So a possible extension of $\widehat{X}$ does not exist.

In the case $\int_{(0,1)} m\{(x, 1]\}\mathrm{d}x < \infty$, the boundary 0 is regular or exit according as $m\{(0, 1]\}$ is finite or infinite. In both cases, the excursion measure $\mathbf{n}_+$ exists as an infinite measure on the path space $\mathcal{W}_0$ and, as was shown by Fitzsimmons and Yano ([4]; cf. also [27]), it can be obtained by a time change from Itô's measure n_+ of Brownian positive excursions. Hence, we have a stationary Poisson point process $\mathbf{p}$ on $\mathcal{W}_0$ with characteristic measure $\mathbf{n}_+$. Let ν_1 be the characteristic measure of the point process $\sigma[\mathbf{p}]$ on $(0, \infty)$ obtained as the image of $\mathbf{p}$ under the map $\sigma : \mathcal{W}_0 \ni \omega \mapsto \sigma(\omega) \in (0, \infty)$. Then we can show that

$$\int_{(0,\infty)} (1 \wedge x)\nu_1(\mathrm{d}x) \text{ is finite or infinite according as the boundary 0 is regular or exit.}$$

Hence, in the regular case, the totality of extensions are described by the same Feller boundary conditions as (3.3) in which, however, we understand that $\mathcal{G}u = \frac{\mathrm{d}}{m(\mathrm{d}x)} \frac{\mathrm{d}u}{\mathrm{d}x}$ and the domain $\mathcal{C}_b^2([0, \infty))$ needs to be modified.

In the exit case, we have, almost surely, $\int_0^{t+} \int_{\mathcal{W}_0} \sigma(\omega) N_{\mathbf{p}}(\mathrm{d}s, \mathrm{d}\omega) = \infty$ for $t > 0$ so we cannot define a subordinator. This fact implies that we must have $p_2 = 0$ in the boundary condition, that is, a continuous entering inside from the boundary is impossible. Entering is possible only by *jumping-in*. The jumping-in measure $p_4(\mathrm{d}x)$ is possible if and only if $\int_0^\infty \left(\left(\int_0^x m\{(y, 1]\}\mathrm{d}y\right) \wedge 1\right) p_4(\mathrm{d}x) < \infty$ ([11]; cf. also [28]).

As we remarked in the Introduction, the study of all possible boundary conditions in this exit boundary case has been a main motivation for Itô in his theory of excursion point processes.

4. The multi-dimensional case: Diffusions with Wentzell's boundary conditions

We shall consider the multi-dimensional extension of the problem which we discussed in the previous section for one-dimensional half-intervals. A diffusion process in a multi-dimensional domain or manifold is described by Kolmogorov's differential equations (i.e., a semielliptic second-order differential operator) and, when the boundary exists, its possible behavior on the boundary is determined by Wentzell's boundary condition (cf. [26]). In the one-dimensional case, the problem was completely settled by Feller and by Itô and McKean, for which we have discussed the construction of path functions in the previous section. After the success of the Itô–McKean theory, the problems concerning Wentzell's boundary condition were studied by many people with various approaches: for example, in an analytical approach, Sato and Ueno [22] laid out a fundamental route and, by following it, Bony, Courrège and Priouret [1] succeeded in constructing diffusions in a very general case; in a probabilistic approach, Ikeda [6] applied Itô's

SDE in a two-dimensional case and Watanabe (cf. [7]) and El Karoui [3] applied the SDE method in a general case.

Here, we shall show that Itô's method based on excursion point processes, as applied above in the one-dimensional case, still works well for this problem; such an approach has been discussed by Watanabe [25] (cf. also [7]), Takanobu [23] and Takanobu and Watanabe [24].

Let $D = (\mathbf{R}^d)^+ = \{x = (x^1, \dots, x^d); x^d \geq 0\}$ be the upper half-space of $\mathbf{R}^d$, $D^\circ = \{x \in D; x^d > 0\}$ be its interior, and so its boundary is a $(d-1)$-dimensional hyperplane given by $\partial D = \{x \in D; x^d = 0\}$. We denote $x \in D$ as $x = (\bar{x}, x^d)$, $\bar{x} = (x^1, \dots, x^{d-1})$, and so $(\bar{x}, 0) \in \partial D$.

Suppose we are given a second-order differential operator $\mathcal{G}$ on D, for $u \in \mathcal{C}_b^2(D)$ (the space of $\mathcal{C}^2$-functions on D, bounded and with bounded derivatives),

$$\mathcal{G}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b^i(x)\frac{\partial u}{\partial x^i}(x), \tag{4.1}$$

and Wentzell's boundary condition:

$$\begin{aligned} Lu((\bar{x},0)) = {} & \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha^{ij}((\bar{x},0))\frac{\partial^2 u}{\partial x^i \partial x^j}((\bar{x},0)) + \sum_{i=1}^{d-1} \beta^i((\bar{x},0))\frac{\partial u}{\partial x^i}((\bar{x},0)) \\ & + \mu((\bar{x},0))\frac{\partial u}{\partial x^d}((\bar{x},0)) - \rho((\bar{x},0)) \cdot (\mathcal{G}u)((\bar{x},0)) \\ & + \int_{\{D\setminus\{0\}\}\cap\{|y|\le 1\}} \left\{ u((\bar{x},0)+y) - u((\bar{x},0)) - \sum_{i=1}^{d-1} y^i \frac{\partial u}{\partial x^i}((\bar{x},0)) \right\} n_{\bar{x}}(\mathrm{d}y) \\ & + \int_{\{D\setminus\{0\}\}\cap\{|y|>1\}} \{u((\bar{x},0)+y) - u((\bar{x},0))\}\, n_{\bar{x}}(\mathrm{d}y), \end{aligned} \tag{4.2}$$

where:

(i) $a^{ij}(x)$ and $b^i(x) i, j = 1, \dots, d$, are bounded Borel-measurable functions on D such that $a^{ij}(x) = a^{ji}(x)$, $\sum_{i,j=1}^d a^{ij}(x)\xi^i\xi^j \geq 0$ for all $\xi \in \mathbf{R}^d$,

(ii) $\alpha^{ij}((\bar{x},0))$, $\beta^i((\bar{x},0))$, $i, j = 1, \dots, d-1$, $\mu((\bar{x},0))$ and $\rho((\bar{x},0))$ are bounded Borel-measurable functions on ∂D such that $\alpha^{ij}((\bar{x},0)) = \alpha^{ji}((\bar{x},0))$, $\sum_{i,j=1}^{d-1} \alpha^{ij}((\bar{x},0))\xi^i\xi^j \geq 0$ for all $\xi \in \mathbf{R}^{d-1}$, $\mu((\bar{x},0)) \geq 0$ and $\rho((\bar{x},0)) \geq 0$,

(iii) for each $(\bar{x},0) \in \partial D$, $n_{\bar{x}}(\mathrm{d}y)$ is a nonnegative Radon measure on $D \setminus \{0\}$ such that

$$\int_{\{D\setminus\{0\}\}\cap\{|y|\le 1\}} \left(|\bar{y}|^2 + y^{\mathrm{d}}\right) n_{\bar{x}}(\mathrm{d}y) + n_{\bar{x}}\left(\{D \setminus \{0\}\} \cap \{|y| > 1\}\right) < \infty, \tag{4.3}$$

and the integral (4.3) is bounded in $\bar{x}$.

Remark 4.1. We often denote a function $f((\bar{x},0))$ on the boundary simply by $f(\bar{x})$, so we write $\alpha^{ij}((\bar{x},0)) = \alpha^{ij}(\bar{x})$, $\beta^i((\bar{x},0)) = \beta^i(\bar{x})$, and so on.

Remark 4.2. We have assumed that the coefficients in $\mathcal{G}$ and L are bounded globally, which may seem too strong. However, by a standard localization argument, more general cases can be reduced to this case and so we do not hesitate to set such assumptions. Also, we treat here only conservative diffusions, although we know well how to treat more general cases; for the one-dimensional case, we treated the general case in the previous section.

Our objective is a diffusion process $\mathbf{X} = (X(t), P_x)$ on D determined by a pair of analytic data $(\mathcal{G}, L)$.

Definition 4.1. By a $(\mathcal{G}, L)$-process $\mathbf{X} = (X(t), P_x)$, we mean a càdlàg process $X(t)$ on D such that, under P_x, $X(0) = x$ a.s., and, for every $u \in \mathcal{C}_b^2(D)$, the process $t \in [0, \infty) \mapsto u(X(t))$ is a semimartingale with the semimartingale decomposition given by

$$u(X(t)) = u(x) + \text{a martingale} + \int_0^t (\mathcal{G}u)(X(s))\mathrm{d}s + \int_0^t (Lu)(X(s))\mathrm{d}\phi(s) \tag{4.4}$$

where $\phi(s)$ is a continuous increasing process such that

$$\int_0^t \mathbf{1}_{\partial D}(X(s))\mathrm{d}\phi(s) = \phi(t) \quad \text{and} \quad \int_0^t \mathbf{1}_{\partial D}(X(s))\mathrm{d}s = \int_0^t \rho(X(s))\mathrm{d}\phi(s). \tag{4.5}$$

It is well-known that, if we can show the uniqueness in law of a $(\mathcal{G}, L)$-process, then it is a diffusion process. We shall extend our method based on excursion point processes given in the previous section to construct $(\mathcal{G}, L)$-process and discuss its uniqueness in law.

4.1. The case of $\mathcal{G} = \frac{1}{2}\Delta$ and Wentzell's boundary operator L with constant coefficients

The simplest case is when $a^{ij}(x)$, $b^i(x)$, $\alpha^{ij}(\bar{x})$, $\beta^i(\bar{x})$, $\mu(\bar{x})$ and $\rho(\bar{x})$ are all constants: $a^{ij}(x) = a^{ij}$, $b^i(x) = b^i$, $\alpha^{ij}(\bar{x}) = \alpha^{ij}$, $\beta^i(\bar{x}) = \beta^i$, $\mu(\bar{x}) = \mu$, $\rho(\bar{x}) = \rho$, and the measure $n_{\bar{x}}(\mathrm{d}y) := n(\mathrm{d}y)$ is independent of $\bar{x}$. We assume, for simplicity, that $a^{ij} = \delta^{ij}$ and $b^i = 0$, and so $\mathcal{G} = \frac{1}{2}\Delta$; probabilistically, the minimal diffusion is a Brownian motion $\widehat{B}(t)$ in D° before it hits the boundary ∂D. So, we are given a symmetric, nonnegative definite $(d-1)\times(d-1)$-matrix α^{ij}, a $(d-1)$-vector β^i, nonnegative constants μ and ρ, and a nonnegative measure $n(\mathrm{d}y)$ on $D \setminus \{0\}$.

Lemma 4.1. *For some positive integer l, there exists a Borel map*

$$g : (\mathbf{R}^l)^+ \ni u \mapsto g(u) = (\bar{g}(u), g^d(u)) \in D \tag{4.6}$$

with $g(0) = 0$ which satisfies the following properties:

(i) *On the set $\{u \in (\mathbf{R}^l)^+; |u| \le 1\}$, the function $g(u)$ is bounded.*
(ii) $\lim_{|u|\to 0} g(u) = 0$.
(iii) *The image of the measure $\frac{\mathrm{d}u}{|u|^{l+1}}$ on $\{u \in (\mathbf{R}^l)^+; |g(u)| > 0\}$ under the map $u \mapsto y = g(u)$ coincides with the measure $n(\mathrm{d}y)$ on $D \setminus \{0\}$. Here, $\mathrm{d}u$ is the l-dimensional Lebesgue measure.*

A proof is omitted. By (4.3), we have

$$\int_{\{0<|u|\le 1\}\cap(\mathbf{R}^l)^+} \left\{|\bar{g}(u)|^2 + g^d(u)\right\} \frac{\mathrm{d}u}{|u|^{l+1}} < \infty. \tag{4.7}$$

So, instead of the Wentzell boundary operator L of the type given by (4.2), we may start with one of the following form:

$$Lu((\bar{x}, 0)) = \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j}((\bar{x}, 0)) + \sum_{i=1}^{d-1} \beta^i \frac{\partial u}{\partial x^i}((\bar{x}, 0))$$
$$+ \mu \frac{\partial u}{\partial x^d}((\bar{x}, 0)) - \rho \cdot (\mathcal{G}u)((\bar{x}, 0))$$

$$+ \int_{\{0<|u|\le 1\}\cap(\mathbf{R}^l)^+} \left\{ u((\bar{x},0)+g(u)) - u((\bar{x},0)) - \sum_{i=1}^{d-1} g(u)^i \frac{\partial u}{\partial x^i}((\bar{x},0)) \right\} \frac{\mathrm{d}u}{|u|^{l+1}}$$

$$+ \int_{\{|u|>1\}\cap(\mathbf{R}^l)^+} \{u((\bar{x},0)+g(u)) - u((\bar{x},0))\} \frac{\mathrm{d}u}{|u|^{l+1}}. \tag{4.8}$$

We introduce several path spaces in D to describe excursions away from the boundary. We set

$$\mathcal{W}(D) = \{w \in \mathbf{C}([0,\infty) \to D);\, w(t \wedge \sigma(w)) = w(t)\} \tag{4.9}$$

where

$$\sigma(w) = \inf\{t > 0;\, w(t) \in \partial D\}. \tag{4.10}$$

We set also

$$\mathcal{W}^+(D) = \{w \in \mathcal{W}(D);\, \sigma(w) > 0\}. \tag{4.11}$$

For $x \in D$, let $\mathcal{W}_x(D)$ (resp. $\mathcal{W}_x^+(D)$) be the subclass of $\mathcal{W}(D)$ (resp, $\mathcal{W}^+(D)$) consisting of all paths w such that $w(0) = x$.

Note that, if $x \in D^\circ$, then $\mathcal{W}_x(D) = \mathcal{W}_x^+(D)$, while, if $x \in \partial D$, then $\mathcal{W}_x(D) = \mathcal{W}_x^+(D) \cup \{\mathbf{x}\}$, where $\mathbf{x}$ is the constant path at x: $\mathbf{x}(t) \equiv x$.

We are now going to construct the $(\frac{1}{2}\Delta, L)$-diffusion process $\mathbf{X}$; we shall define its path function $X^a(t)$ starting at $a \in D$, i.e., $X^a(0) = a$. First of all, we take a sufficiently large probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$ on which we can realize the following objects:

(i) A filtration $(\mathcal{G}_t)$ on Ω such that $\mathcal{G}_t \subset \mathcal{F}_0$ for every $t > 0$ and a d-dimensional $(\mathcal{G}_t)$-Brownian motion $\widehat{\mathbf{B}}^a = (\widehat{B}^a(t))$ such that $\widehat{B}^a(0) = a$.

(ii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_1$ with values in the product space $W_0^{(d-1)} \times \mathcal{W}_0$, having characteristic measure $P_0^{(d-1)} \times n_+$. Here, $W_0^{(d-1)} = \{w \in \mathbf{C}([0,\infty) \to \mathbf{R}^{d-1});\, w(0) = 0\}$ and $P_0^{(d-1)}$ is the $(d-1)$-dimensional Wiener measure on $W_0^{(d-1)}$. The one-dimensional path space $\mathcal{W}_0$ and the measure n_+ on $\mathcal{W}_0$ are the same as in Section 3.1; n_+ is the excursion measure of one-dimensional positive Brownian excursions on $\mathcal{W}_0$ (cf. Example 2.2).

(iii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_2$ with values in the product space $W^d_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})$ with characteristic measure given by the product measure $P^{(d)}_{(\bar{0},1)} \times \left(\frac{\mathrm{d}u}{|u|^{l+1}}\right)$. Here, $W^d_{(\bar{0},1)} = \{w \in \mathbf{C}([0,\infty) \to \mathbf{R}^d);\, w(0) = (\bar{0},1)\}$ ($\bar{0}$ is the origin in $\mathbf{R}^{d-1}$), and $P^{(d)}_{(\bar{0},1)}$ is d-dimensional Wiener measure on $W^d_{(\bar{0},1)}$, $u \in (\mathbf{R}^l)^+ \setminus \{0\}$ and $\mathrm{d}u$ is the l-dimensional Lebesgue measure.

(iv) A continuous $(\mathcal{F}_t)$-Lévy process (a Gaussian diffusion) $\eta(t)$ on $\partial D \cong \mathbf{R}^{d-1}$ with $\eta(0) = 0$, associated with the covariance matrix α^{ij}, $i, j = 1, \dots, d-1$, and the drift vector $\tilde{\beta}^i$, $i = 1, \dots, d-1$, given by

$$\tilde{\beta}^i = \beta^i - \int_{(\mathbf{R}^l)^+\cap\{0<|u|\le 1\}} g^i(u) \cdot P^{(d)}_{(\bar{0},1)}[\sigma(w) > h(u)^{-2}] \frac{\mathrm{d}u}{|u|^{l+1}}, \tag{4.12}$$

where, for $w = (\bar{w}, w^d) \in W^d_{(\bar{0},1)}$, we set $\sigma(w) = \min\{s \ge 0;\, w^d(s) = 0\}$. Here, the function $h(u)$ is defined by (4.19) below.

We set up the point processes $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *as mutually independent*: Then all the random elements $\widehat{\mathbf{B}}^a$, $\mathbf{p}_1$, $\mathbf{p}_2$ and η are mutually independent.

An element of the product space $W_0^{(d-1)} \times \mathcal{W}_0$ is denoted by $[\bar{w}, \omega]$ with $\bar{w} = (\bar{w}(t)) \in W_0^{(d-1)}$ and $\omega = (\omega(t)) \in \mathcal{W}_0$. Also, an element in the product space $W^d_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})$ is denoted by $[w, u]$ with $w = (w(t)) \in W^d_{(\bar{0},1)}$ and $u \in (\mathbf{R}^l)^+ \setminus \{0\}$.

We define the following mappings (in the following, when there is no fear of confusion, we often identify $\bar{x} \in \mathbf{R}^{d-1}$ with $(\bar{x}, 0) \in \partial D$, for simplicity):

$$\Phi : \partial D \times \{W_0^{(d-1)} \times \mathcal{W}_0\} \ni (\bar{x}, [\bar{w}, \omega]) \mapsto \Phi(\bar{x}, [\bar{w}, \omega]) \in \mathcal{W}_{\bar{x}}(D)$$

and

$$\Psi : \partial D \times \{W^d_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})\} \ni (\bar{x}, [w, u]) \mapsto \Psi(\bar{x}, [w, u]) \in \mathcal{W}_{(\bar{x},0)+g(u)}(D).$$

The definition of Φ *is as follows*:
$\Phi(\bar{x}, [\bar{w}, \omega]) = (\bar{\Phi}(\bar{x}, [\bar{w}, \omega]), \Phi^d(\bar{x}, [\bar{w}, \omega]))$,

$$\bar{\Phi}(\bar{x}, [\bar{w}, \omega])(t) = \begin{cases} \bar{x} + \mu \cdot \bar{w}\left(\dfrac{t}{\mu^2} \wedge \sigma(\omega)\right), & \text{in the case } \mu > 0, \\ \bar{x}, & \text{in the case } \mu = 0, \end{cases} \tag{4.13}$$

and

$$\Phi^d(\bar{x}, [\bar{w}, \omega])(t) = \begin{cases} \mu \cdot \omega\left(\dfrac{t}{\mu^2}\right), & \text{in the case } \mu > 0, \\ 0, & \text{in the case } \mu = 0. \end{cases} \tag{4.14}$$

The definition of Ψ *is as follows*: Denoting $\Psi(\bar{x}, [w, u]) = (\bar{\Psi}(\bar{x}, [w, u]), \Psi^d(\bar{x}, [w, u]))$ with $w = (\bar{w}, w^d)$ and, remembering $\sigma(w) = \min\{s \geq 0; w^d(s) = 0\}$,

$$\bar{\Psi}(\bar{x}, [w, u])(t) = \begin{cases} \bar{x} + \bar{g}(u) + g^d(u) \cdot \bar{w}\left(\dfrac{t}{(g^d(u))^2} \wedge \sigma(w)\right), & \text{in the case } g^d(u) > 0, \\ \bar{x} + \bar{g}(u), & \text{in the case } g^d(u) = 0, \end{cases} \tag{4.15}$$

and

$$\Psi^d(\bar{x}, [w, u])(t) = \begin{cases} g^d(u) \cdot w^d\left(\dfrac{t}{(g^d(u))^2} \wedge \sigma(w)\right), & \text{in the case } g^d(u) > 0, \\ 0, & \text{in the case } g^d(u) = 0. \end{cases} \tag{4.16}$$

Note that $\Phi(\bar{x}, [\bar{w}, \omega])(0) = (\bar{x}, 0) \in \partial D$, while $\Psi(\bar{x}, [w, u])(0) = (\bar{x}, 0) + g(u)$, which is in $D \setminus \{(\bar{x}, 0)\}$ when $g(u) \neq 0$.

Then $\sigma\{\Phi(\bar{x}, [\bar{w}, \omega])\}$ and $\sigma\{\Psi(\bar{x}, [w, u])\}$ are given, respectively, by $\mu^2\sigma(\omega)$ and $(g^d(u))^2\sigma(w)$, which are independent of $\bar{x}$ and we denote them by $\widehat{\sigma}([\bar{w}, \omega])$ and $\widehat{\sigma}([w, u])$, respectively. Then it is easy to deduce the following:

$$\int_{W_0^{(d-1)} \times \mathcal{W}_0} \widehat{\sigma}([\bar{w}, \omega]) \cdot \mathbf{1}_{\{\sigma(\omega) \leq 1\}} P_0^{(d-1)}(\mathrm{d}\bar{w}) n_+(\mathrm{d}\omega) = \int_{\mathcal{W}_0} \mu^2 \sigma(\omega) \cdot \mathbf{1}_{\{\sigma(\omega) \leq 1\}} n_+(\mathrm{d}\omega) < \infty \tag{4.17}$$

and

$$\int_{W_0^{(d-1)}\times\mathcal{W}_0}\mathbf{1}_{\{\sigma(\omega)>1\}}P_0^{(d-1)}(\mathrm{d}\bar{w})n_+(\mathrm{d}\omega)=\int_{\mathcal{W}_0}\mathbf{1}_{\{\sigma(\omega)>1\}}n_+(\mathrm{d}\omega)<\infty. \tag{4.18}$$

We set

$$h(u)=|\bar{g}(u)|^2+g^d(u),\quad u\in(\mathbf{R}^l)^+\setminus\{0\}, \tag{4.19}$$

$$\begin{aligned}&\int_{W_{(\bar{0},1)}^{(d)}\times(\mathbf{R}^l)^+}\widehat{\sigma}([w,u])\cdot\mathbf{1}_{\{\sigma(w)\le h(u)^{-2},\ 0<|u|\le 1\}}P_{(\bar{0},1)}^{(d)}(\mathrm{d}w)\frac{\mathrm{d}u}{|u|^{l+1}}\\ &\quad=\int_{(\mathbf{R}^l)^+}E_{(\bar{0},1)}^{(d)}\left[(g^d(u))^2\sigma(w);\sigma(w)\le h(u)^{-2}\right]\cdot\mathbf{1}_{\{0<|u|\le 1\}}\frac{\mathrm{d}u}{|u|^{l+1}}<\infty,\end{aligned} \tag{4.20}$$

and

$$\int_{W_{(\bar{0},1)}^{(d)}\times(\mathbf{R}^l)^+}\mathbf{1}_{\{\sigma(w)>h(u)^{-2}\text{ or }|u|>1\}}P_{(\bar{0},1)}^{(d)}(\mathrm{d}w)\frac{\mathrm{d}u}{|u|^{l+1}}<\infty. \tag{4.21}$$

For example, (4.20) and (4.21) can be verified easily from (4.7) if we note simple estimates like $P_{(\bar{0},1)}^{(d)}(\sigma(w)>h(u)^{-2})=O(h(u))$ and $E_{(\bar{0},1)}^{(d)}(\sigma(w)\wedge h(u)^{-2})=O(h(u)^{-1})$ as $|u|\to 0$.

We define two ∂D-valued functions:

$$\varphi(\bar{x},[\bar{w},\omega])=\Phi(\bar{x},[\bar{w},\omega])(\sigma[\Phi(\bar{x},[\bar{w},\omega])])-\bar{x} \tag{4.22}$$

and

$$\psi(\bar{x},[w,u])=\Psi(\bar{x},[w,u])(\sigma[\Psi(\bar{x},[w,u])])-\bar{x}. \tag{4.23}$$

Then we see immediately that

$$\varphi(\bar{x},[\bar{w},\omega])=\begin{cases}\mu\cdot\bar{w}(\sigma(\omega)), & \text{in the case }\mu>0,\\ \bar{0}, & \text{in the case }\mu=0,\end{cases} \tag{4.24}$$

and, writing $w=(\bar{w},w^d)$,

$$\psi(\bar{x},[w,u])=\begin{cases}\bar{g}(u)+g^d(u)\cdot\bar{w}(\sigma(w)), & \text{in the case }g^d(u)>0,\\ \bar{g}(u), & \text{in the case }g^d(u)=0,\end{cases} \tag{4.25}$$

so that both are independent of $\bar{x}$. We denote them by $\varphi([\bar{w},\omega])$ and $\psi([w,u])$, respectively. Then, we can easily obtain the following estimates:

$$\int_{W_0^{(d-1)}\times\mathcal{W}_0}|\varphi([\bar{w},\omega])|^2\cdot\mathbf{1}_{\{\sigma(\omega)\le 1\}}P_0^{(d-1)}(\mathrm{d}\bar{w})n_+(\mathrm{d}\omega)<\infty \tag{4.26}$$

and

$$\int_{W_{(\bar{0},1)}^{(d)}\times(\mathbf{R}^l)^+}|\psi([w,u])|^2\cdot\mathbf{1}_{\{\sigma(w)\le h(u)^{-2},\ 0<|u|\le 1\}}P_{(\bar{0},1)}^{(d)}(\mathrm{d}w)\frac{\mathrm{d}u}{|u|^{l+1}}<\infty. \tag{4.27}$$

We define an $(\mathcal{F}_t)$-Lévy process $\xi=(\xi(t))$ on $\partial D\cong\mathbf{R}^{d-1}$ by setting

$$\xi(t)=\widehat{B}^a(m_{\partial D}^{\widehat{B}^a})+\eta(t)+\int_0^{t+}\int_{W_0^{(d-1)}\times\mathcal{W}_0}\varphi([\bar{w},\omega])\cdot\mathbf{1}_{\{\sigma(\omega)\le 1\}}\widetilde{N}_{\mathbf{p}_1}(\mathrm{d}s,\mathrm{d}([\bar{w},\omega]))$$

$$+ \int_0^{t+} \int_{W_0^{(d-1)} \times \mathcal{W}_0} \varphi([\bar{w}, \omega]) \cdot \mathbf{1}_{\{\sigma(\omega) > 1\}} N_{\mathbf{p}_1}(\mathrm{d}s, \mathrm{d}([\bar{w}, \omega]))$$
$$+ \int_0^{t+} \int_{W_{(\bar{0},1)}^{(d)} \times (\mathbf{R}^l)^+} \psi([w, u]) \cdot \mathbf{1}_{\{\sigma(w) \le h(u)^{-2},\ 0 < |u| \le 1\}} \widetilde{N}_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w, u]))$$
$$+ \int_0^{t+} \int_{W_{(\bar{0},1)}^{(d)} \times (\mathbf{R}^l)^+} \psi([w, u]) \cdot \mathbf{1}_{\{\sigma(w) > h(u)^{-2} \text{ or } |u| > 1\}} N_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w, u])), \tag{4.28}$$

where

$$m_{\partial D}^{\widehat{B}^a} = \inf\{s \ge 0;\ \widehat{B}^a(s) \in \partial D\}. \tag{4.29}$$

By (4.18), (4.21), (4.26) and (4.27), the right-hand side (RHS) of (4.28) is well-defined and it yields an $(\mathcal{F}_t)$-Lévy process.

Also, we define an increasing $(\mathcal{F}_t)$-Lévy process $A = (A(t))$ by setting

$$A(t) = m_{\partial D}^{\widehat{B}^a} + \rho t + \int_0^{t+} \int_{W_0^{(d-1)} \times \mathcal{W}_0} \widehat{\sigma}([\bar{w}, \omega]) N_{\mathbf{p}_1}(\mathrm{d}s, \mathrm{d}([\bar{w}, \omega]))$$
$$+ \int_0^{t+} \int_{W_{(\bar{0},1)}^{(d)} \times ((\mathbf{R}^l)^+ \setminus \{0\})} \widehat{\sigma}([w, u]) N_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w, u])). \tag{4.30}$$

By (4.17), (4.18), (4.20) and (4.21), the RHS of (4.30) is well-defined and yields an increasing $(\mathcal{F}_t)$-Lévy process. Since $\widehat{\sigma}([\bar{w}, \omega]) = \mu^2 \sigma(\omega)$ and $\widehat{\sigma}([w, u]) = (g^d(u))^2 \sigma(w)$, it is easy to see that $\lim_{t \to \infty} A(t) = \infty$ a.s., if and only if $\mu + \rho + \int_{(\mathbf{R}^l)^+} \mathbf{1}_{\{g^d(u) > 0\}} \mathrm{d}u > 0$ (including the value infinity). It is also easy to see that $t \mapsto A(t)$ *is strictly increasing a.s., if and only if the following condition*(H_1) *is satisfied.*

(H_1) At least one of the following three conditions is satisfied:

$$\text{(i) } \mu > 0, \quad \text{(ii) } \rho > 0, \quad \text{(iii) } \int_{(\mathbf{R}^l)^+} \mathbf{1}_{\{0 < |u| \le 1,\ g^d(u) > 0\}} \frac{\mathrm{d}u}{|u|^{l+1}} = \infty.$$

Note that we have $\lim_{t \to \infty} A(t) = \infty$ a.s., when (H_1) is satisfied.
In the following, we assume that(H_1) *is satisfied.*

Now we can define the path function $X^a(t)$. Setting $A(0-) = 0$, the following holds with probability 1: For any given $t \in [0, \infty)$, there exists a unique $s \in [0, \infty)$, denoted by $s = \phi(t)$, such that $A(s-) \le t \le A(s)$.

If $s = 0$, then this implies that $0 \le t \le m^{\widehat{B}^a}$ and we set $X^a(t) = \widehat{B}^a(t)$.

If $s > 0$ and if $A(s) > A(s-)$, then this implies that $s \in \mathbf{D}_{\mathbf{p}_1} \cup \mathbf{D}_{\mathbf{p}_2}$. By the independence of $\mathbf{p}_1$ and $\mathbf{p}_2$, we have $\mathbf{D}_{\mathbf{p}_1} \cap \mathbf{D}_{\mathbf{p}_2} = \emptyset$ a.s., so that only one case, $s \in \mathbf{D}_{\mathbf{p}_1}$ or $s \in \mathbf{D}_{\mathbf{p}_2}$, can occur. In the first case, we set

$$X^a(t) = \Phi(\xi(s-), \mathbf{p}_1(s))(t - A(s-)).$$

In the second case, we set

$$X^a(t) = \Psi(\xi(s-), \mathbf{p}_2(s))(t - A(s-)).$$

If $s > 0$ and if $A(s) = A(s-)$, then this implies that either $s \in \mathbf{D}_{\mathbf{p}_2}$ and $\widehat{\sigma}(\mathbf{p}_2(s)) = 0$, or $s \notin \mathbf{D}_{\mathbf{p}_1} \cup \mathbf{D}_{\mathbf{p}_2}$.

In the first case, we set

$$X^a(t) = \xi(s)(=\Psi(\xi(s-), \mathbf{p}_2(s))(0)) \;= \xi(s-) + \bar{g}(u_s)$$

where u_s denotes the "u-component" of $\mathbf{p}_2(s) \in W^{(d)}_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})$.
In the second case, we have $\xi(s) = \xi(s-)$ and we set

$$X^a(t) = \xi(s).$$

In this way, we have completely defined the path function $\{X^a(t)\}$. We can identify this as a $(\mathcal{G}, L)$-process and show its uniqueness in law.

4.2. The case of $\mathcal{G} = \frac{1}{2}\Delta$ and with Wentzell's boundary operator L of variable coefficients

Here, we consider $(\mathcal{G}, L)$-diffusion processes with $\mathcal{G} = \frac{1}{2}\Delta$ as in the previous subsection but the boundary operator L has variable coefficients. So we treat the case of L given in the same form as (4.8) in which, however, α^{ij}, β^i, μ, ρ, and $g(u) = (\bar{g}(u), g^d(u))$ may depend on the boundary points $\bar{x} \in \partial D$, so that they are replaced by $\alpha^{ij}(\bar{x})$, $\beta^i(\bar{x})$, $\mu(\bar{x})$, $\rho(\bar{x})$, $g(\bar{x}, u) = (\bar{g}(\bar{x}, u), g^d(\bar{x}, u))$. Of course, these functions satisfy, at each fixed $\bar{x} \in \partial D$, the same conditions as were given in the previous section and we assume, for simplicity, that these functions are bounded on ∂D. As for $g(\bar{x}, u) = (\bar{g}(\bar{x}, u), g^d(\bar{x}, u))$, we assume, more precisely, that there exists a positive function $h(u)$, of $u \in (\mathbf{R}^l)^+$, having the property that $h(0) = 0$, $\lim_{|u|\to 0} h(u) = 0$, bounded on $\{0 < |u| \le 1\}$ and

$$\int_{(\mathbf{R}^l)^+\cap\{0<|u|\le 1\}} h(u)\frac{\mathrm{d}u}{|u|^{l+1}} < \infty, \tag{4.31}$$

which satisfies

$$|\bar{g}(\bar{x}, u)|^2 + g^d(\bar{x}, u) \le h(u) \quad \text{if } \bar{x} \in \partial D \text{ and } 0 < |u| \le 1. \tag{4.32}$$

Now we define the mappings

$$\Phi : \partial D \times \{W_0^{(d-1)} \times \mathcal{W}_0\} \to \mathcal{W}(D)$$

and

$$\Psi : \partial D \times \{W^d_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})\} \to \mathcal{W}(D)$$

in the same way as (4.13) and (4.14) and as (4.15) and (4.16), respectively, in which, however, we replace μ and $g(u) = (\bar{g}(u), g^d(u))$ by $\mu(\bar{x})$ and $g(\bar{x}, u) = (\bar{g}(\bar{x}, u), g^d(\bar{x}, u))$, respectively. Then,

$$\widehat{\sigma}(\bar{x}, [\bar{w}, \omega]) := \sigma\{\Phi(\bar{x}, [\bar{w}, \omega])\}, \quad \text{and} \quad \widehat{\sigma}(\bar{x}, [w, u]) := \sigma\{\Psi(\bar{x}, [w, u])\}$$

now depend on $\bar{x}$.

Also, ∂D-valued functions $\varphi(\bar{x}, [\bar{w}, \omega])$ and $\psi(\bar{x}, [w, u])$ are defined by (4.22) and (4.23), respectively, which now really depend on $\bar{x}$. Then, (4.24) and (4.25) hold in which, however, μ and $g(u) = (\bar{g}(u), g^d(u))$ are replaced by $\mu(\bar{x})$ and $g(\bar{x}, u) = (\bar{g}(\bar{x}, u), g^d(\bar{x}, u))$, respectively.

Now, (4.18) and (4.21) hold and estimates like (4.17), (4.20), (4.26) and (4.27) still hold by modifying the dependence in $\bar{x}$; in these estimates, the integrals in the LHS's are uniformly bounded in $\bar{x}$.

The path function of a $(\mathcal{G}, L)$-diffusion can be defined in the same way as in the previous section from a process $(\xi(t))$ on the boundary ∂D and an increasing process $(A(t))$. However, they are no longer Lévy processes. Eq. (4.28) for the process $(\xi(t))$ now turns out to be a SDE of jump type (cf. (4.36) below).

We add further sufficient conditions on the coefficients of L so that the SDE has a pathwise unique solution (in the following, K denotes some positive constant):

(1) There exist bounded functions $\tau_k^i(\bar{x})$, $i = 1, \ldots, d-1$, $k = 1, \ldots, r$, such that

$$\alpha^{ij}(\bar{x}) = \sum_{k=1}^{r} \tau_k^i(\bar{x})\tau_k^j(\bar{x})$$

and

$$|\tau_k^i(\bar{x}) - \tau_k^i(\bar{y})| \leq K|\bar{x} - \bar{y}|.$$

(2)

$$|\beta^i(\bar{x}) - \beta^i(\bar{y})| \leq K|\bar{x} - \bar{y}|.$$

(3)

$$|\mu(\bar{x}) - \mu(\bar{y})| \leq K|\bar{x} - \bar{y}|.$$

(4) For every $u \in (\mathbf{R}^l)^+$ such that $0 < |u| \leq 1$,

$$|\bar{g}(\bar{x}, u) - \bar{g}(\bar{y}, u)|^2 \leq K \cdot h(u)|\bar{x} - \bar{y}|^2,$$

and

$$|g^d(\bar{x}, u) - g^d(\bar{y}, u)| \leq K \cdot h(u)|\bar{x} - \bar{y}|.$$

We can deduce the following estimates from (3) and (4):

$$\int_{W_0^{(d-1)} \times \mathcal{W}_0} |\varphi(\bar{x}, [\bar{w}, \omega]) - \varphi(\bar{y}, [\bar{w}, \omega])|^2 \cdot \mathbf{1}_{\{\sigma(\omega) \leq 1\}} P_0^{(d-1)}(\mathrm{d}\bar{w}) n_+(\mathrm{d}\omega)$$
$$\leq K|\bar{x} - \bar{y}|^2 \tag{4.33}$$

and

$$\int_{W_{(\bar{0},1)}^{(d)} \times (\mathbf{R}^l)^+} |\psi(\bar{x}, [w, u]) - \psi(\bar{y}, [w, u])|^2 \cdot \mathbf{1}_{\{\sigma(w) \leq h(u)^{-2},\ 0 < |u| \leq 1\}} P_{(\bar{0},1)}^{(d)}(\mathrm{d}w) \frac{\mathrm{d}u}{|u|^{l+1}}$$
$$\leq K|\bar{x} - \bar{y}|^2. \tag{4.34}$$

We are now going to construct a path function $(X^a(t))$ of the $(\mathcal{G} = \frac{1}{2}\Delta, L)$-diffusion process starting at $a \in D$. First of all we set up the following on some probability space with a filtration $(\mathcal{F}_t)$.

(i) A filtration $(\mathcal{G}_t)$ on Ω such that $\mathcal{G}_t \subset \mathcal{F}_0$ for every $t > 0$ and a d-dimensional $(\mathcal{G}_t)$-Brownian motion $\widehat{\mathbf{B}}^a = (\widehat{B}^a(t))$ such that $\widehat{B}^a(0) = a$.

(ii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_1$ with values in the product space $W_0^{(d-1)} \times \mathcal{W}_0$, having characteristic measure $P_0^{(d-1)} \times n_+$. Here, $W_0^{(d-1)} = \{w \in \mathbf{C}([0, \infty) \to \mathbf{R}^{d-1});\ w(0) = 0\}$ and $P_0^{(d-1)}$ is the $(d-1)$-dimensional Wiener measure on $W_0^{(d-1)}$. The one-dimensional path space $\mathcal{W}_0$ and the measure n_+ on $\mathcal{W}_0$ are the same as in Section 3.1; n_+ is the positive Brownian excursion measure on $\mathcal{W}_0$ (cf. Example 2.2).

(iii) A stationary $(\mathcal{F}_t)$-Poisson point process $\mathbf{p}_2$ with values in the product space $W^d_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})$ with characteristic measure given by the product measure $P^{(d)}_{(\bar{0},1)} \times \left(\frac{\mathrm{d}u}{|u|^{l+1}}\right)$. Here, $W^d_{(\bar{0},1)} = \{w \in \mathbf{C}([0,\infty) \to \mathbf{R}^d); w(0) = (\bar{0},1)\}$ ($\bar{0}$ is the origin in $\mathbf{R}^{d-1}$), and $P^{(d)}_{(\bar{0},1)}$ is d-dimensional Wiener measure on $W^d_{(\bar{0},1)}$, $u \in (\mathbf{R}^l)^+ \setminus \{0\}$ and $\mathrm{d}u$ is the l-dimensional Lebesgue measure.

(iv) An r-dimensional $(\mathcal{F}_t)$-Brownian motion $B = (B^k(t))_{k=1}^r$ with $B(0) = 0$.

Define the functions $\tilde{\beta}^i(\bar{x})$, $i = 1, \ldots, d-1$, on ∂D by

$$\tilde{\beta}^i(\bar{x}) = \beta^i - \int_{(\mathbf{R}^l)^+ \cap \{0<|u|\le 1\}} g^i(\bar{x},u) \cdot P^{(d)}_{(\bar{0},1)}[\sigma(w) > h(u)^{-2}] \frac{\mathrm{d}u}{|u|^{l+1}}. \tag{4.35}$$

It is easy to see that $\tilde{\beta}^i(\bar{x})$ is also Lipschitz continuous. Define the ∂D-valued functions $\bar{\tau}_k(\bar{x})$, $k = 1, \ldots, r$, and $\check{\beta}(\bar{x})$ by setting

$$\bar{\tau}_k(\bar{x}) = (\tau_k^i(\bar{x}))_{i=1}^{d-1}, \qquad \check{\beta}(\bar{x}) = (\tilde{\beta}^i(\bar{x}))_{i=1}^{d-1}.$$

Consider the following SDE of jump type:

$$\begin{aligned}
\xi(t) = {} & \widehat{B}^a(m^{\widehat{B}^a}_{\partial D}) + \sum_{k=1}^r \int_0^t \bar{\tau}_k(\xi(s))\mathrm{d}B^k(s) + \int_0^t \check{\beta}(\xi(s))\mathrm{d}s \\
& + \int_0^{t+} \int_{W_0^{(d-1)} \times \mathcal{W}_0} \varphi(\xi(s-), [\bar{w},\omega]) \cdot \mathbf{1}_{\{\sigma(\omega)\le 1\}} \widetilde{N}_{\mathbf{p}_1}(\mathrm{d}s, \mathrm{d}([\bar{w},\omega])) \\
& + \int_0^{t+} \int_{W_0^{(d-1)} \times \mathcal{W}_0} \varphi(\xi(s-), [\bar{w},\omega]) \cdot \mathbf{1}_{\{\sigma(\omega)> 1\}} N_{\mathbf{p}_1}(\mathrm{d}s, \mathrm{d}([\bar{w},\omega])) \\
& + \int_0^{t+} \int_{W^{(d)}_{(\bar{0},1)} \times (\mathbf{R}^l)^+} \psi(\xi(s-), [w,u]) \cdot \mathbf{1}_{\{\sigma(w)\le h(u)^{-2},\ 0<|u|\le 1\}} \widetilde{N}_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w,u])) \\
& + \int_0^{t+} \int_{W^{(d)}_{(\bar{0},1)} \times (\mathbf{R}^l)^+} \psi(\xi(s-), [w,u]) \cdot \mathbf{1}_{\{\sigma(w)> h(u)^{-2} \text{ or } |u|>1\}} N_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w,u])).
\end{aligned} \tag{4.36}$$

By the estimates (4.33) and (4.34) combined with the Lipschitz continuity of $\bar{\tau}_k$ and $\check{\beta}$, we can apply Theorem 2.1 to conclude that $\xi(t)$ is determined uniquely as the pathwise unique solution of (4.36).

Then $A(t)$ can be defined by

$$\begin{aligned}
A(t) = {} & m^{\widehat{B}^a}_{\partial D} + \int_0^t \rho(\xi(s))\mathrm{d}s + \int_0^{t+} \int_{W_0^{(d-1)} \times \mathcal{W}_0} \widehat{\sigma}(\xi(s-), [\bar{w},\omega]) N_{\mathbf{p}_1}(\mathrm{d}s, \mathrm{d}([\bar{w},\omega])) \\
& + \int_0^{t+} \int_{W^{(d)}_{(\bar{0},1)} \times ((\mathbf{R}^l)^+ \setminus \{0\})} \widehat{\sigma}(\xi(s-), [w,u]) N_{\mathbf{p}_2}(\mathrm{d}s, \mathrm{d}([w,u])).
\end{aligned} \tag{4.37}$$

We assume that the following condition (H_2) is satisfied:

(H_2) For every $\bar{x} \in \partial D$, one of the following three conditions holds:

$$\text{(i) } \mu(\bar{x}) > 0, \qquad \text{(ii) } \rho(\bar{x}) > 0, \qquad \text{(iii) } \int_{(\mathbf{R}^l)^+} \mathbf{1}_{\{0<|u|\le 1,\ g^d(\bar{x},u)>0\}} \frac{\mathrm{d}u}{|u|^{l+1}} = \infty.$$

Then we can conclude that, with probability 1, $t \mapsto A(t)$ is strictly increasing and $\lim_{t\to\infty} A(t) = \infty$. Now, $X^a(t)$ can be constructed in exactly the same way as in the previous subsection.

4.3. The case of $\mathcal{G}$ being a general second-order elliptic differential operator

We consider the case of $\mathcal{G}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b^i(x)\frac{\partial u}{\partial x^i}(x)$. We assume that there exist bounded functions $\theta_q^i(x)$, $i = 1, \ldots, d$, $q = 1, \ldots, m$, such that

$$a^{ij}(x) = \sum_{q=1}^{m} \theta_q^i(x)\theta_q^j(x).$$

We also assume the Lipschitz conditions:

$$|\theta_q^i(x) - \theta_q^i(y)| + |b^i(x) - b^i(y)| \le K|x-y|.$$

We assume further that $a^{dd}(x) = 1$ on D and

$$\theta_q^d(x) = 0, \quad q = 1, \ldots, m-1 \quad \text{and} \quad \theta_m^d(x) = 1. \tag{4.38}$$

This assumption is not so restrictive; if $a^{dd}(x) > 0$ everywhere on D, then, at least locally, the problem can be reduced to this case by a method of time change or a transformation of coordinates. As for the boundary operator L described by functions $\alpha^{ij}(\bar{x})$, $\beta^i(\bar{x})$, $\mu(\bar{x})$, $\rho(\bar{x})$ and $g(\bar{x}, u) = (\bar{g}(\bar{x}, u), g^d(\bar{x}, u))$, we make all the same assumptions as we did in the previous subsection.

We shall apply the same method as in the previous section: A necessary modification is that, instead of the product spaces $W_0^{(d-1)} \times \mathcal{W}_0$ and $W_{(\bar{0},1)}^d \times ((\mathbf{R}^l)^+ \setminus \{0\})$, we take spaces $W_0^{(m-1)} \times \mathcal{W}_0$ and $W_{(\bar{0},1)}^m \times ((\mathbf{R}^l)^+ \setminus \{0\})$, respectively, whose elements are denoted by the same notation $[\bar{w}, \omega]$ and $[w, u]$ as above, with $\bar{w} \in W_0^{(m-1)}$, $w \in W_{(\bar{0},1)}^m$ and $\bar{0}$ is the origin in $\mathbf{R}^{m-1}$. We define the mappings

$$\Phi : \partial D \times \{W_0^{(m-1)} \times \mathcal{W}_0\} \ni (\bar{x}, [\bar{w}, \omega]) \mapsto \Phi(\bar{x}, [\bar{w}, \omega]) \in \mathcal{W}_{\bar{x}}(D)$$

and

$$\Psi : \partial D \times \{W_{(\bar{0},1)}^m \times ((\mathbf{R}^l)^+ \setminus \{0\})\} \ni (\bar{x}, [w, u]) \mapsto \Psi(\bar{x}, [w, u]) \in \mathcal{W}_{(\bar{x},0)+g(u)}(D).$$

To define them, we first consider the following d-dimensional SDE for $Y(t) = (Y^i(t))$ on the measure space $\{W_0^{(m-1)} \times \mathcal{W}_0, P_0^{(m-1)} \times \mathbf{n}^+\}$: for given $\bar{x} = (x^1, \ldots, x^{d-1})$,

$$\begin{aligned} Y^i(t) &= x^i + c\sum_{q=1}^{m-1}\int_0^t \theta_q^i(Y(s))\mathrm{d}w^q(s) \\ &\quad + c\int_0^t \theta_m^i(Y(s))d\omega(s) + c^2\int_0^t b^i(Y(s))\mathrm{d}s, \quad i = 1, \ldots, d-1, \\ Y^d(t) &= c\omega(t), \end{aligned} \tag{4.39}$$

where c is a given nonnegative constant. Since n_+ is the Brownian excursion measure, the stochastic integral for $\mathrm{d}\omega(s)$ can be defined as in ordinary Itô calculus and the above SDE has a pathwise unique solution. We denote this solution as $Y^{(c)} = (Y^{(c)}(t; \bar{x}, [\bar{w}, \omega]))$.

Next, we consider the following d-dimensional SDE for $Z(t) = (Z^i(t))$ on the Wiener space $\{W_{(\bar{0},1)}^{(m)}, P_{(\bar{0},1)}^{(m)}\}$: for given $\bar{x} = (x^1, \ldots, x^{d-1})$,

$$Z^i(t) = x^i + c\sum_{q=1}^{m}\int_0^t \theta_q^i(Z(s))\mathrm{d}w^q(s) + c^2\int_0^t b^i(Z(s))\mathrm{d}s, \quad i = 1,\ldots,d-1,$$
$$Z^d(t) = cw^m(t), \tag{4.40}$$

where c is a given nonnegative constant. We denote its pathwise unique solution by $Z^{(c)} = (Z^{(c)}(t;\bar{x},w))$.

Now the definition of $\Phi(\bar{x},[\bar{w},\omega])$ is as follows:

$$\Phi(\bar{x},[\bar{w},\omega])(t) = \begin{cases} Y^{(\mu(\bar{x}))}\left(\dfrac{t}{\mu^2(\bar{x})}\wedge\sigma(\omega);\bar{x},[\bar{w},\omega]\right), & \text{in the case } \mu(\bar{x}) > 0, \\ (\bar{x},0), & \text{in the case } \mu(\bar{x}) = 0. \end{cases} \tag{4.41}$$

The definition of $\Psi(\bar{x},[w,u])$ is as follows: letting $\sigma(w) = \inf\{s \geq 0; w^m(s) = 0\}$,

$$\Psi(\bar{x},[w,u])(t) = \begin{cases} Z^{(g^d(\bar{x},u))}\left(\dfrac{t}{(g^d)^2(\bar{x},u)}\wedge\sigma(w);\bar{x},w\right), & \text{in the case } g^d(\bar{x},u) > 0, \\ (\bar{x}+\bar{g}(\bar{x},u),0), & \text{in the case } g^d(\bar{x},u) = 0. \end{cases} \tag{4.42}$$

Note that

$$\Phi(\bar{x},[\bar{w},\omega])(0) = (\bar{x},0) \quad \text{and} \quad \Psi(\bar{x},[w,u])(0) = (\bar{x},0) + g(\bar{x},u).$$

Using these Φ and Ψ, $\widehat{\sigma}(\bar{x},[\bar{w},\omega])$, $\widehat{\sigma}(\bar{x},[w,u])$, $\varphi(\bar{x},[\bar{w},\omega])$ and $\psi(\bar{x},[w,u])$ are defined in the same way as above and the construction of the path function $X^a(t)$ can be carried out in the same way under the same condition (H_2). A necessary modification is just that the d-dimensional $(\mathcal{G}_t)$-Brownian motion $(\widehat{B}^a(t))$, which was used to construct the part of the diffusion before hitting the boundary, should be replaced by a $(\mathcal{G}_t)$-measurable continuous process $\widehat{X}^a(t)$ on $\mathbf{R}^d$ which starts at $a \in D$. This process is obtained as the pathwise unique solution of the SDE

$$\widehat{X}^i(t) = a^i + \sum_{q=1}^{m}\int_0^t \theta_q^i(\widehat{X}(s))\mathrm{d}B^q(s) + \int_0^t b^i(\widehat{X}(s))\mathrm{d}s, \quad i = 1,\ldots,d,$$

where $(B^q(t))$ is an m-dimensional $(\mathcal{G}_t)$-Brownian motion which we first set up on the probability space.

Now, we can identify this process $(X^a(t))$ as the unique $(\mathcal{G},L)$-diffusion in law under the conditions on the coefficients that we imposed above. The details appear in [24], though some modifications should be made which we omit in this paper.

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