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Eriksson's numbers game and finite Coxeter groups

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Abstract

The numbers game is a one-player game played on a finite simple graph with certain "amplitudes" assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. This game and its interactions with Coxeter/Weyl group theory and Lie theory have been studied by many authors. In particular, Eriksson connects certain geometric representations of Coxeter groups with games on graphs with certain real number amplitudes. Games played on such graphs are "E-games". Here we investigate various finiteness aspects of E-game play: We extend Eriksson's work relating moves of the game to reduced decompositions of elements of a Coxeter group naturally associated to the game graph. We use Stembridge's theory of fully commutative Coxeter group elements to classify what we call here the "adjacency-free" initial positions for finite E-games. We characterize when the positive roots for certain geometric representations of finite Coxeter groups can be obtained from E-game play. Finally, we provide a new Dynkin diagram classification result of E-game graphs meeting a certain finiteness requirement.

1. Introduction

The numbers game is a one-player game played on a finite simple graph with weights (which we call "amplitudes") on its edges and with an initial assignment of real numbers to its nodes. Each of the two edge amplitudes (one for each direction) will be certain negative real numbers. The move a player can make is to "fire" one of the nodes with a positive number. This move transforms the number at the fired node by changing its sign, and it also transforms the number at each adjacent node in a certain way using an amplitude along the incident edge. The player

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fires the nodes in some sequence of the player's choosing, continuing until no node has a positive number.

The numbers game has been an object of interest for many authors. For graphs with integer amplitudes the game is attributed to Mozes [23]. Eriksson has studied the game extensively, see for example [11–16,9]. Eriksson's numbers game allows for certain real number amplitudes. Particularly important for this paper is his ground-breaking work in [12,15,16] analysing convergence of numbers games and of the connection between the numbers game and Coxeter groups. Much of the numbers game discussion in Section 4.3 of the book [5] by Björner and Brenti can be found in [12,15]. The game has also been studied by Proctor [24,25], Björner [4], and Wildberger [28–30]. Wildberger studies a dual version which he calls the "mutation game." See Alon et al. [3] for a brief and readable treatment of the numbers game on "unweighted" cyclic graphs. The numbers game facilitates computations with Coxeter groups and their geometric representations (e.g. see Section 4.3 of [5] or Sections 3 and 4). Proctor developed this process in [24] to compute Weyl group orbits of weights with respect to the fundamental weight basis. Here we use his perspective of firing nodes with positive, as opposed to negative, numbers. In [10], we use data from certain numbers games to obtain distributive lattice models for families of semisimple Lie algebra representations and their Weyl characters.

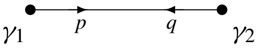
This paper extends Eriksson's work, focusing on play from "dominant" positions where all numbers are either fireable or zero, for which the connection to Coxeter groups turns out to be quite explicit. We will let J denote the set of nodes where the numbers are zero, and J^c denotes its complement. The main results can be summarized as follows: In Section 3 we show how, under certain finiteness assumptions, legal play sequences from a J^c-dominant position correspond to reduced words in the quotient W^J. In Section 4 we relate Stembridge's notion of full commutativity for finite quotients W^J to the J^c-dominant positions for which no game results in positive numbers on adjacent nodes. We then use a result of Stembridge to classify these "adjacency-free" positions. In Section 5 we say precisely when all positive roots in the root system for a geometric representation of a finite Coxeter group can be obtained from a legal play sequence. In Section 6 we show that playing from a dominant position, the game will terminate if and only if it is played on a graph corresponding to a finite Coxeter group. (Another proof of this result based on ideas from [12] is given in [9].) The geometric representations which connect the numbers game and Coxeter groups were introduced in [12,15] and studied further in Sections 4.1–4.3 of [5,8]. Definitions and results about these representations which are needed here are given in Section 2. There we also record several key results that are used throughout Sections 3–6: Eriksson's Strong Convergence Theorem, Eriksson's Comparison Theorem, and Eriksson's Reduced Word Result.

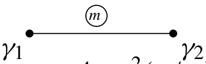
2. Definitions and preliminary results

Fix a positive integer n and a totally ordered set I_n with n elements (usually $I_n := \{1 < \cdots < n\}$). An *E-generalized Cartan matrix* or *E-GCM*[2] is an $n \times n$ matrix $M = (M_{ij})_{i,j \in I_n}$ with real entries satisfying the requirements that each main diagonal matrix entry is 2, that all other

[2] Motivation for terminology: E-GCMs with integer entries are just generalized Cartan matrices, which are the starting point for the study of Kac–Moody algebras: beginning with a GCM, one can write down a list of the defining relations for a Kac–Moody algebra as well as its associated Weyl group [21,22]. Here we use the modifier "E" because of the relationship between these matrices and the combinatorics of Eriksson's E-games. Eriksson uses "E" for edge; he also allows for "N-games" where, in addition, nodes can be weighted.

matrix entries are nonpositive, that if a matrix entry M_{ij} is nonzero then its transpose entry M_{ji} is also nonzero, and that if $M_{ij}M_{ji}$ is nonzero then $M_{ij}M_{ji} \geq 4$ or $M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij})$ for some integer $k_{ij} \geq 3$. These peculiar constraints on products of transpose pairs of matrix entries are precisely those required in order to guarantee "strong convergence" for E-games, cf. Theorem 2.1, Theorem 3.6 of [12], Theorem 3.1 of [16]. To an $n \times n$ E-generalized Cartan matrix $M = (M_{ij})_{i,j \in I_n}$ we associate a finite graph Γ (which has undirected edges, no loops, and no multiple edges) as follows: The nodes $(\gamma_i)_{i \in I_n}$ of Γ are indexed by the set I_n, and an edge is placed between nodes γ_i and γ_j if and only if $i \neq j$ and the matrix entries M_{ij} and M_{ji} are nonzero. We call the pair (Γ, M) an *E-GCM graph*. We depict a generic two-node E-GCM graph as follows:



In this graph, $p = -M_{12}$ and $q = -M_{21}$. We use  for the collection of all two-node E-GCM graphs for which $M_{12}M_{21} = pq = 4\cos^2(\pi/m)$ for an integer $m > 3$; we use $m = \infty$ if $M_{12}M_{21} = pq \geq 4$. When $m = 3$ (i.e. $pq = 1$), we use an unlabelled edge . An *E-Coxeter graph* will be any E-GCM graph whose connected components come from one of the collections of Fig. 2.1.

For the remainder of the paper the notation (Γ, M) refers to an arbitrarily fixed E-GCM graph with nodes indexed by I_n, unless (Γ, M) is otherwise specified. A *position* $\lambda = (\lambda_i)_{i \in I_n}$ is an assignment of real numbers to the nodes of (Γ, M). The position λ is *dominant* (respectively, *strongly dominant*) if $\lambda_i \geq 0$ (respectively $\lambda_i > 0$) for all $i \in I_n$; λ is *nonzero* if at least one $\lambda_i \neq 0$. For $i \in I_n$, the *fundamental position* ω_i is the assignment of the number 1 at node γ_i and the number 0 at all other nodes. Given a position λ for (Γ, M), to *fire* a node γ_i is to change the number at each node γ_j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node γ_i is positive. Otherwise, node γ_i is not allowed to be fired. In view of this transformation we think of entries of the E-GCM as *amplitudes*, and we sometimes refer to E-GCMs as *amplitude matrices*. The *numbers game* is the one-player game on (Γ, M) in which the player (1) Assigns an initial position to the nodes of Γ; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number.[3] Consider now the E-Coxeter graph in the $\mathcal{I}_2^{(4)}$ family depicted in Fig. 2.2. As we can see in Fig. 2.2, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position λ, a *game sequence for* λ is the (possibly empty, possibly infinite) sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$, where γ_{i_j} is the jth node that is fired in some numbers game with initial position λ. More generally, a *firing sequence* from some position λ is an initial portion of some game sequence played from λ. The phrase *legal firing sequence* is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is of finite length l (possibly with $l = 0$) if the number is nonpositive at each

[3] Mozes studied numbers games on E-GCM graphs with integer amplitudes and for which the amplitude matrix M is *symmetrizable* (i.e. there is a nonsingular diagonal matrix D such that $D^{-1}M$ is symmetrical). In [23] he obtained strong convergence results and a geometric characterization of the initial positions for which the game terminates.

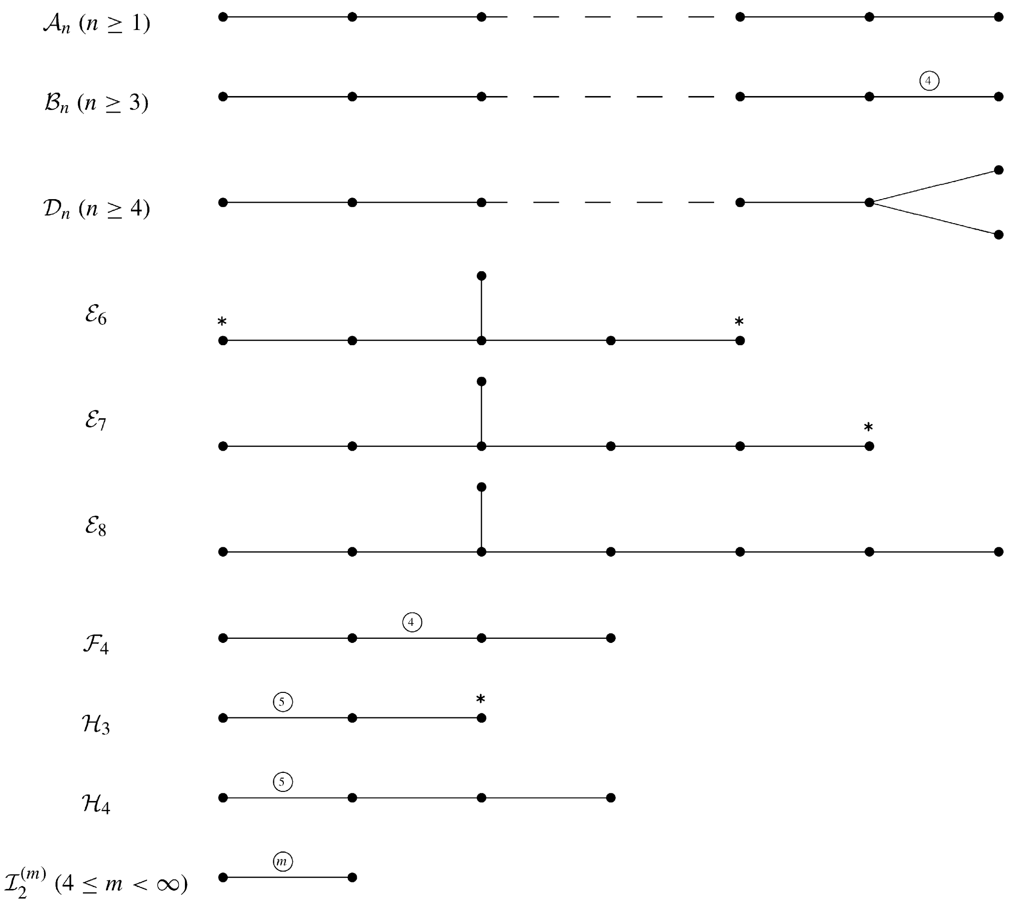


Fig. 2.1. Families of connected E-Coxeter graphs. (For adjacent nodes, the notation ⓜ means that the amplitude product on the edge is $4\cos^2(\pi/m)$; for an unlabelled edge take $m = 3$. The asterisks for $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{H}_3$ pertain to Theorem 4.2.).

node after the lth firing. In this case we say the game sequence is *convergent* and the resulting position is the *terminal position* for the game sequence.

Following [12,16], we say the numbers game on an E-GCM graph (Γ, M) is *strongly convergent* if given any initial position, every game sequence either diverges or converges to the same terminal position in the same number of steps. The next result follows from Theorem 3.1 of [16] (or see Theorem 3.6 of [12]).

Theorem 2.1 (*Eriksson's Strong Convergence Theorem*). *The numbers game on a connected E-GCM graph is strongly convergent.*

The following weaker result also applies when the E-GCM graph is not connected:

Lemma 2.2. *For any E-GCM graph, if a game sequence for an initial position λ diverges, then all game sequences for λ diverge.*

The next result is an immediate consequence of Theorem 4.3 of [12] or Theorem 4.5 of [15]. Eriksson's proof of this result in [12] uses only combinatorial and linear algebraic methods.

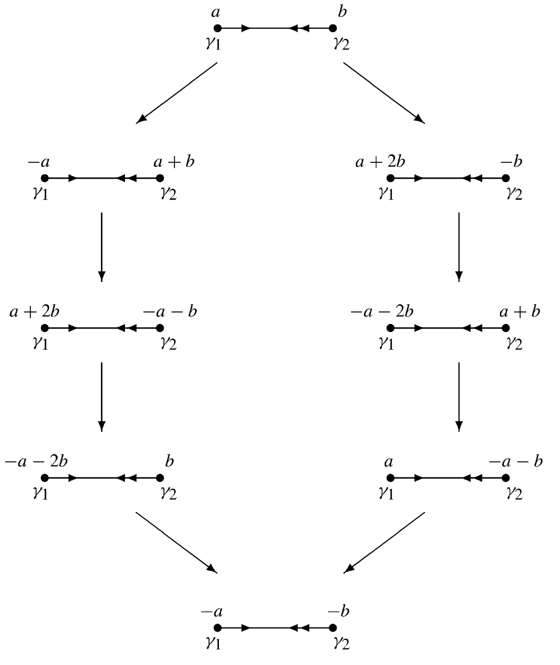


Fig. 2.2. The numbers game for an E-Coxeter graph in the $\mathcal{I}_2^{(4)}$ family.

Theorem 2.3 (*Eriksson's Comparison Theorem*). *Given an E-GCM graph, suppose that a game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$ converges. Suppose that a position $\lambda' := (\lambda'_i)_{i \in I_n}$ has the property that $\lambda'_i \leq \lambda_i$ for all $i \in I_n$. Then some game sequence for the initial position λ' also converges.*

Let r be a positive real number. Observe that if $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$, then $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for the initial position $r\lambda := (r\lambda_i)_{i \in I_n}$. This observation and Theorem 2.3 imply the following result:

Lemma 2.4. *Let $\lambda = (\lambda_i)_{i \in I_n}$ be a dominant initial position such that $\lambda_j > 0$ for some $j \in I_n$. Suppose that a game sequence for λ converges. Then some game sequence for the fundamental position ω_j also converges.*

The following is an immediate consequence of Lemmas 2.2 and 2.4:

Lemma 2.5. *An E-GCM graph is not admissible if for each fundamental position there is a divergent game sequence.*

The following is proved easily with an induction argument on the number of nodes.

Lemma 2.6. *Suppose (Γ, M) is connected with nonzero dominant position λ. Then in any convergent game sequence for λ, every node of Γ is fired at least once.*

If I'_m is a subset of the node set I_n of (Γ, M), then let Γ' be the subgraph of Γ with node set I'_m and the induced set of edges, and let M' be the corresponding submatrix of the amplitude matrix M. We call (Γ', M') an *E-GCM subgraph* of (Γ, M). In light of Lemmas 2.2 and 2.6, the following result amounts to an observation.

Lemma 2.7. *If a connected E-GCM graph is admissible, then any connected E-GCM subgraph is also admissible.*

Define the associated Coxeter group $W = W(\Gamma, M)$ to be the Coxeter group with identity denoted ε, generators $\{s_i\}_{i \in I_n}$, and defining relations $s_i^2 = \varepsilon$ for $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$, where the m_{ij} are determined as follows:

$$m_{ij} = \begin{cases} k_{ij} & \text{if } M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } M_{ij}M_{ji} \geq 4. \end{cases}$$

(Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j.) Throughout the paper, W denotes the Coxeter group $W(\Gamma, M)$ associated to an arbitrarily fixed E-GCM graph (Γ, M) with index set I_n. One can think of the E-GCM graph as a refinement of the information from the Coxeter graph for the associated Coxeter group. Observe that any Coxeter group on a finite set of generators is isomorphic to the Coxeter group associated to some E-GCM graph. The Coxeter group W is *irreducible* if Γ is connected. Let ℓ denote the length function for the W. An expression $s_{i_p} \cdots s_{i_2} s_{i_1}$ for an element of W is *reduced* if $\ell(s_{i_p} \cdots s_{i_2} s_{i_1}) = p$. An empty product in W is taken as ε. For a firing sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ from some initial position on (Γ, M), the corresponding element of W is taken to be $s_{i_p} \cdots s_{i_2} s_{i_1}$. Parts (1) and (2) of what we call Eriksson's Reduced Word Result follow respectively from Propositions 4.1 and 4.2 of [15].

Theorem 2.8 (*Eriksson's Reduced Word Result*). (1) *If $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ is a legal sequence of node firings in a numbers game played from some initial position on (Γ, M), then $s_{i_p} \cdots s_{i_2} s_{i_1}$ is a reduced expression for the corresponding element of W.* (2) *If $s_{i_p} \cdots s_{i_2} s_{i_1}$ is a reduced expression for an element of W, then $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ is a legal sequence of node firings in a numbers game played from any given strongly dominant position on (Γ, M).*

To conclude this section we summarize results from [8] concerning certain geometric representations of Coxeter groups introduced by Eriksson in [12,15]. Let V be a real n-dimensional vector space freely generated by $(\alpha_i)_{i \in I_n}$ (elements of this ordered basis are *simple roots*). Equip V with a possibly asymmetrical bilinear form $B : V \times V \to \mathbb{R}$ defined on the basis $(\alpha_i)_{i \in I_n}$ by $B(\alpha_i, \alpha_j) := \frac{1}{2} M_{ij}$. For each $i \in I_n$ define an operator $S_i : V \to V$ by the rule $S_i(v) := v - 2B(\alpha_i, v)\alpha_i$ for each $v \in V$. One can check that S_i^2 is the identity transformation, so $S_i \in GL(V)$.

As can be seen, for example, in [5] Theorem 4.2.2, there is a unique homomorphism $\sigma_M : W \to GL(V)$ for which $\sigma_M(s_i) = S_i$. Theorem 4.2.7 of [5] shows that σ_M is injective. We call σ_M a *geometric representation* of W. We now have W acting on V, and for all $w \in W$ and $v \in V$ we write $w.v$ for $\sigma_M(w)(v)$. Define $\Phi_M := \{\alpha \in V \mid \alpha = w.\alpha_i$ for some $i \in I_n$ and $w \in W\}$. For each $w \in W$, $\sigma_M(w)$ permutes Φ_M, so σ_M induces an action of W on Φ_M. Evidently, $\Phi_M = -\Phi_M$. Elements of Φ_M are *roots* and are necessarily nonzero. If $\alpha = \sum c_i \alpha_i$ is a root with all c_i nonnegative (respectively nonpositive), then say α is a *positive* (respectively *negative*) root. Let Φ_M^+ and Φ_M^- denote the collections of positive and negative roots respectively. Let $w \in W$ and $i \in I_n$. Proposition 4.2.5 of [5] states: *If $\ell(ws_i) > \ell(w)$ then $w.\alpha_i \in \Phi_A^+$, and if $\ell(ws_i) < \ell(w)$ then $w.\alpha_i \in \Phi_A^-$*. It follows that Φ_M is partitioned by Φ_M^+ and Φ_M^-.

We say two adjacent nodes γ_i and γ_j in (Γ, M) are *odd-neighbourly* if m_{ij} is odd, *even-neighbourly* if $m_{ij} \geq 4$ is even, and ∞-*neighbourly* if $m_{ij} = \infty$. When m_{ij} is odd and $M_{ij} \neq M_{ji}$, we say that the adjacent nodes γ_i and γ_j form an *odd asymmetry*. For odd m_{ij}, let v_{ji} be the element $(s_i s_j)^{(m_{ij}-1)/2}$, and set $K_{ji} := \frac{-M_{ji}}{2\cos(\pi/m_{ij})}$, which is positive. It is a consequence of Lemma 3.1 of [8] that $v_{ji}.\alpha_i = K_{ji}\alpha_j$. Observe that $K_{ij}K_{ji} = 1$ and moreover that $v_{ij} = v_{ji}^{-1}$. A *path of odd neighbours* (or *ON-path*, for short) in (Γ, M) is a sequence $\mathcal{P} := [\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_p}]$ of nodes from Γ for which consecutive pairs are odd-neighbourly. This ON-path has length p, and we allow ON-paths to have length zero. We say γ_{i_0} and γ_{i_p} are the *start* and *end* nodes of the ON-path, respectively. Let $w_{\mathcal{P}} \in W$ be the Coxeter group element $v_{i_p i_{p-1}} \cdots v_{i_2 i_1} v_{i_1 i_0}$, and let $\Pi_{\mathcal{P}} := K_{i_p i_{p-1}} \cdots K_{i_2 i_1} K_{i_1 i_0}$, where $w_{\mathcal{P}} = \varepsilon$ with $\Pi_{\mathcal{P}} = 1$ when $\mathcal{P}$ has length zero. Note that $w_{\mathcal{P}}.\alpha_{i_0} = \Pi_{\mathcal{P}}\alpha_{i_p}$. The next result follows from Theorem 3.3 of [8].

Proposition 2.9. *Let $w \in W$ and $i \in I_n$. (1) Then $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K > 0$ if and only if $w.\alpha_i = w_{\mathcal{P}}.\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$, in which case $K = \Pi_{\mathcal{P}}$. (2) Similarly $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K < 0$ if and only if $w.\alpha_i = (w_{\mathcal{P}}s_i).\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$, in which case $K = -\Pi_{\mathcal{P}}$.*

An ON-path $\mathcal{P} = [\gamma_{i_0}, \ldots, \gamma_{i_p}]$ is an *ON-cycle* if $\gamma_{i_p} = \gamma_{i_0}$. It is a *unital* ON-cycle if $\Pi_{\mathcal{P}} = 1$. For ON-paths $\mathcal{P}$ and $\mathcal{Q}$, write $\mathcal{P} \sim \mathcal{Q}$ and say $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* if these ON-paths have the same start and end nodes and $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}}$. This is an equivalence relation on the set of all ON-paths. An ON-path $\mathcal{P}$ is *simple* if it has no repeated nodes with the possible exception that the start and end nodes may coincide. We say (Γ, M) is *unital ON-cyclic* if and only if $\Pi_{\mathcal{C}} = 1$ for all ON-cycles $\mathcal{C}$. Note that (Γ, M) is unital ON-cyclic if and only if $\mathcal{P} \sim \mathcal{Q}$ whenever $\mathcal{P}$ and $\mathcal{Q}$ are ON-paths with the same start and end nodes. The property that (Γ, M) has no odd asymmetries is sufficient but not necessary to imply that (Γ, M) is unital ON-cyclic. An E-GCM graph is *ON-connected* if any two nodes can be joined by an ON-path. An *ON-connected component* of (Γ, M) is an E-GCM subgraph (Γ', M') whose nodes form a maximal collection of nodes in (Γ, M) which can be pairwise joined by ON-paths. For any $\alpha \in \Phi_M$, set $\mathfrak{S}_M(\alpha) := \{K\alpha \mid K \in \mathbb{R}\} \cap \Phi_M^+$. The next result is Theorem 3.6 of [8].

Proposition 2.10. *Choose any ON-connected component (Γ', M') of (Γ, M), and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Then (Γ', M') is unital ON-cyclic if and only if $|\mathfrak{S}_M(\alpha_x)| < \infty$ for some $x \in J$ if and only if $|\mathfrak{S}_M(\alpha_x)| < \infty$ for all $x \in J$, in which case we have $|\mathfrak{S}_M(\alpha_x)| = |\mathfrak{S}_M(\alpha_y)|$ for all $x, y \in J$.*

For any $w \in W$, set $N_M(w) := \{\alpha \in \Phi_M^+ \mid w.\alpha \in \Phi_M^-\}$. The following is Lemma 3.8 of [8]:

Lemma 2.11. *For any $i \in I_n$, $s_i(\Phi_M^+ \setminus \mathfrak{S}_M(\alpha_i)) = \Phi_M^+ \setminus \mathfrak{S}_M(\alpha_i)$. Now let $w \in W$. If $w.\alpha_i \in \Phi_M^+$, then $N_M(ws_i) = s_i(N_M(w)) \dot\cup \mathfrak{S}_M(\alpha_i)$, a disjoint union. If $w.\alpha_i \in \Phi_M^-$, then $N_M(ws_i) = s_i(N_M(w) \setminus \mathfrak{S}_M(\alpha_i))$.*

When (Γ, M) is ON-connected and unital ON-cyclic, let $f_{\Gamma,M} := |\mathfrak{S}_M(\alpha_x)|$ for any fixed $x \in I_n$. For $J \subseteq I_n$, let $\mathfrak{C}(J)$ denote the set of all ON-connected components of (Γ, M) containing some node from the set $\{\gamma_x\}_{x \in J}$. The next result is Theorem 3.9 of [8].

Proposition 2.12. *Let $w \in W$ with $p = \ell(w) > 0$. (1) Then $N_M(w)$ is finite if and only if w has a reduced expression $s_{i_1} \cdots s_{i_p}$ for which $\mathfrak{S}_M(\alpha_{i_q})$ is finite for all $1 \le q \le p$ if and only if every reduced expression $s_{i_1} \cdots s_{i_p}$ for w has $\mathfrak{S}_M(\alpha_{i_q})$ finite for all $1 \le q \le p$. (2) Now suppose $w = s_{i_1} \cdots s_{i_p}$ and $N_M(w)$ is finite. Let $J := \{i_1, \ldots, i_p\}$. In view of (1), let f_1 be the min and f_2 the max of all integers in the set $\{f_{\Gamma',M'} \mid (\Gamma', M') \in \mathfrak{C}(J)\}$. Then $f_1\,\ell(w) \le |N_M(w)| \le f_2\,\ell(w)$.*

We have the natural pairing $\langle \lambda, v \rangle := \lambda(v)$ for elements λ in the dual space V^* and vectors v in V. We think of V^* as the space of positions for numbers games played on (Γ, M): For $\lambda \in V^*$, the numbers for the corresponding position are $(\lambda_i)_{i \in I_n}$ where for each $i \in I_n$ we have $\lambda_i := \langle \lambda, \alpha_i \rangle$. Regard the fundamental positions $(\omega_i)_{i \in I_n}$ to be the basis for V^* dual to the basis $(\alpha_j)_{j \in I_n}$ for V relative to the natural pairing $\langle \cdot, \cdot \rangle$, so $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. Given $\sigma_M : W \to GL(V)$, the contragredient representation $\sigma_M^* : W \to GL(V^*)$ is determined by $\langle \sigma_M^*(w)(\lambda), v \rangle = \langle \lambda, \sigma_M(w^{-1})(v) \rangle$. From here on, when $w \in W$ and $\lambda \in V^*$, write $w.\lambda$ for $\sigma_M^*(w)(\lambda)$. Then $s_i.\lambda$ is the result of firing node γ_i when the E-GCM graph is assigned position λ, whether the firing is legal or not. We have a one-to-one correspondence between roots and certain elements of V^{**}: Given a root α, the *root functional* $\phi_\alpha : V^* \to \mathbb{R}$ is given by $\phi_\alpha(\mu) = \langle \mu, \alpha \rangle$, and ϕ_α is *positive* (resp. *negative*) if $\alpha \in \Phi_M^+$ (resp. Φ_M^-).

Remark 2.13. From the definitions one sees that the following are equivalent: (1) $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is legally played from some position λ, (2) $\langle s_{i_{q-1}} \cdots s_{i_1}.\lambda, \alpha_{i_q} \rangle > 0$ for $1 \leq q \leq p$, (3) $\langle \lambda, \beta_q \rangle > 0$ where $\beta_q := s_{i_1} \cdots s_{i_{q-1}}.\alpha_{i_q}$ for $1 \leq q \leq p$, (4) $\phi_{\beta_q}(\lambda) > 0$ for $1 \leq q \leq p$. That $\beta_q \in \Phi_M^+$ for $1 \leq q \leq p$ follows from [5] Proposition 4.2.5 and the fact that $\ell(s_{i_1} \cdots s_{i_{q-1}}) < \ell(s_{i_1} \cdots s_{i_{q-1}} s_{i_q})$. □

Let D be the set of dominant positions. The *Tits cone* is $U_M := \cup_{w \in W} wD$. The next result is Theorem 4.3 of [8].

Proposition 2.14. *Suppose (Γ, M) is connected and unital ON-cyclic. If the Coxeter group W is infinite, then $U_M \cap (-U_M) = \{0\}$.*

In Section 4 of [15], Eriksson characterizes the set of initial positions for which the game converges. In contrast to [15], here we fire at nodes with positive rather than negative numbers, so we have $-U_M$ instead of U_M in the following statement:

Theorem 2.15 (*Eriksson*). *The set of initial positions for which the numbers game on the E-GCM graph (Γ, M) converges is precisely $-U_M$.*

3. Extensions of Eriksson's Reduced Word Result for dominant positions

In this section we consider legal play sequences from dominant positions with a specified set J of nodes where the numbers are zero. This leads to certain extensions of Eriksson's Reduced Word Result in Proposition 3.2 and Corollary 3.4. Eriksson's Strong Convergence Theorem is used in deriving two corollaries to Proposition 3.2. For any $J \subseteq I_n$, W_J is the subgroup generated by $\{s_i\}_{i \in J}$, a *parabolic* subgroup, and $W^J := \{w \in W \mid \ell(ws_j) > \ell(w)$ for all $j \in J\}$ is the set of *minimal coset representatives* (see [5] Ch. 2). When $J = \emptyset$, W_J is the one-element group and $W^J = W$. If W is finite, we may choose the (unique) longest element w_0 in W. Since we must have $\ell(w_0 s_i) < \ell(w_0)$ for all $i \in I_n$, it follows that $w_0.\alpha_i \in \Phi_M^-$ for all i. So if $\alpha = \sum c_i \alpha_i \in \Phi_M^+$, then $w_0.\alpha \in \Phi_M^-$, i.e. $N_M(w_0) = \Phi_M^+$. More generally, for any W (not necessarily finite) and for any subset J of I_n, we let $(w_0)_J$ denote the longest element of W_J when W_J is finite.

Lemma 3.1. *Let $J \subseteq I_n$, and suppose W_J is finite. Suppose $\alpha = \sum_{j \in J} c_j \alpha_j$ is a root in Φ_M^+. Then $(w_0)_J.\alpha \in \Phi_M^-$.*

Proof. Note that any element of W_J preserves the subspace $V_J := \mathrm{span}_{\mathbb{R}}\{\alpha_j\}_{j \in J}$. As seen just above, $(w_0)_J$ will send each simple root α_j for $j \in J$ to some root in Φ_M^-. Then $(w_0)_J.\alpha \in \Phi_M^-$. $\square$

In what follows, for any subset J of I_n, a position λ is J^c*-dominant* if its zeros are precisely on the nodes in set J, i.e. $\lambda = \sum_{i \in I_n \setminus J} \lambda_i \omega_i$ with $\lambda_i > 0$ for all $i \in I_n \setminus J$. Part (2) of Eriksson's Reduced Word Result and the "if" direction of Theorem 4.3.1.iv of [5] are the $J = \emptyset$ case of our next result.

Proposition 3.2. *Let $J \subseteq I_n$ and let λ be J^c-dominant. Suppose W_J is finite. Let $s_{i_p} \cdots s_{i_2} s_{i_1}$ be any reduced expression for an element of W^J. Then $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal sequence of node firings from initial position λ. That is, the root $s_{i_1} s_{i_2} \cdots s_{i_{q-1}}.\alpha_{i_q}$ is positive for $1 \le q \le p$.*

Proof. By Remark 2.13, we must show that $\langle \lambda, \beta_q \rangle > 0$ for $1 \le q \le p$, where $\beta_q := s_{i_1} \cdots s_{i_{q-1}}.\alpha_{i_q}$. Suppose $s_{j_r} \cdots s_{j_2} s_{j_1}$ is a reduced expression for some $v_J \in W_J$. Since $s_{i_p} \cdots s_{i_2} s_{i_1} s_{j_1} \cdots s_{j_r}$ is reduced (cf. Proposition 2.4.4 of [5]), it follows that $\ell(v_J s_{i_1} \cdots s_{i_{q-2}} s_{i_{q-1}}) < \ell(v_J s_{i_1} \cdots s_{i_{q-1}} s_{i_q})$. In particular $v_J.\beta_q \in \Phi_M^+$ for all $v_J \in W_J$. We wish to show that β_q cannot be contained in $\mathrm{span}_{\mathbb{R}}\{\alpha_j\}_{j \in J}$. Suppose otherwise, so $\beta_q = \sum_{j \in J} c_j \alpha_j$. Remark 2.13 shows that $\beta_q \in \Phi_M^+$ for $1 \le q \le p$. But now the finiteness of W_J and Lemma 3.1 imply that $(w_0)_J.\beta_q \in \Phi_M^-$, a contradiction. Then it must be the case that $\beta_q = \sum_{i \in I_n} c_i \alpha_i$ with $c_k > 0$ for some $k \in I_n \setminus J$. So $\langle \lambda, \beta_q \rangle = \langle \lambda, \sum_{i \in I_n} c_i \alpha_i \rangle = \sum_{i \in I_n} c_i \lambda_i$, which is positive since all c_i's are nonnegative, $\lambda_k > 0$, and $c_k > 0$. $\square$

It is an open question whether the finiteness hypothesis of Proposition 3.2 can be relaxed. See Section 6 for comments on a possible connection between Proposition 3.2 and Theorem 6.1. Let $\mathfrak{P}(\lambda)$ denote the set of all positions obtainable from legal firing sequences in numbers games with initial position λ. Clearly $\mathfrak{P}(\lambda) \subseteq W\lambda$, where the latter is the orbit of λ under the W-action on V^*. Since the statement of Theorem 5.13 of [20] holds for geometric representations, then W_J is the full stabilizer of any J^c-dominant λ, so $W\lambda$ and W^J can be identified. So from Proposition 3.2 we see that for J^c-dominant λ with W_J finite, then $\mathfrak{P}(\lambda) = W\lambda$. The $J = \emptyset$ version of the previous statement is part (ii) of Theorem 4.3.1 of [5].

For finite W, we use $(w_0)^J$ to denote the minimal coset representative for $w_0 W_J$.

Corollary 3.3. *Suppose W is finite. Let $J \subseteq I_n$. Then all game sequences for any J^c-dominant λ have length $\ell((w_0)^J) = \ell(w_0) - \ell((w_0)_J)$.*

Proof. Proposition 3.2 implies that there is a game sequence for λ with length $\ell((w_0)^J) = \ell(w_0) - \ell((w_0)_J)$. By Eriksson's Strong Convergence Theorem, this must be the length of any game sequence for λ. $\square$

For finite Coxeter groups, the next result strengthens Part (1) of Eriksson's Reduced Word Result. At this time it is an open question whether the finiteness hypothesis for W can be relaxed.

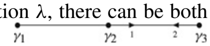
Corollary 3.4. *Let $J \subseteq I_n$ and let λ be any J^c-dominant position. Suppose W is finite. Suppose $\mathbf{s} := (\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal firing sequence for played from λ. Then $w := s_{i_p} \cdots s_{i_2} s_{i_1}$ is a reduced expression for an element of W^J. Moreover, $\mathbf{s}$ is a game sequence if and only if $w = (w_0)^J$.*

Proof. By Corollary 3.3, we may extend the legal firing sequence $\mathbf{s}$ to some game sequence $\mathbf{s}' := (\gamma_{i_1}, \ldots, \gamma_{i_p}, \gamma_{i_{p+1}}, \ldots, \gamma_{i_L})$ with $L = \ell(w_0) - \ell((w_0)_J) \ge p$. Let $v := s_{i_L} \cdots s_{i_{p+2}} s_{i_{p+1}}$, and $u := vw$. By Part (1) of Eriksson's Reduced Word Result, w, v, and u are reduced. In particular, $\ell(u) = L$. Write $u = u^J u_J$ for $u^J \in W^J$ and $u_J \in W_J$. By Proposition 3.2, we may take a legal firing sequence $\mathbf{t} := (\gamma_{j_1}, \ldots, \gamma_{j_K})$ from λ corresponding to some reduced expression for u^J. Now $u.\lambda$ is the terminal position for the game sequence $\mathbf{s}'$ played from λ. Since $u.\lambda = u^J.\lambda$, then $\mathbf{t}$ is a game sequence terminating at this same position. By Eriksson's Strong Convergence Theorem, it must be the case that $\ell(u^J) = K = L = \ell(u)$. Hence $u_J = \varepsilon$ and $u = u^J \in W^J$. Now write $w = w^J w_J$ for $w^J \in W^J$ and $w_J \in W_J$. If $w_J \neq \varepsilon$, then w_J has a reduced expression ending in s_j for some $j \in J$. Then $\ell(us_j) = \ell(vws_j) = \ell(vw^J w_J s_j) < \ell(u)$. But this contradicts the fact that $u \in W^J$. Hence $w_J = \varepsilon$, so $w = w^J \in W^J$. By Proposition 2.4.4 of [5], $\ell(u(w_0)_J) = L + \ell((w_0)_J)$. Since $L + \ell((w_0)_J) = \ell(w_0)$, then $u(w_0)_J = w_0 = (w_0)^J(w_0)_J$, so $u = (w_0)^J$. It now follows that $w = (w_0)^J$ if and only if $\mathbf{s}$ is a game sequence. □

4. Adjacency-free positions and full commutativity of Coxeter group elements

In this section we study dominant positions whose numbers games are all equivalent up to a notion of interchanging moves. We say these positions are "adjacency-free." For finite W, we classify the adjacency-free positions by showing how they correspond with quotients W^J whose elements are fully commutative in the sense of [26] (see also [27]; see [18,19] for full commutativity in a different context). Adjacency-free positions have other connections to the literature. In what follows, a Weyl group is a Coxeter group for which each $m_{ij} \in \{2, 3, 4, 6, \infty\}$. In Proposition 3.1 of [24], Proctor shows that for finite irreducible Weyl groups W, those quotients for which the Bruhat order $(W^J, \leq)$ (see [5]) is a lattice have $|J^c| = 1$ and correspond precisely to the adjacency-free fundamental positions for the connected "Dynkin diagrams of finite type" (E-Coxeter graphs with integer amplitudes). In Proposition 3.2 of that paper, he shows that these lattices are, in fact, distributive. In [10] we use information obtained from numbers games played from adjacency-free fundamental positions on Dynkin diagrams of finite type to construct certain "fundamental" posets. We show that the distributive lattices of order ideals obtained from certain combinations of our fundamental posets can be used to produce Weyl characters and in some cases explicit constructions irreducible representations of the corresponding semisimple Lie algebra. For rank two versions of these posets and distributive lattices, see [1,2]. When an adjacency-free fundamental position for a Dynkin diagram of finite type corresponds to a "minuscule" fundamental weight (see [24–26]), then our fundamental poset coincides with the corresponding "wave" poset of [25] and "heap" of [26].

For a firing sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ from a position λ, any position $s_{i_j} \cdots s_{i_1}.\lambda$ (including λ itself) is an *intermediate position* for the sequence. A game sequence played from λ is *adjacency-free* if no intermediate position for the sequence has positive numbers on a pair of adjacent nodes. A position λ is *adjacency-free* if every game sequence played from λ is adjacency-free.[4] Following Section 1.1 of [26] and Section 8.1 of [20], we let $\mathcal{W} = I_n^*$ be the free monoid on the set I_n. Elements of $\mathcal{W}$ are *words* and will be viewed as finite sequences of elements from I_n. The binary operation is concatenation, and the identity is the empty word. Fix a word $\mathbf{s} := (i_1, \ldots, i_r)$. Then $\ell_{\mathcal{W}}(\mathbf{s}) := r$ is the *length* of $\mathbf{s}$. A *subword* of $\mathbf{s}$ is any subsequence $(i_p, i_{p+1}, \ldots, i_q)$ of consecutive elements of $\mathbf{s}$. For a nonnegative integer m and $x, y \in I_n$, let

[4] For a dominant position λ, there can be both adjacency-free and nonadjacency-free game sequences. For example, for the E-Coxeter graph  from the $\mathcal{B}_3$ family, the game sequence $(\gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3, \gamma_1, \gamma_2)$ played from the fundamental position ω_2 is adjacency-free while the game sequence $(\gamma_2, \gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_3, \gamma_2)$ is not adjacency-free. Then the position ω_2 for this E-Coxeter graph is not adjacency-free.

$\langle x, y\rangle_m$ denote the sequence $(x, y, x, y, \ldots) \in \mathcal{W}$ so that $\ell_{\mathcal{W}}(\langle x, y\rangle_m) = m$. We employ several types of "elementary simplifications" in $\mathcal{W}$. An *elementary simplification of braid type* replaces a subword $\langle x, y\rangle_{m_{xy}}$ with the subword $\langle y, x\rangle_{m_{xy}}$ if $2 \leq m_{xy} < \infty$. An *elementary simplification of length-reducing type* replaces a subword (x, x) with the empty subword. We let $\mathcal{S}(\mathbf{s})$ be the set of all words that can be obtained from $\mathbf{s}$ by some sequence of elementary simplifications of braid or length-reducing type. Since s_i in W is its own inverse for each $i \in I_n$, there is an induced mapping $\mathcal{W} \to W$. We compose this with the mapping $W \to W$ for which $w \mapsto w^{-1}$ to get $\psi : \mathcal{W} \to W$ given by $\psi(\mathbf{s}) = s_{i_r} \cdots s_{i_1}$. Tits' Theorem for the word problem on Coxeter groups (cf. Theorem 8.1 of [20]) implies that: *For words* $\mathbf{s}$ *and* $\mathbf{t}$ *in* $\mathcal{W}$, $\psi(\mathbf{s}) = \psi(\mathbf{t})$ *if and only if* $\mathcal{S}(\mathbf{s}) \cap \mathcal{S}(\mathbf{t}) \neq \emptyset$. (This theorem is the basis for Part (1) of Eriksson's Reduced Word Result.) We say $\mathbf{s}$ is a *reduced* word for $w = \psi(\mathbf{s})$ if $\ell_{\mathcal{W}}(\mathbf{s}) = \ell(w)$ (assume this is the case for the remainder of the paragraph). Let $\mathcal{R}(w) \subseteq \mathcal{W}$ denote the set of all reduced words for w. Suppose that $\mathbf{t} \in \mathcal{R}(w)$. By Tits' Theorem, $\mathcal{S}(\mathbf{s}) \cap \mathcal{S}(\mathbf{t}) \neq \emptyset$, so that $\mathbf{t}$ can be obtained from $\mathbf{s}$ by a sequence of elementary simplifications of braid or length-reducing type. Since $\ell_{\mathcal{W}}(\mathbf{s}) = \ell(w) = \ell_{\mathcal{W}}(\mathbf{t})$, then no elementary simplifications of length-reducing type can be used to obtain $\mathbf{t}$ from $\mathbf{s}$. Then any member of $\mathcal{R}(w)$ can be obtained from any other member by a sequence of elementary simplifications of braid type. An *elementary simplification of commuting type* replaces a subword (x, y) with the subword (y, x) if $m_{xy} = 2$. The *commutativity class* $\mathcal{C}(\mathbf{s})$ of the word $\mathbf{s}$ is the set of all words that can be obtained from $\mathbf{s}$ by a sequence of elementary simplifications of commuting type. Clearly $\mathcal{C}(\mathbf{s}) \subseteq \mathcal{R}(w)$. In fact there is a decomposition of $\mathcal{R}(w)$ into commutativity classes: $\mathcal{R}(w) = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_k$, a disjoint union. If $\mathcal{R}(w)$ has just one commutativity class, then w is *fully commutative*. Proposition 1.1 of [26] states: *An element* $w \in W$ *is fully commutative if and only if for all* $x, y \in I_n$ *such that* $3 \leq m_{xy} < \infty$, *there is no member of* $\mathcal{R}(w)$ *that contains* $\langle x, y\rangle_{m_{xy}}$ *as a subword.*

Proposition 4.1. *Let* $J \subseteq I_n$. (1) *Suppose* W_J *is finite. Suppose an adjacency-free position* λ *is* J^c*-dominant. Then every element of* W^J *is fully commutative.* (2) *Suppose* W *is finite. Suppose each element of* W^J *is fully commutative. Then any* J^c*-dominant position is adjacency-free.*

Proof. Our proof of (1) is by induction on the lengths of elements in W^J. It is clear that the identity element is fully commutative. Now suppose that for all v^J in W^J with $\ell(v^J) < k$, it is the case that v^J is fully commutative, and consider w^J in W^J such that $\ell(w^J) = k$. Suppose that for some adjacent γ_x and γ_y in Γ with $3 \leq m_{xy} < \infty$, we have $\langle x, y\rangle_{m_{xy}}$ as a subword of some reduced word $\mathbf{s} = (i_1, \ldots, i_k) \in \mathcal{R}(w^J)$. Since $(i_1, \ldots, i_{k-1})$ is a reduced word and $s_{i_{k-1}} \cdots s_{i_1}$ is in W^J, then $\langle x, y\rangle_{m_{xy}}$ cannot be a subword of $(i_1, \ldots, i_{k-1})$. Therefore it must be the case that $\mathbf{s} = (i_1, \ldots, i_p, \langle x, y\rangle_{m_{xy}})$ for $p = k - m_{xy}$. Then, $\mathbf{s}' = (i_1, \ldots, i_p, \langle y, x\rangle_{m_{xy}})$ is also a reduced word for w^J. Since both $\mathbf{s}$ and $\mathbf{s}'$ correspond to legal firing sequences from λ (Proposition 3.2), it must be the case that there are positive numbers at adjacent nodes γ_x and γ_y after the first p firings. But this contradicts the hypothesis that λ is adjacency-free. Hence no reduced word for w^J can have a subword of the form $\langle x, y\rangle_{m_{xy}}$ for nodes γ_x and γ_y with $3 \leq m_{xy} < \infty$. By Proposition 1.1 of [26] it follows that w^J is fully commutative, which completes the proof of part (1).

For part (2), assume every member of W^J is fully commutative, and let λ be any J^c-dominant position. Let $L := \ell(w_0) - \ell((w_0)_J) = \ell((w_0)^J)$. Suppose an intermediate position $s_{i_k} \cdots s_{i_1}.\lambda$ for some game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_L})$ has positive numbers on adjacent nodes γ_x and γ_y. Then by Eriksson's Strong Convergence Theorem, there is a game sequence of length L from λ corresponding to a reduced word $\mathbf{s} = (i_1, \ldots, i_k, \langle x, y\rangle_{m_{xy}}, j_{k+m_{xy}+1}, \ldots, j_L)$ for $u := \psi(\mathbf{s})$.

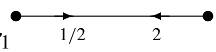
By Corollary 3.4, $u = (w_0)^J$. So $(w_0)^J$ is fully commutative (by hypothesis) and has reduced word **s**, in violation of Proposition 1.1 of [26]. Therefore λ must be adjacency-free. □

In Theorem 5.1 of [26], Stembridge classifies those W^J for irreducible Coxeter groups W such that every member of W^J is fully commutative. In view of Proposition 4.1 and the classification of finite Coxeter groups, we may apply this result here to conclude that when W is finite and irreducible, then the adjacency-free dominant positions of (Γ, M) are exactly those specified in the following theorem. Observe that a dominant position λ is adjacency-free if and only if $r\lambda := (r\lambda_i)_{i \in I_n}$ is adjacency-free for all positive real numbers r. Call any such $r\lambda$ a *positive multiple* of λ.

Theorem 4.2. *Suppose (Γ, M) is connected. If W is finite, then an adjacency-free dominant position is a positive multiple of a fundamental position. All fundamental positions for any E-Coxeter graph of type $\mathcal{A}_n$ are adjacency-free. The adjacency-free fundamental positions for any graph of type $\mathcal{B}_n$, $\mathcal{D}_n$, or $\mathcal{I}_2(m)$ are precisely those corresponding to end nodes. The adjacency-free fundamental positions for any graph of type $\mathcal{E}_6$, $\mathcal{E}_7$, or $\mathcal{H}_3$ are precisely those corresponding to the nodes marked with asterisks in Fig. 2.1. Any graph of type $\mathcal{E}_8$, $\mathcal{F}_4$, or $\mathcal{H}_4$ has no adjacency-free fundamental positions.* □

For finite irreducible Coxeter groups W, it is a consequence of Theorems 5.1 and 6.1 of [26] that the Bruhat order $(W^J, \leq)$ is a lattice if and only if $(W^J, \leq)$ is a distributive lattice if and only if each element of W^J is fully commutative. In these cases $|J^c| = 1$ and all such J^c's correspond to the adjacency-free fundamental positions from Theorem 4.2. Proposition 4.1 adds to these equivalences the property that each element of W^J is fully commutative if and only if for any associated E-GCM graph, any J^c-dominant position is adjacency-free. The adjacency-free viewpoint is similar to Proctor's original viewpoint (cf. Lemma 3.2 of [24]).

5. Generating positive roots from E-game play

The results of this section expand on Remark 4.6 of [15]. The goal here is to characterize when all positive roots can be obtained from a single game sequence, as in the following example: In Fig. 2.2 with amplitude matrix $M = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, assume the initial position $\lambda = (a, b)$ is strongly dominant. For the game sequence $(\gamma_2, \gamma_1, \gamma_2, \gamma_1)$, notice that the respective numbers at the fired nodes are b, $a + 2b$, $a + b$, and a. Thought of now as root functionals, the latter are in one-to-one correspondence with the positive roots $\Phi_M^+ = \{\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1\}$. For $M = \begin{pmatrix} 2 & -1/2 \\ -2 & 2 \end{pmatrix}$ with E-Coxeter graph in the $\mathcal{A}_2$ family (cf. Exercise 4.9 of [5]), the situation is different. From a strongly dominant position $\lambda = (a, b)$ on γ_1 

γ_2, the game sequence $(\gamma_2, \gamma_1, \gamma_2)$ has respective numbers b, $a + 2b$, and $\frac{1}{2}a$ at the fired nodes. However, the positive roots are $\Phi_M^+ = \{\alpha_2, \alpha_1 + 2\alpha_2, \frac{1}{2}\alpha_1, \alpha_1, \frac{1}{2}\alpha_1 + \alpha_2, 2\alpha_2\}$.

In general, for $p \geq 1$ suppose $\mathbf{s} := (\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal firing sequence from some initial position λ on (Γ, M). After $(\gamma_{i_1}, \ldots, \gamma_{i_{q-1}})$ is played $(1 \leq q \leq p)$, the number at node γ_{i_q} is $\langle s_{i_{q-1}} \cdots s_{i_1}.\lambda, \alpha_{i_q} \rangle = \langle \lambda, s_{i_1} \cdots s_{i_{q-1}}.\alpha_{i_q} \rangle = \phi_{\beta_q}(\lambda)$ with $\beta_q := s_{i_1} \cdots s_{i_{q-1}}.\alpha_{i_q}$. With **s** and λ understood, then we say ϕ_{β_q} is the *root functional at node* γ_{i_q}.[5] By Part (1) of Eriksson's Reduced

[5] It follows from Part (2) of Eriksson's Reduced Word Result and Remark 2.13 that for any given strongly dominant position λ and any positive root α, there is a legal firing sequence $(\gamma_{i_1}, \ldots, \gamma_{i_{q-1}})$ played from λ such that ϕ_α is the root functional at node γ_{i_q}.

Word Result, $w := s_{i_p} \cdots s_{i_2} s_{i_1}$ is reduced. This is exactly the situation of Exercise 5.6.1 of [20], where the representation is the "standard" geometric representation of W. There, one concludes that the β_q's are distinct and precisely all of the positive roots β for which $w.\beta$ is a negative root. In our more general setting we have:

Lemma 5.1. *Let $w = s_{i_p} \cdots s_{i_2} s_{i_1}$ with $\ell(w) = p \geq 1$. Let $\beta_q := s_{i_1} s_{i_2} \cdots s_{i_{q-1}}.\alpha_{i_q}$ for $1 \leq q \leq p$. Then $\beta_q \neq \beta_r$ for $q \neq r$ and $\{\beta_q\}_{q=1}^{p} \subseteq N_M(w)$. Moreover, $\{\beta_q\}_{q=1}^{p} = N_M(w)$ if and only if for $1 \leq q \leq p$ the ON-connected component (Γ', M') containing γ_{i_q} is unital ON-cyclic with $f_{\Gamma',M'} = 1$.*

Proof. Each $\beta_q \in \Phi_M^+$ by Remark 2.13. Also, $w.\beta_q = s_{i_p} s_{i_{p-1}} \cdots s_{i_q}.\alpha_{i_q} \in \Phi_M^-$ follows from the fact that $\ell(s_{i_p} s_{i_{p-1}} \cdots s_{i_{q+1}} s_{i_q} s_{i_q}) < \ell(s_{i_p} s_{i_{p-1}} \cdots s_{i_{q+1}} s_{i_q})$. Hence $\beta_q \in N_M(w)$. For $q < r$, suppose $\beta_q = \beta_r$. Then one can see that $s_{i_q} \cdots s_{i_{r-1}}.\alpha_{i_r} = \alpha_{i_q}$, and so $s_{i_{q+1}} \cdots s_{i_{r-1}}.\alpha_{i_r} = -\alpha_{i_q} \in \Phi_M^-$. Then $\ell(s_{i_{q+1}} \cdots s_{i_{r-1}} s_{i_r}) < \ell(s_{i_{q+1}} \cdots s_{i_{r-1}})$. But $s_{i_{q+1}} \cdots s_{i_{r-1}} s_{i_r}$ is reduced and longer than $s_{i_{q+1}} \cdots s_{i_{r-1}}$, a contradiction. So $\beta_q \neq \beta_r$. For the "if" direction of the last assertion of the lemma, by Proposition 2.12 $N_M(w)$ is finite. Since $f_1 = f_2 = 1$, then $\ell(w) = |N_M(w)| = p$. For the "only if" direction, $N_M(w)$ has finite order $p = \ell(w)$. Then by Proposition 2.12, each $\mathfrak{S}_M(\alpha_{i_q})$ is finite, so by Proposition 2.10 the ON-connected component (Γ', M') containing γ_{i_q} is unital ON-cyclic. Combining $\ell(w) = |N_M(w)|$ and $f_1 \ell(w) \leq |N_M(w)| \leq f_2 \ell(w)$ gives $f_1 = f_2 = 1$. Therefore $f_{\Gamma',M'} = 1$. □

From this lemma, it is apparent now why the game sequence exhibited in the above $\mathcal{A}_2$ example failed to generate all of the positive roots: the E-GCM graph has an odd asymmetry which results in some positive roots which are nontrivial multiples of simple roots. In this case, $f_{\Gamma,M} = 2 = |\mathfrak{S}_M(\alpha_i)|$ for $i = 1, 2$. The positive roots $\{\alpha_2, \alpha_1 + 2\alpha_2, \frac{1}{2}\alpha_1\}$ associated with the root functionals of the game sequence $(\gamma_2, \gamma_1, \gamma_2)$ are a proper subset of $N_M(w_0) = \Phi_M^+$ where $w_0 = s_2 s_1 s_2$. In general, if W is finite and (Γ, M) has odd asymmetries then not every positive root will be encountered as a positive root functional in a given game sequence, as the next result shows. However, if the amplitude matrix M is integral, then (Γ, M) has no odd asymmetries and thus enjoys the equivalent properties of the following theorem:

Theorem 5.2. *Suppose W is finite. Let $s_{i_l} \cdots s_{i_2} s_{i_1}$ be any reduced expression for w_0. For $1 \leq j \leq l$, set $\beta_j := s_{i_1} s_{i_2} \cdots s_{i_{j-1}}.\alpha_{i_j}$. Then the following are equivalent:*

(1) *(Γ, M) has no odd asymmetries;*
(2) *Each ON-connected component (Γ', M') of (Γ, M) is unital ON-cyclic with $f_{\Gamma',M'} = 1$;*
(3) *$\{\beta_j\}_{j=1}^{l} = \Phi_M^+$;*
(4) *$\ell(w_0) = |\Phi_M^+|$;*
(5) *Each positive root appears as the root functional ϕ_{β_j} at some node γ_{i_j} for the game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ played from any strongly dominant position.*

Proof. For (1) ⇔ (2), note that by Proposition 2.9 a nontrivial positive multiple of some simple root is itself a root if and only if there are odd asymmetries. For (2) ⇒ (3), recall from Section 3 that $N_M(w_0) = \Phi_M^+$. Lemma 5.1 shows that that $\{\beta_j\}_{j=1}^{l} = N_M(w_0)$, so (3) follows. (4) follows immediately from (3). For (4) ⇒ (5), first note that by Part (2) of Eriksson's Reduced Word Result, the firing sequence $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is legal from any strongly dominant position, and by Corollary 3.3 this is a game sequence. Lemma 5.1 and comments preceding that lemma show that for this game sequence the positive roots in the set $\{\beta_j\}_{j=1}^{l}$ appear precisely once each as root functionals. The hypothesis $\ell(w_0) = |\Phi_M^+|$ means that $\{\beta_j\}_{j=1}^l = \Phi_M^+$, from which (5) follows. To show (5) $\Rightarrow$ (2), choose an ON-connected component (Γ', M'). Propositions 2.9 and 2.10 show that (Γ', M') must be unital ON-cyclic, else W will be infinite. Let J be the subset of I_n corresponding to the nodes of the subgraph Γ'. For notational convenience set $w = w_0$, $w_J = (w_0)_J$, and $w^J = (w_0)^J$. Set $w_J = s_{j_k} \cdots s_{j_2} s_{j_1}$, a reduced expression. Using Lemma 2.11, we see that

$$|N_M(ws_{j_1})| = |N_M(w)| - f_{\Gamma',M'},$$
$$|N_M(ws_{j_1}s_{j_2})| = |N_M(ws_{j_1})| - f_{\Gamma',M'} = |N_M(w)| - 2f_{\Gamma',M'},$$

so that eventually $|N_M(w)| = |N_M(w^J)| + \ell(w_J) f_{\Gamma',M'}$. Now by hypothesis each positive root functional appears once and therefore, by Lemma 5.1, exactly once. Then $l = \ell(w) = |\Phi_M^+| = |N_M(w)|$. By Proposition 2.12, $|N_M(w^J)| \geq \ell(w^J)$. Summarizing, $\ell(w^J) + \ell(w_J) = \ell(w) = |N_M(w)| = |N_M(w^J)| + \ell(w_J) f_{\Gamma',M'} \geq \ell(w^J) + \ell(w_J) f_{\Gamma',M'}$, from which $f_{\Gamma',M'} = 1$. □

6. A Dynkin diagram classification of E-GCM graphs meeting a certain finiteness requirement

We say a connected E-GCM graph is *admissible* if there exists a nonzero dominant initial position with a convergent game sequence. In this section we prove the following Dynkin diagram classification result.

Theorem 6.1. *A connected E-GCM graph is admissible if and only if it is a connected E-Coxeter graph. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.*

Our proof of Theorem 6.1 given at the end of this section uses the classification of finite Coxeter groups. Another proof based on ideas from [12] is given in [9]. That proof uses combinatorial reasoning together with a result from the Perron–Frobenius theory for eigenvalues of nonnegative real matrices, and it does not require the classification of finite Coxeter groups. Before proceeding toward our proof of Theorem 6.1, we record two closely related results. In [12], Eriksson establishes the following result. (For an "A-D-E" version, see [11].) The statement we give here essentially combines his Theorems 6.5 and 6.7. An E-GCM graph is *strongly admissible* if every nonzero dominant position has a convergent game sequence.

Theorem 6.2 (*Eriksson*). *A connected E-GCM graph is strongly admissible if and only if it is a connected E-Coxeter graph.*

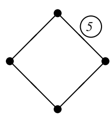
Using this result Eriksson rederives in Section 8.4 of [12] the well-known classification of finite irreducible Coxeter groups, which we state as: *An irreducible Coxeter group $W(\Gamma, M)$ is finite if and only if the connected E-GCM graph (Γ, M) is an E-Coxeter graph from* Fig. 2.1. In Propositions 4.1 and 4.2 of [6], Deodhar gives a number of statements equivalent to the assertion that a given irreducible Coxeter group is finite. As an immediate consequence of Theorems 6.1 and 6.2 and the classification of finite irreducible Coxeter groups, we add to that list the following equivalence.

Corollary 6.3. *An irreducible Coxeter group W is finite if and only if there is an admissible E-GCM graph whose associated Coxeter group is W if and only if any E-GCM graph is strongly admissible when its associated Coxeter group is W.* □

Extending Proposition 3.2 to all subsets $J \subseteq I_n$ would yield a simple proof of the first assertion of Theorem 6.1: For any given proper subset $J \subset I_n$, the E-GCM graph (Γ, M) would have a convergent game sequence for some J^c-dominant λ if and only if W^J is finite if and only if W is finite (by Proposition 4.2 of [6]). Observe that the "if" direction of the first assertion in Theorem 6.1 follows from Theorem 6.2. The second assertion in Theorem 6.1 follows from Eriksson's Strong Convergence Theorem. So our effort in the proof of Theorem 6.1 will be mainly concerned with demonstrating the "only if" part of the first assertion. Our proof of this part is by induction on the number of nodes. The main idea of our proof is to use reductions effected by the preliminary results of Section 2 together with some further results derived here. The lemmas that follow use Lemma 2.5, which depends crucially on Eriksson's Comparison Theorem. We say an n-node graph Γ is a *loop* if the nodes can be numbered $\gamma_1, \ldots, \gamma_n$ in such a way that for all $1 \le i \le n$, γ_i is adjacent precisely to γ_{i+1} and γ_{i-1}, understanding that $\gamma_0 = \gamma_n$ and $\gamma_{n+1} = \gamma_1$.

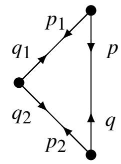
Lemma 6.4. *Suppose that the underlying graph Γ of an E-GCM graph (Γ, M) is a loop and that for any edge in (Γ, M) the amplitude product is unity. Then (Γ, M) is not admissible.*

Proof. We find a divergent game sequence starting from the fundamental position ω_1. Then by renumbering the nodes, we see that every fundamental position will have a divergent game sequence, and by Lemma 2.5 it then follows that (Γ, M) is not admissible. Let the ON-cycle $\mathcal{C}$ be $[\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1]$. From initial position ω_1 we propose starting with the firing sequence $(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots \gamma_2)$. One can check that all of these node firings are legal and that the resulting numbers are zero at all nodes other than γ_1, γ_2, and γ_n. The numbers at the latter nodes are, respectively, $1 + \Pi_{\mathcal{C}} + \Pi_{\mathcal{C}}^{-1}$, $M_{12}(\Pi_{\mathcal{C}}^{-1})$, and $M_{1n}(\Pi_{\mathcal{C}})$. By repeating the proposed firing sequence $(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots \gamma_2)$ from this position we obtain zero at all nodes except at γ_1, γ_2 and γ_n, which are now $1 + \Pi_{\mathcal{C}} + \Pi_{\mathcal{C}}^{-1} + \Pi_{\mathcal{C}}^{2} + \Pi_{\mathcal{C}}^{-2}$, $M_{12}(\Pi_{\mathcal{C}}^{-1} + \Pi_{\mathcal{C}}^{-2})$ and $M_{1n}(\Pi_{\mathcal{C}} + \Pi_{\mathcal{C}}^{2})$ respectively. After k applications of the proposed firing sequence we have numbers $1 + \sum_{j=1}^{k} \Pi_{\mathcal{C}}^{j} + \Pi_{\mathcal{C}}^{-j}$, $M_{12}(\sum_{j=1}^{k} \Pi_{\mathcal{C}}^{-j})$, and $M_{1n}(\sum_{j=1}^{k} \Pi_{\mathcal{C}}^{j})$ at nodes γ_1, γ_2 and γ_n with zeros elsewhere. Thus we have exhibited a divergent game sequence. $\square$

Lemma 6.5. *An E-GCM graph in the family*  *is not admissible.* $\square$

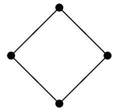
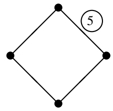
Proof. Let (Γ, M) be an E-GCM graph in the given family. Label the nodes γ_1, γ_2, γ_3, and γ_4 clockwise from the top. Our strategy is to show that the repeating firing sequence $\mathbf{r} := (\mathbf{s}, \mathbf{s}, \ldots)$ can be legally applied to some position obtained from E-game play starting with any given fundamental position, where $\mathbf{s}$ is the subsequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. This will give us a divergent game sequence from each fundamental position, so by Lemma 2.5 it will follow that (Γ, M) is not admissible. For adjacent nodes γ_1 and γ_2, set $p := -M_{12}$, $q := -M_{21}$. Note that $pq = (3 + \sqrt{5})/2$. Set $r := -M_{23}$, $s := -M_{32}$, $t := -M_{34}$, $u := -M_{43}$, $v := -M_{41}$, and $w := -M_{14}$. We have $rs = tu = vw = 1$. Note that p, q, r, s, t, u, v, w are the absolute values of the amplitudes read in alphabetical order clockwise from the top. We say a position (a, b, c, d) meets condition $(*)$ if $a > 0$, $b \ge 0$, $c \ge 0$, $d \le 0$, $aw + d \ge 0$, and $aprt + brt + ct + d > 0$. One can easily check that from any such position the firing sequence $\mathbf{s} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is legal: The positive numbers at the fired nodes are respectively a, $ap + b$, $apr + br + c$ and $aw + aprt + brt + ct + d$. The resulting position is (A, B, C, D) with $A =$

$\frac{3+\sqrt{5}}{2}a+bq+v(aprt+brt+ct+d)$, $B = sc$, $C = u(aw+d)$, and $D = -aw-aprt-brt-ct-d$. Clearly $A > 0$, $B \geq 0$, $C \geq 0$, and $D < 0$. Also, $Aw + D = (\frac{3+\sqrt{5}}{2} - 1)aw + bqw > 0$, and $Aprt + Brt + Ct + D = (\frac{3+\sqrt{5}}{2} - 1)aprt + (\frac{3+\sqrt{5}}{2} - 1)brt + prtv(aprt + brt + ct + d) > 0$. So, (A, B, C, D) meets condition $(*)$. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(*)$, so it follows that the divergent firing sequence $\mathbf{r}$ can be legally played from this initial position. Play the legal sequence $(\gamma_2, \gamma_3, \gamma_4)$ from the fundamental position $\omega_2 = (0, 1, 0, 0)$ to obtain the position $(q + rtv, 0, 0, -rt)$. It is easily checked that the latter position meets condition $(*)$. It follows that the divergent firing sequence $(\gamma_2, \gamma_3, \gamma_4, \mathbf{r})$ can be legally played from ω_2. Similarly see that the divergent firing sequence $(\gamma_3, \gamma_4, \mathbf{r})$ can be legally played from ω_3 and that the divergent firing sequence $(\gamma_4, \mathbf{r})$ can be legally played from ω_4. □

Lemma 6.6. *Suppose (Γ, M) is the following three-node E-GCM graph:*  *Assume that all node pairs are odd-neighbourly. Then (Γ, M) is not admissible.*

Notes on the proof. As in the proofs of Lemmas 6.4 and 6.5 we apply Lemma 2.5 after showing that from each fundamental position there is a legal firing sequence that can be repeated indefinitely. However, the variable amplitude products on edges of this graph make this argument a little more delicate than our arguments for the previous lemmas. A key part of the argument in this case is an explicit computation of matrix representations of powers of $\sigma_M(s_i s_j)$ with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of simple roots. These computations are used to understand positions resulting from alternating sequences of firings on adjacent nodes. For complete details, see [7].

We can now prove Theorem 6.1.

Proof of Theorem 6.1. First we use induction on n, the number of nodes, to show that any connected admissible E-GCM graph must be from one of the families of Fig. 2.1. Clearly a one-node E-GCM graph is admissible. For some $n \geq 2$, suppose the result is true for all connected admissible E-GCM graphs with fewer than n nodes. Let (Γ, M) be a connected, admissible, n-node E-GCM graph. Suppose (Γ, M) is unital ON-cyclic. Then by Proposition 2.14 and Theorem 2.15, we must have W finite. Then by the classification of finite irreducible Coxeter groups, (Γ, M) must be in one of the families of graphs in Fig. 2.1. Now suppose (Γ, M) is not unital ON-cyclic. First we show that any cycle (ON or otherwise) in (Γ, M) must use all n nodes. Indeed, the (connected) E-GCM subgraph (Γ', M') whose nodes are the nodes of a cycle must be admissible by Lemma 2.7. If (Γ', M') has fewer than n nodes, then the induction hypothesis applies. But E-Coxeter graphs have no cycles (ON or otherwise), so (Γ', M') must be all of (Γ, M). Second, (Γ, M) has an ON-cycle $\mathcal{C}$ for which $\Pi_{\mathcal{C}} \neq 1$. We can make the following choice for $\mathcal{C}$: Choose $\mathcal{C}$ to be a simple ON-cycle with $\Pi_{\mathcal{C}} \neq 1$ whose length is as small as possible. This smallest length must therefore be n. We wish to show that the underlying graph Γ is a loop. Let the numbering of the nodes of Γ follow $\mathcal{C}$, so $\mathcal{C} = [\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1]$. If Γ is not a loop, then there are adjacencies amongst the γ_i's besides those of consecutive elements of $\mathcal{C}$. But this in turn means that (Γ, M) has a cycle that uses fewer than n nodes. So Γ is a loop. Of course we must have $n \geq 3$. Lemma 6.6 rules out the possibility that $n = 3$. Any connected E-GCM subgraph (Γ', M') obtained from (Γ, M) by removing a single node must now be a "branchless" E-Coxeter graph from Fig. 2.1 whose adjacencies are all odd. So if $n = 4$, (Γ, M) must be in one of the families  or 

, which are ruled out by Lemmas 6.4 and 6.5 respectively. If $n \geq 5$, the only possibility is that (Γ, M) meets the hypotheses of Lemma 6.4 and therefore is not admissible. In all cases, we see that if (Γ, M) is not unital ON-cyclic, then it is not admissible. This completes the induction step, so we have shown that a connected admissible E-GCM graph must be in one of the families of Fig. 2.1.

On the other hand, if (Γ, M) is from Fig. 2.1, then the Coxeter group W is finite (again by the classification), so there is an upper bound on the length of any element in W. So by Part (1) of Eriksson's Reduced Word Result, the numbers game converges for any initial position. The remaining claims of Theorem 6.1 now follow from Eriksson's Strong Convergence Theorem. □

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References

[1] L.W. Alverson II, R.G. Donnelly, S.J. Lewis, M. McClard, R. Pervine, R.A. Proctor, N.J. Wildberger, Distributive lattices defined for representations of rank two semisimple Lie algebras, preprint, arXiv:0707.2421.

[2] L.W. Alverson II, R.G. Donnelly, S.J. Lewis, R. Pervine, Constructions of representations of rank two semisimple Lie algebras with distributive lattices, Electron J. Combin. 13 (#R109) (2006) 44 pp.

[3] N. Alon, I. Krasikov, Y. Peres, Reflection sequences, Amer. Math. Monthly 96 (1989) 820–823.

[4] A. Björner, On a combinatorial game of S. Mozes, 1988, preprint.

[5] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Springer, New York, 2005.

[6] V.V. Deodhar, On the root system of a Coxeter group, Comm. Algebra 10 (1982) 611–630.

[7] R.G. Donnelly, Eriksson's numbers game on certain edge-weighted three-node cyclic graphs, preprint, arXiv:0708.0880.

[8] R.G. Donnelly, Roots of asymmetric geometric representations of Coxeter groups, preprint, arXiv:0707.3310.

[9] R.G. Donnelly, K. Eriksson, The numbers game and Dynkin diagram classification results, preprint.

[10] R.G. Donnelly, N.J. Wildberger, Distributive lattices and Weyl characters for certain families of irreducible semisimple Lie algebra representations (in preparation).

[11] K. Eriksson, Convergence of Mozes's game of numbers, Linear Algebra Appl. 166 (1992) 151–165.

[12] K. Eriksson, Strongly Convergent Games and Coxeter Groups, Ph.D. Thesis, KTH, Stockholm, 1993.

[13] K. Eriksson, Node firing games on graphs, in: Jerusalem Combinatorics'93, in: Contemp. Math., vol. 178, Amer. Math. Soc., Providence, RI, 1994, pp. 117–127.

[14] K. Eriksson, Reachability is decidable in the numbers game, Theoret. Comput. Sci. 131 (1994) 431–439.

[15] K. Eriksson, The numbers game and Coxeter groups, Discrete Math. 139 (1995) 155–166.

[16] K. Eriksson, Strong convergence and a game of numbers, European J. Combin. 17 (1996) 379–390.

[17] J.P. Eveland, Using Pictures to Classify Solutions to Certain Systems of Matrix Equations, Undergraduate Honors Thesis, Murray State University, 2001.

[18] C.K. Fan, A Hecke algebra quotient and some combinatorial applications, J. Algebraic Combin. 5 (1996) 175–189.

[19] C.K. Fan, Structure of a Hecke algebra quotient, J. Amer. Math. Soc. 10 (1997) 139–167.

[20] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.
[21] V.G. Kac, Infinite-dimensional Lie Algebras, 3rd edition, Cambridge University Press, Cambridge, 1990.
[22] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Birkhäuser Boston Inc., Boston, MA, 2002.
[23] S. Mozes, Reflection processes on graphs and Weyl groups, J. Combin. Theory Ser. A 53 (1990) 128–142.
[24] R.A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, European J. Combin. 5 (1984) 331–350.
[25] R.A. Proctor, Minuscule elements of Weyl groups, the numbers game, and d-complete posets, J. Algebra 213 (1999) 272–303.
[26] J.R. Stembridge, On the fully commutative elements of Coxeter groups, J. Algebraic Combin. 5 (1996) 353–385.
[27] J.R. Stembridge, The enumeration of fully commutative elements of Coxeter groups, J. Algebraic Combin. 7 (1998) 291–320.
[28] N.J. Wildberger, A combinatorial construction for simply-laced Lie algebras, Adv. in Appl. Math. 30 (2003) 385–396.
[29] N.J. Wildberger, Minuscule posets from neighbourly graph sequences, European J. Combin. 24 (2003) 741–757.
[30] N.J. Wildberger, The mutation game, Coxeter graphs, and partially ordered multisets, preprint.