Fixed Point Theorems and Dissipative Processes

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1. INTRODUCTION

Suppose $X$ is a Banach space, $T: X \rightarrow X$ is a continuous mapping. The map $T$ is said to be dissipative if there is a bounded set $B$ in $X$ such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n \geq N(x)$. In his study of ordinary differential equations in $n$-dimensional Euclidean space (which were $\omega$-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with $T x$ representing the solution of the differential equation at time $\omega$ which started at $x$ at time zero. The basic problem is to give information about the limiting behavior of orbits of $T$ and to discuss the existence of fixed points of $T$. Since 1944 a tremendous literature has accumulated on this subject, and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] as references. Levinson [12] showed that some iterate of $T$ has a fixed point, and he characterized the maximal compact invariant set of $T$. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of $T$ has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space $X$ arising in retarded functional differential equations, and $T$ completely continuous, Jones [9] and Yoshizawa [16] showed that $T$ has a fixed point by using Browder's theorem. For an arbitrary Banach space $X$ and $T$ completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently, Billotti and LaSalle [1]

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have obtained the same result with $T$ completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when $T$ is condensing on balls in $X$, in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subset X$ and $\alpha$ is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties but said nothing about fixed points of $T$. More recently, Hale, LaSalle, and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators $T$ which includes $\alpha$-contractions or $k$-set contractions; that is, there is a constant $k$, $0 \leq k < 1$, such that $\alpha(TB) \leq k \alpha(B)$ for any bounded $B \subset X$. They have characterized the maximal compact invariant set of $T$, shown that it is asymptotically stable, and proved that some iterate of $T$ has a fixed point.

There are a number of deficiencies in these theories, two of which follow: First, in the applications to $\omega$-periodic retarded functional differential equations, the hypothesis that $T$ is completely continuous implies that the period $\omega$ in the equation is greater than or equal to the delay $r$ in the differential system. In particular, this implies that the previous theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see, for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral functional differential equations, the operator $T$ is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an $\alpha$-contraction. However, the theory for this case implies only that some iterate of $T$ has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle, and Slemrod [7] and to impose an additional condition on $T$ which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for $T$ condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. DISSIPATIVE SYSTEMS

The $\varepsilon$-neighborhood of a set $K \subset X$ will be denoted by $\mathcal{B}_\varepsilon(K)$, the closure by $\text{Cl}(K)$ and the convex closure by $\overline{\text{co}}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness$^1$ of a bounded set $K$ in $X$ (see [3]). Suppose $T$ is a continuous map $T : X \to X$. The map $T$ is said to be weak condensing if for any

$^1 \alpha(K) = \inf_0 \max (\text{diam } U_i)$, where $U = \{U_i\}$ is a finite cover of $K$. 

bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map $T$ is said to be a weak $\alpha$-contraction if there is a constant $k$, $0 \leq k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$. If $T$ takes bounded sets into bounded sets, then a weak $\alpha$-contraction is an $\alpha$-contraction. The map $T^{n_0}$ is said to be weak completely continuous if there is an integer $n_0$ such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^{n_0}x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If $T$ is weak completely continuous it is weak condensing. If $T$ is completely continuous then $T$ is weak completely continuous. The map $T$ is said to be weak asymptotically smooth if for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^{n_0}x \in B$ for $n \geq 0$ implies $T^n x \in B_{\varepsilon}(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subset X$ attracts a set $H \subset X$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $T^n(H) \subset B_{\varepsilon}(K)$ for $n \geq N(H, \varepsilon)$. We say $K$ attracts compact sets of $X$ if $K$ attracts each compact set $H \subset X$. We say $K$ attracts neighborhoods of compact sets of $X$ if for any compact set $H \subset X$, there is a neighborhood $H_0$ of $H$ such that $K$ attracts $H_0$.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$. If $B$ satisfies the property that for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n(A) \subset B$ for $n \geq N(A)$, then $T$ is said to be compact dissipative. If $B$ satisfies the property that for any $x \in X$, there is an open neighborhood $O_x$ and an integer $N(x)$ such that $T^n O_x \subset B$, $n \geq N(x)$, then $T$ is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

**Lemma 1.** (a) Hale, LaSalle, Slemrod [7]. If $T$ is continuous, local dissipative, and asymptotically smooth, then there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$.

(b) Billotti and LaSalle [1]. If $T$ is continuous, point dissipative and $T^{n_0}$ is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood $H_0$ of $H$ and an integer $N(H)$ such that $\bigcup_{j \geq 0} T^j H_0$ is bounded and $T^n H_0 \subset K$ for $n \geq N(H)$. In particular, $T$ is local dissipative and $T$ asymptotically smooths.

**Lemma 2.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of $X$, then
(a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of $H$, such that $\bigcup_{n \geq 0} T^n H_1$ is bounded;

(b) $\bigcup_{i \geq 0} T^i B$ is precompact if $B$ is compact.

Proof. (a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H, \mathcal{E})$ is the number occurring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where $T$ is bounded. Define $\Omega_N = \mathcal{B}_\mathcal{E}(K)$, $\Gamma_N = \Omega_N$, $\Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

(b) The set $A = \bigcup_{j \geq 0} T^j B$ is bounded. Since $T^j(B)$ is compact for any $j$ we have $\alpha(A) = \alpha(\bigcup_{j \geq n} T^j(B))$ for any $n$. But given $\mathcal{E} > 0$, if $n \geq n_1(B, \mathcal{E})$, we have $\bigcup_{j \geq n} T^j B \subset \mathcal{B}_\mathcal{E}(B)$, and, thus, $\alpha(A) \leq 2\mathcal{E}$. Thus, $\alpha(A) = 0$ and $A$ is precompact. This proves the lemma.

If we use Lemmas 1 and 2, we will get the following theorem which was proved in [7].

**Theorem 1.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$, then $J = \bigcap_{i \geq 0} T^i(K)$ is independent of the sets $K$ satisfying the above property, $J$ is the maximal compact invariant set of $T$, is asymptotically stable and a global attractor.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood $U$ of $K$ and an integer $n$ such that $T^n(U) \subset U$. Thus, if $T$ possesses the fixed point property, then some iterate of $T$ has a fixed point (see [7]).

Regarding fixed points of $T$, it is known (see [1, 5, 8, 9, 14]) that $T$ completely continuous and point dissipative implies $T$ has a fixed point. Later, we give some weaker conditions which assert that $T$ has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

**Theorem 2.** (a) If $T: X \to X$ is continuous, weak condensing, and compact dissipative, then there is a compact invariant set $K$ which attracts compact sets of $X$ and $T$ is local dissipative.

(b) If $T$ is weak condensing and point dissipative then there is a compact invariant set $K$ that attracts points of $X$.

Proof. (a) It is an easy matter to prove the following fact: If $H$ is a compact set such that $T: H \to H$, then the set $A = \bigcap_n T^n(H)$ is compact,
nonempty, \(T(A) = A\), and \(T^n(H)\) tends to \(A\) in the Hausdorff metric. Now, for any compact set \(L\) of \(X\), let \(L_1 = \bigcup_{j \geq 0} T^j(L)\). Since \(L_1\) is bounded, \(L_1 = L \cup T(L_1)\) and \(T\) is weak condensing, it follows that \(\alpha(L_1) = 0\), and, thus, \(H = \text{Cl}(L_1)\) is compact. Also \(T(H) \subset H\). Let \(A_L = \bigcap_{n \geq 0} T^n(H)\). But, by hypothesis, there is a closed bounded set \(B \subset X\) such that \(A_L \subset B\) for each compact set \(L\). Since \(T(\bigcup A_L) = \bigcup A_L\), where the union is taken over all compact sets \(L \subset X\), it follows that the set \(K = \text{Cl}(\bigcup A_L)\) is compact, \(T(K) \subset K\), and \(K\) attracts compact sets of \(X\).

Nussbaum [13] has shown that if a nonempty invariant set attracts compact sets then it attracts neighborhoods of points and so if \(T\) is weak condensing and compact dissipative it is local dissipative. This proves (a) and the proof of (b) is the same.

With a slight change in the argument above, we can prove the following.

**Lemma 3.** If \(T\) is a weak \(\alpha\)-contraction, then \(T\) asymptotically smooths.

**Proof.** If \(B\) is a bounded set, then \(B^* = \text{Cl}(\bigcup A_\alpha)\), where \(A_\alpha\) is constructed as above for the elements \(x \in B\) such that \(T^n x \in B\), for any \(n \geq 0\).

**Corollary.** If \(T\) is a weak \(\alpha\)-contraction and compact dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

**Proof.** Use Lemma 3, Theorem 2(a) and Lemma 1(a).

### 3. Fixed Point Theorems

In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious lemma.

**Lemma 4.** If \(A\) is a compact set of \(X\) and \(F \subset X\) contains a sequence \(\{x_n\}\) such that \(d(x_n, A) \to 0\) as \(n \to \infty\), then \(A \cap \text{Cl}(F) \neq \emptyset\).

**Theorem 3.** Suppose \(K \subset B \subset S \subset X\) are convex subsets with \(K\) compact, \(S\) closed, bounded, and \(B\) open in \(S\). If \(T: S \to X\) is continuous, \(T^j B \subset S\), \(j \geq 0\), and \(K\) attracts points of \(B\), then there is a convex, closed bounded subset \(A\) of \(S\) such that

\[
A = \overline{\bigcup_{j \geq 1} T^j(B \cap A)}, \quad A \cap K \neq \emptyset.
\]

**Proof.** Let \(\mathcal{F}\) be the set of convex, closed, bounded subsets \(L\) of \(S\) such that \(T^j(B \cap L) \subset L\) for \(j \geq 1\) and \(L \cap K \neq \emptyset\). The family \(\mathcal{F}\) is not empty.
because $S \in \mathcal{F}$. If $L \in \mathcal{F}$, let $L_1 = \overline{c_0[\bigcup_{j \geq 1} T^j(B \cap L)]}$. By Lemma 4, $L_1 \cap K \neq \emptyset$. Also, $L_1$ is convex, closed, and contained in $S$. Since $L \in \mathcal{F}$, we have $L \supset L_1$ and $L_1 \supset T^j(B \cap L) \supset T^j(B \cap L_1)$ for all $j \geq 1$. Thus, $L_1 \in \mathcal{F}$. It follows that a minimal element $A$ of $\mathcal{F}$ will satisfy the conditions of the theorem.

To prove such a minimal element exists, let $(L_\alpha)_{\alpha \in I}$ be a totally ordered family of sets in $\mathcal{F}$. The set $L = \bigcap_{\alpha \in I} L_\alpha$ is closed, convex and contained in $S$. Also, $T^j(B \cap L) \subsetneq T^j(B \cap L_\alpha)$ for any $\alpha \in I$ and $j \geq 1$. Thus, $T^j(B \cap L) \subset L$ for $j \geq 1$. If $J$ is any finite subset of $I$, we have $K \cap (\bigcap_{\alpha \in J} L_\alpha) \neq \emptyset$ and, from compactness, it follows that $K \cap (\bigcap_{\alpha \in I} L_\alpha) \neq \emptyset$. Thus, $L \cap \mathcal{F}$ and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following theorems and lemma.

**Theorem 4.** The set $A$ of Theorem 3 is compact if and only if there is a compact set $Q = Q(B)$ such that $Q \cap B \neq \emptyset$ and $T^j(Q \cap B) \subset Q$ for all $j \geq 0$.

**Lemma 5.** [Horn 8]. Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space $X$ with $S_0$, $S_2$ compact and $S_1$ open in $S_2$. Let $T : S_2 \to X$ be a continuous mapping such that for some integer $m > 0$, $T^j(S_1) \subset S_2$, $0 \leq j \leq m - 1$, $T^j(S_1) \subset S_2$, $m < j < 2m - 1$. Then $T$ has a fixed point.

**Theorem 5.** Suppose $K \subset B \subset S \subset X$ are convex subsets with $K$ compact, $S$ closed bounded and $B$ open in $S$. If $T : S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, $K$ attracts compact sets of $B$ and the set $A$ of Theorem 3 is compact, then $T$ has a fixed point.

**Proof.** Since $K$ is compact and convex, the set $B$ can be taken as $S \cap \mathcal{B}_{\delta}(K)$ for some $\delta > 0$. Let $Q$ be as in Theorem 4, $S_0 = \text{Cl}(\mathcal{B}_{\delta/2}(K)) \cap Q$, $S_1 = \mathcal{B}_{\delta}(K) \cap Q$ and $S_2 = S \cap Q$. Then $S_0 \subset S_1 \subset S_2$, $S_0$, $S_2$ compact and $S_1$ is open in $S_2$. Also, $T^j(S_1) \subset S_2$, $0 \leq j \leq n_1(K, \delta)$ and $T^j(S_1) \subset S_0$ for $j \geq n_1(K, \delta)$ for some integer $n_1(K, \delta)$. An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to Horn's theorem.

Any additional conditions on the map $T$ which will ensure that the set $A$ in Theorem 3 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is Theorem 6.

**Theorem 6.** If $T$ is weak condensing, then the set $A$ in Theorem 5 is compact.

**Proof.** If $\bar{A} = \bigcup_{j \geq 1} T^j(B \cap A)$, then $\bar{A} = T(B \cap A) \cup T(\bar{A})$ and $\alpha(\bar{A}) = \alpha(\bar{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\bar{A})))$. Since $\alpha(T(\bar{A})) < \alpha(\bar{A})$ if $\alpha(\bar{A}) > 0$, it follows that $\alpha(\bar{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then
\( \alpha(A) = \alpha(B \cap A) < \alpha(B \cap A) \leq \alpha(A) \), and this is a contradiction. Thus, \( \alpha(B \cap A) = 0 \). However, this implies \( \alpha(A) = 0 \) and \( A \) is compact, proving the theorem.

**Corollary 1.** If the sets \( K, B, \) and \( S \) in Theorem 5 exist and \( K \) attracts the compact sets of \( B \) and \( T \) is weak condensing, then \( T \) has a fixed point.

**Proof.** This is immediate from Theorems 5 and 6.

**Corollary 2.** If \( T: X \to X \) is continuous, pointwise dissipative, and \( T \) is weak completely continuous, then \( T \) has a fixed point.

**Proof.** This is immediate from Lemma 1(b) and Corollary 1.

**Corollary 3.** If \( T \) is a weak \( \alpha \)-contraction and there are sets \( K, B, \) and \( S \) as in Corollary 1, then \( T \) has a fixed point.

**Corollary 4.** If \( T \) is weak condensing and compact dissipative, then \( T \) has a fixed point.

**Proof.** From Theorem 2(a), \( T \) is a local dissipative system. Thus \( \overline{K} \) has an open convex neighborhood \( B \) with bounded orbit. The result now follows from Theorems 2, 5, and 6.

For \( \alpha \)-contractions, this result is contained in [13].

**Corollary 5.** If \( T^{n_0} \) is weak completely continuous, \( T \) is weak condensing and point dissipative, then \( T \) has a fixed point.

**Proof.** This follows from Lemma 1(b) and Corollary 4.

**Lemma 6.** If \( S: X \to X \) is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, \( \| \cdot \|_1 \), in \( X \) such that \( \| S \|_1 < 1 \).

**Proof.** Define \( \| x \|_1 = \| x \| + \| Sx \| + \cdots + \| S^n x \| + \cdots \). The assumption on the spectrum implies there is an \( 0 < r \leq 1 \) such that \( \| S^n \| < r^n \) if \( n \) is sufficiently large. Thus, there is a constant \( K \) such that \( \| x \| \leq \| x \|_1 \leq K \| x \| \). Also, for \( x \neq 0 \),

\[
\frac{\| Sx \|_1}{\| x \|_1} = 1 - \left[ 1 + \frac{\| Sx \|}{\| x \|} + \frac{\| S^2x \|}{\| x \|} + \cdots \right]^{-1} \leq 1 - \frac{1}{K}.
\]

The lemma is proved.

**Corollary 6.** If \( T \) is compact dissipative, \( T = S + U \), where \( S \) is linear
and continuous with spectrum contained in the open unit ball and \( \Omega \), \( T(\Omega) \) bounded implies \( \text{Cl}(U(\Omega)) \) compact for any \( \Omega \subset X \), then \( T \) has a fixed point. If, in addition, \( S^{no} \) is completely continuous and \( T \) is only point dissipative, then \( T \) has a fixed point.

**Proof.** The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that \( T^{no} \) is \( S^{no} \) plus a weak completely continuous operator.

The next result generalizes an asymptotic fixed point theorem of Browder [2].

**Theorem 7.** Suppose \( S_0, S_1, S_2 \) are subsets of a Banach space, \( S_0, S_2 \) convex, closed, \( S_1 \) open, \( S_2 \) bounded, \( S_0 \subset S_1 \subset S_2 \). Assume \( T: S_2 \rightarrow X \) is condensing in the following sense: if \( \Omega, T(\Omega) \) are contained in \( S_2 \) and \( \alpha(\Omega) > 0 \), then \( \alpha(T(\Omega)) < \alpha(\Omega) \). Assume also that \( T \) satisfies: for any compact set \( H \subset S_2 \), \( T^j(H) \subset S_2 \), \( j > 0 \), and there is a number \( N(H) \) such that \( T^j(H) \subset S_0 \) for \( j \geq N(H) \). Then \( T \) has a fixed point.

**Proof.** Following the proof of Theorem 2, there is a compact set \( K \) which attracts the compact sets of \( S_1 \). Since \( K \subset S_0 \), it follows that \( \overline{S} \subset K \subset S_0 \). Let \( B \) be a closed, convex neighborhood of \( \overline{S} \), \( B \subset S_2 \). Theorems 4 and 5 complete the proof.

4. DISSIPATIVE FLOWS

A family \( \{T(t), t \geq 0\} \) of mappings from \( X \) into \( X \) is an \( \omega \)-periodic (autonomous) flow if \( T(t)x \) is continuous, \( T(0)x = x \) and there is an \( \omega > 0 \) (for every \( \omega > 0 \)), \( T(t + \omega)x = T(t)T(\omega)x \) for all \( t, x \). A point \( x_0 \) corresponds to an \( \omega \)-periodic orbit (equilibrium point) if there is an \( \omega > 0 \) (for every \( \omega > 0 \)), \( T(t + \omega)x_0 = T(t)x_0 \) for all \( t \geq 0 \). For \( \omega \)-periodic flows, these \( x_0 \) coincide with the fixed points of \( T(\omega) \). The concepts of dissipativeness and attractors for flows are defined as obvious generalizations of the ones for discrete flows in Section 2.

**Lemma 7.** Let \( \{T(t), t \geq 0\} \) be an autonomous flow and \( \omega > 0 \) be arbitrary. If there is a sequence \( x_n \) satisfying \( T(\omega, n)x_n \rightarrow x_n \), \( \omega_n \rightarrow \omega/n \), and some subsequence converges to \( x_0 \) as \( n \rightarrow \infty \), then \( x_0 \) is an equilibrium point.

**Proof.** Changing the notation if necessary, we may assume that \( x_n \) converges to \( x_0 \). Let \( k_n(t) \) be the integer defined by: \( k_n(t)\omega_n \leq t < (k_n(t) + 1)\omega_n \). Then, \( T(k_n(t)\omega_n)x_n = x_n \) and so; \( |T(t)x_0 - x_0| \leq |T(t)x_0 - T(k_n(t)\omega_n)x_n| + |T(k_n(t)\omega_n)x_n - T(k_n(t)\omega_n)x_0| + |x_n - x_0| \).
Since \( k_n(t) \omega_n \) tends to \( t \) as \( n \to \infty \), the right hand side of the expression goes to zero, and this proves the lemma.

Let \( \{T(t), t \geq 0\} \) be either an \( \omega \)-periodic or an autonomous flow. We say that \( T(t) : X \to X, t \geq 0 \) fixed, is weak condensing if for any bounded set \( A \) for which \( \alpha(A) > 0 \) and \( \bigcup_{0 < t < \infty} T(t, A) \) is bounded it follows that \( \alpha(T(r, A)) < \alpha(A) \).

The proof of the next theorem is exactly the same as the one given for discret dynamical systems.

**Theorem 8.** If the \( \omega \)-periodic flow \( \{T(t), t \geq 0\} \) is compact dissipative and \( T(\omega) \) is weak condensing, then it is local dissipative. In particular, there are convex bounded sets \( K \subset S_1 \subset S_2 \), \( K \) compact, \( S_2 \) closed and \( S_1 \) open with the following property: \( T(t, S_1) \subset S_2 \) for \( t \geq 0 \) and any compact set \( H \subset S_1 \) is attracted by \( K \). If the flow is point dissipative and, for some integer \( N \), \( T(N \omega) \) is weak completely continuous, then the flow is local dissipative.

As an application of Theorem 7, Lemma 7, and Theorem 8 we get the following.

**Theorem 9.** If the \( \omega \)-periodic flow \( \{T(t), t \geq 0\} \) is weak condensing, then any of the following conditions is sufficient to guarantee the existence of an \( \omega \)-periodic orbit: (a) the flow is compact dissipative; (b) the flow is point dissipative and \( T(N \omega) \) is weak completely continuous for some integer \( N \). If the flow is autonomous and, for some \( \omega > 0 \), \( T(t) \) is weak condensing for any \( t \) in \((0, \omega]\), then the same assumptions imply the existence of an equilibrium point.

**Corollary 7.** Let \( \{T(t), t \geq 0\} \) be an \( \omega \)-periodic flow. If \( T(t) = S(t) + U(t) \), where \( S(t) \) is a bounded linear operator such that \( S^n(\omega) = S(n\omega) \) for any integer \( n \), \( S(\omega) \) has spectral radius less than one and \( \bigcup_{0 < s \leq \omega} T(s, A) \) is bounded it follows that \( U(\omega, A) \) has compact closure, then the conclusions of Theorem 9 for compact dissipative hold. Furthermore, if for some \( N \), \( S(N \omega) \) is completely continuous, then the conclusions for point dissipative also hold.

### 5. Functional Differential Equations

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let \( r \geq 0 \) be a given real number, \( E^n \) be an \( n \)-dimensional linear vector space with norm \( | \cdot | \), \( C([a, b], E^n) \) be the space of continuous functions from \([a, b]\) to \( E^n \) with the uniform topology and let \( C = C([-r, 0], E^n) \). For \( \varphi \in C, | \varphi | = \sup_{-r < t < 0} | \varphi(t) \) for any \( x \in C([-r, a], E^n), a \geq 0 \), let \( x_t \in C, t \in [0, A], \)
be defined by \( x(t + \theta) = x(t), -r \leq \theta \leq 0 \). Suppose \( D: R \times C \to E^n \) is a continuous linear operator \( D\varphi = \varphi(0) - g(\varphi) \),

\[
g(\varphi) = \int_{-r}^{0} [d\mu(\theta)] \varphi(\theta)
\]

for \( s \geq 0 \), \( \varphi \in C \) where \( \mu \) is an \( n \times n \) matrix function of bounded variation, \( \gamma \) is continuous and nondecreasing on \([0, r]\), \( \gamma(0) = 0 \). If \( f: R \times C \to E^n \) is continuous, then a NFDE is a relation

\[
(d/dt) D(x_t) = f(t, x_t).
\]

A solution \( x = x(\varphi) \) through \( \varphi \) at time \( \sigma \) is a continuous function defined on \([\sigma - r, \sigma + A]\), \( A > 0 \), such that \( x_\sigma = \varphi, D(x_t) \) is continuously differentiable on \([\sigma, \sigma + A]\) and \( (2) \) is satisfied on \([\sigma, \sigma + A]\). We assume a solution \( x(\varphi) \) of \( (2) \) through any \( \varphi \in C \) exists on \([\sigma - r, \infty)\), is unique and \( x(\varphi)(t) \) depends continuously on \((\varphi, t) \in C \times [\sigma - r, \infty)\).

In the following, we let \( T_p(t): C \to C, t \geq 0 \), be the continuous linear operator defined by \( T_p(t)\varphi = y_\varphi(t), t \geq 0 \), where \( y = y(\varphi) \) is the solution of

\[
(d/dt) D(y_t) = 0, \quad y_0 = \varphi.
\]

If \( D \) is \( \omega \)-periodic in \( t, C_D = \{ \varphi \in C: D(\varphi) = 0 \} \), then \( C_D \) is a Banach space with the topology of \( C \), \( T_p(\omega, 0): C_D \to C_D \), and \( T_p(n\omega, 0) = T_p^n(\omega, 0) \).

The operator \( D \) is said to be uniformly stable if there exist constants \( K > 1, \alpha > 0 \), such that

\[
|T_p(t)\varphi| \leq Ke^{-\alpha t} |\varphi|, \quad \varphi \in C_D, \quad t \geq \sigma.
\]

Notice the operator \( D\varphi = \varphi(0) \) corresponding to retarded functional differential equations is always stable.

**Remark.** The conclusion of the main theorem below is valid for the more general \( D(t, \varphi) = D_0(t, \varphi) + \int_{-r}^{0} A(t, \varphi) \varphi(\theta) d\theta \), where \( D_0 \) is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that \( D \) uniformly stable implies there exists an \( n \times n \) matrix function \( B(t) \) defined and of bounded variation on \([-r, \infty)\), continuous from the left, \( B(t) = 0, -r \leq t \leq 0 \), and a constant \( M_1 \) such that

\[
|T_p(t)\varphi| \leq M_1 |\varphi|, \quad t \geq 0, \quad \varphi \in C, \quad \sup_{t \in (-r, 0]} |B(t)| \leq M_1.
\]
and, for any continuous function \( h: [0, \infty) \to \mathbb{R}^n \), the solution of the problem,

\[
D(x_t) = D(\varphi) + \int_0^t h(s) \, ds, \quad x_0 = \varphi,
\]

is given by

\[
x_t = T_D(t)\varphi - \int_0^t B_{t-s}h(s) \, ds.
\]

Furthermore, there exist \( n \) functions \( \varphi_1, \ldots, \varphi_n \) in \( C \) such that \( D(\Phi) = I \), the identity, where \( \Phi = (\varphi_1, \ldots, \varphi_n) \).

Let \( \psi: C \to C_D \) be the continuous linear operator defined by \( \psi(\varphi) = \varphi - \Phi \, D(\varphi) \).

**Lemma 8.** If \( D \) is uniformly stable and \( f \) maps bounded sets of \( R \times C \) into bounded sets of \( \mathbb{R}^n \), then there is a family of continuous transformations \( T_1(t): C \to C, t \geq 0 \) which are weak completely continuous and

\[
T(t)\varphi \overset{\text{def}}{=} x_t(\varphi) = T_D(t)\psi(\varphi) + T_1(t)\varphi.
\]

If \( D \varphi = \varphi(0) \), then \( T(t) \) is weak completely continuous for \( t \geq r \).

**Proof.** Equation (2) with initial value \( x_0 = \varphi \) is equivalent to

\[
D(x_t) = D(\varphi) + \int_0^t f(s, x_s) \, ds, \quad t \geq 0, \quad x_0 = \varphi,
\]

which from (7) is equivalent to

\[
T(t)\varphi \overset{\text{def}}{=} x_t = T_D(t)\psi(\varphi) + T_D(t)D(\varphi) - \int_0^t B_{t-s}f(s, x_s) \, ds
\]

\[
= T_D(t)\psi(\varphi) + T_1(t).
\]

It is now an easy matter to verify the assertions in the theorem.

Since the condition that \( D \) is uniformly stable implies the linear operator \( S(\omega) = T_D(\omega)\psi \) has spectrum contained inside the unit ball, Corollary 6, Lemma 1(b) and Corollary 7 imply.

**Theorem 4.** If there exists an \( \omega > 0 \) such that \( f(t + \omega, \varphi) = f(t, \varphi) \) for all \( \varphi \in C \), \( f \) takes bounded sets of \( R \times C \to \mathbb{R}^n \) and system (2) is compact dissipative, then there is an \( \omega \)-periodic solution of (2). If \( f \) satisfies the same hypotheses and is independent of \( t \), then there is a constant function \( c \) in \( C \) such that \( f(c) = 0 \); that is, an equilibrium point of (2). If \( D(\varphi) = \varphi(0) \), then the same conclusions are true for point dissipative.
REFERENCES