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Some more examples of monotonically Lindelöf and not monotonically Lindelöf spaces

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Abstract

A space is monotonically Lindelöf (mL) if one can assign to every open cover $\mathcal{U}$ a countable open refinement $r(\mathcal{U})$ (still covering the space) so that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. Some examples of mL and non-mL spaces are considered. In particular, it is shown that the product of a mL space and the convergent sequence need not be mL, that some L-spaces are mL, and that $C_p(X)$ is mL only for countable X.

MSC: 54D20

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1. Introduction

Recall that X is *monotonically Lindelöf* (mL) if there is an operator assigning to every open cover $\mathcal{U}$ a countable open refinement $r(\mathcal{U})$ (still covering the space) in such a way that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$ [9]. Here, by saying that a family of sets $\mathcal{A}$ refines a family of sets $\mathcal{B}$ we only mean that every element of $\mathcal{A}$ is a subset of an element of $\mathcal{B}$.

Not many examples of mL spaces are known. Basically, these are all separable metrizable spaces (see [2]), the one point Lindelöfication of the discrete space of cardinality ω_1, all separable GO spaces, in particular, the Sorgenfrey line [2], some non-separable GO spaces, for example, the lexicographic square of [0, 1] [2], (consistently) some non-metrizable countable spaces [8]. On the other hand, such "good" Lindelöf spaces as the one point Lindelöfication of the discrete space of cardinality ω_2, the one point compactification of the discrete space of cardinality ω_1, or a dense countable subset in 2^{ω_1} are not mL. The Alexandroff Duplicate of X is mL iff X is second countable (Jerry Vaughan, unpublished).

In this paper we extend the list of spaces known to be (or not to be) mL.

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Notation. For a family $\mathcal{U}$ of subsets of a space X, and for a subset $Y \subset X$, we let $\mathcal{U}|Y = \{U \cap Y\colon U \in \mathcal{U}\}$. For families of sets $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{U} \wedge \mathcal{V} = \{U \cap V\colon U \in \mathcal{U}$ and $V \in \mathcal{V}\}$. It is clear that $\mathcal{U} \wedge \mathcal{V}$ refines both $\mathcal{U}$ and $\mathcal{V}$, and that $\mathcal{U}_1 \wedge \mathcal{V}_1$ refines $\mathcal{U}_2 \wedge \mathcal{V}_2$ whenever $\mathcal{U}_1$ refines $\mathcal{U}_2$, and $\mathcal{V}_1$ refines $\mathcal{V}_2$. If $\mathcal{U}$ is a family of sets and V a set, we write $V \prec \mathcal{U}$ if V is a subset of some element of $\mathcal{U}$.

2. Some (Michael line)-like spaces are mL

Recall that a space X *concentrates on* $A \subset X$ if every neighborhood of A contains all but countably many points of X. For a space $(X, \mathcal{T})$ and $B \subset X$, denote by $\mathcal{T}_B$ the topology on X generated by the base $\mathcal{T} \cup \{\{p\}\colon p \in X \setminus B\}$. This generalized Michael line construction is mentioned in [7]. It is well-known that if $(X, \mathcal{T})$ concentrates on a Lindelöf subspace B, then both $(X, \mathcal{T})$ and $(X, \mathcal{T}_B)$ are Lindelöf. The following is straightforward:

Proposition 1. *If a second countable space $(X, \mathcal{T})$ concentrates on B, then $(X, \mathcal{T}_B)$ is mL.*

Indeed, having a countable base $\mathcal{B}$ for $(X, \mathcal{T})$ and an open cover $\mathcal{U}$ of $(X, \mathcal{T}_B)$, one can put $r_0(\mathcal{U}) = \{O \in \mathcal{B}\colon O \prec \mathcal{U}\}$ and $r(\mathcal{U}) = r_0(\mathcal{U}) \cup \{\{p\}\colon p \notin \bigcup r_0(\mathcal{U})\}$. Then r is a mL operator for $(X, \mathcal{T}_B)$.

Even if we are going to use only this proposition, here is a formal generalization. Say that $B \subset X$ is *relatively mL in* X if there is an operator r that assigns to every cover $\mathcal{U}$ of B by open subsets of X a countable open cover $r(\mathcal{U})$ of B by open subsets of X in such a way that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$.

Proposition 2. *If $(X, \mathcal{T})$ concentrates on $B \subset X$, and B is relatively mL in $(X, \mathcal{T})$, then $(X, \mathcal{T}_B)$ is mL.*

The proof is straightforward.

Under CH, there is an uncountable $X \subset \mathbb{R}$ that concentrates on $\mathbb{Q}$ [1,10]. Moreover, one can get nontrivial examples without additional assumptions. Recall that $B \subset X$ is called a *Bernstein* set in X if every uncountable closed subset of X has points both in B and not in B. Every complete separable metrizable space contains a Bernstein set. It is clear that every space having a Bernstein set concentrates on it.

Proposition 3. *Let $B \subset X$ be a Bernstein set in $(X, \mathcal{T})$. If B is relatively mL in $(X, \mathcal{T})$ (in particular, if X is second countable), then $(X, \mathcal{T}_B)$ is mL.*

This gives a nontrivial example even for the real line $\mathbb{R}$ with the usual Euclidean topology $\mathcal{E}$.

Corollary 4. *Let B be a Bernstein subset of the real line $\mathbb{R}$. Then $(\mathbb{R}, \mathcal{E}_B)$ is mL.*

In [10], E. Michael showed that the product $(\mathbb{R}, \mathcal{E}_B) \times (\mathbb{R} \setminus B, \mathcal{E}|_{\mathbb{R}\setminus B})$ is not normal. This implies

Corollary 5. *There is a mL space X and a separable metrizable space Y such that the product $X \times Y$ is not normal.*

Proposition 6. *The square of $(\mathbb{R}, \mathcal{E}_B)$ is Lindelöf.*

Proof. We call the topology on $\mathbb{R} \times \mathbb{R}$ generated by the base $\mathcal{E}_B \times \mathcal{E}_B$ (and restrictions of this topology to subspaces) *new*.

Claim 1. *$\mathbb{R} \times \mathbb{R}$ concentrates on $(\mathbb{R} \times B) \cup (B \times \mathbb{R})$ (in the new topology).*

Proof. Let U be a neighborhood (in the new topology) of $(\mathbb{R} \times B) \cup (B \times \mathbb{R})$ in $\mathbb{R} \times \mathbb{R}$. Suppose $H = \mathbb{R} \times \mathbb{R} \setminus U$ is uncountable. Since every horizontal or vertical line intersects H on at most a countable set, one can pick pairwise distinct x_α and y_α in $\mathbb{R}$, $0 \leqslant \alpha < \omega_1$, so that $(x_\alpha, y_\alpha) \in H$. The sets C and D of complete accumulation points of the sets $\{x_\alpha\colon \alpha < \omega_1\}$ and $\{y_\alpha\colon \alpha < \omega_1\}$ (in the Euclidean topology) are closed and uncountable. Thus there are $c \in C \cap B$ and $d \in D \cap B$. So (c, d) is a complete accumulation point for the set $K = \{(x_\alpha, y_\alpha)\colon \alpha < \omega_1\}$ in the Euclidean topology. But $(c, d) \in B \times B$, and points of $B \times B$ have the same basic neighborhoods in the new topology as in the Euclidean one. Therefore (c,d) is a complete accumulation point for K in the new topology as well. This is a contradiction since $(c,d) \in U$ while $K \subset H$. □

Now it suffices to show that $\mathbb{R} \times B$ (in the new topology) is Lindelöf. Clearly, we will get this if we prove the following:

Claim 2. *Let $B \times B \subset U \subset \mathbb{R} \times B$ where U is open in the new topology. Then $\pi_1(\mathbb{R} \times B \setminus U)$ is at most countable (where π_1 is the projection of the product $\mathbb{R} \times B$ onto the first factor).*

Proof. Suppose that the projection is uncountable. Since the intersection of $(\mathbb{R} \times B) \setminus U$ with any horizontal line is at most countable, one can pick by induction points $(x_\alpha, y_\alpha) \in (\mathbb{R} \times B) \setminus U$, for all $\alpha < \omega_1$ so that x_α are pairwise distinct, and so are also y_α. The sets C and D of complete accumulation points of the sets $\{x_\alpha\colon \alpha < \omega_1\}$ and $\{y_\alpha\colon \alpha < \omega_1\}$ (in the Euclidean topology) are closed and uncountable. Thus there are $c \in C \cap B$ and $d \in D \cap B$. So (c,d) is a complete accumulation point for the set $K = \{(x_\alpha, y_\alpha)\colon \alpha < \omega_1\}$ in the Euclidean topology. But $(c,d) \in B \times B$, and points of $B \times B$ have the same basic neighborhoods in the new topology as in the Euclidean one. Therefore (c,d) is a complete accumulation point for K in the new topology as well. This is a contradiction since $(c,d) \in U$ while $K \subset (\mathbb{R} \times B) \setminus U$. □

Recall that an uncountable space X is called *Lusin* if every nowhere dense subset of X is countable.

Proposition 7. *(See [6].)* (CH) *Every uncountable CCC Baire space without isolated points and of π-weight at most $\mathfrak{c}$ contains a dense Lusin subspace.*

In fact, the condition in Proposition 7 is equivalent to CH, see [6]. It is clear that a Lusin space concentrates on every dense subspace.

Proposition 8. *Let B be a dense subspace in a Lusin space $(X, \mathcal{T})$. If B is relatively mL in $(X, \mathcal{T})$ (in particular, if $(X, \mathcal{T})$ second countable), then $(X, \mathcal{T}_B)$ is mL.*

Corollary 9. *Let B be a dense countable subspace in a Lusin space $(X, \mathcal{T})$, and let $(X, \mathcal{T})$ be first countable at all points of B. Then $(X, \mathcal{T}_B)$ is mL.*

In contrast with Proposition 6, mL spaces obtained from Lusin spaces need not, in general, have Lindelöf square. Let $(\mathbb{R}, \mathcal{S})$ denote the Sorgenfrey line.

Proposition 10. (CH) *There is a dense Lusin subspace B of $(\mathbb{R}, \mathcal{S})$ such that the square of $(B, \mathcal{S}|_B)$ is not Lindelöf.*

Proof. Pick a dense Lusin subspace $B_R \subset (\mathbb{R}^+, \mathcal{S}|_{\mathbb{R}^+})$. Put $B_L = \{-b\colon b \in B\}$ and $B = B_L \cup B_R$. Then the square of B contains $\{(b, -b)\colon b \in B\}$. By the Jones' lemma argument, it is not normal.

Proposition 11. $(\mathbb{R}, \mathcal{E}_B) \times (\omega + 1)$ *is not mL.*

Proof. Suppose r were a mL operator on $\mathbb{R} \times (\omega + 1)$.

For a function $f : \mathbb{R} \to [0, \infty)$, we denote by U_f the set of all points $(x, n) \in \mathbb{R} \times (\omega + 1)$ such that $\frac{1}{n} < f(x)$. (In this arithmetic, $1/\omega = 0$.) Denote $\mathcal{U}_f = \{U_f\} \cup \{\mathbb{R} \times \{n\}\colon n \in \omega\} \cup \{\{p\} \times (\omega + 1)\colon p \in \mathbb{R} \setminus B\}$. (Naturally, we are going to consider only those f for which $\mathcal{U}_f$ covers $\mathbb{R} \times (\omega + 1)$.)

For $x \in \mathbb{R} \setminus B$ and $t \in \mathbb{R}$, put $f_x(t) = |x - t|$. It is clear that, for $p \in \mathbb{R} \setminus B$, $r(\mathcal{U}_{f_p})$ must contain a set O such that the projection of O on $\mathbb{R}$ is $\{p\}$, and O contains $\{p\} \times [n, \omega]$ for some n. Moreover, for uncountably many p, this n is the same. Denote the set of such p by A_n. There is a point $z \in \mathbb{R}$ every neighborhood of which contains uncountably many points of A_n. Pick $\varepsilon \ll 1/n$. Consider the function g_z defined by $g_z(t) = \max\{2|z - t|, \varepsilon\}$ (see Fig. 1).

Denote $B_n = \{p \in A_n\colon g_z(t) > f_p(t) \text{ for all } t \in \mathbb{R}\}$. It is clear from the picture that B_n contains all points of A_n that are close enough to z, so B_n is uncountable. Therefore, the cover $\mathcal{U}_{g_z}$ is coarser than each of the covers $\mathcal{U}_{f_p}$, $p \in B_n$.

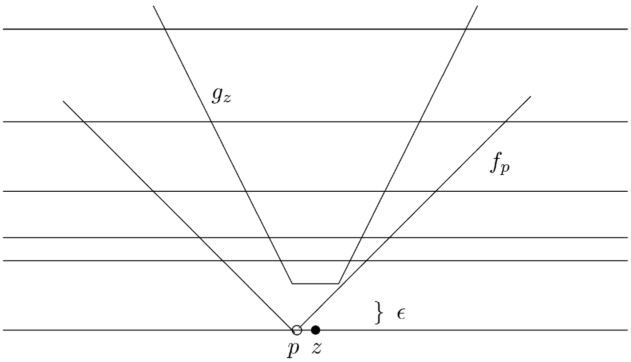

Fig. 1.

Then $r(\mathcal{U}_{g_z})$ must contain, for each $p \in B_n$, an element including the set $\{p\} \times [n, \omega]$. But the only element of $\mathcal{U}_{g_z}$ that includes $\{p\} \times [n, \omega]$ is $\{p\} \times (\omega + 1)$. So, $r(\mathcal{U}_{g_z})$ must contain uncountably many one point wide elements, and thus $r(\mathcal{U}_{g_z})$ must be uncountable which is a contradiction. □

The following is a formal generalization:

Proposition 12. *If the cellularity of $(X, \mathcal{T})$ is uncountable, $\mathcal{T}$ contains a weaker metrizable topology, and Y is first countable at at least one nonisolated point, then $(X, \mathcal{T}) \times Y$ is not mL.*

As we will see from the next proposition, the assumption of something like first countability in the previous one is essential; B still denotes a Bernstein set.

Proposition 13. *The product of $(\mathbb{R}, \mathcal{E}_B)$ and the one point Lindelöfication of the discrete space of cardinality ω_1 is mL.*

Proof. Let $D = \{d_\alpha\colon \alpha < \omega_1\}$ be a discrete space of cardinality ω_1, and let $L = D \cup \{d_{\omega_1}\}$ be the one point Lindelöfication of D. Let $\mathcal{O}$ be a countable base of $\mathcal{E}$. For an open cover $\mathcal{U}$ of $(\mathbb{R}, \mathcal{E}_B) \times L$ and $O \in \mathcal{O}$, put

$$\alpha_{\mathcal{U}}(O) = \begin{cases} \min\{\alpha < \omega_1\colon (\exists U \in \mathcal{U}) \text{ such that } O \times \{d_\beta\colon \alpha \leqslant \beta \leqslant \omega_1\} \subset U\} \\ \qquad \text{if such } U \text{exists,} \\ \omega_1 \quad \text{otherwise.} \end{cases}$$

Put

$$s(\mathcal{U}) = \bigl\{O \in \mathcal{O}\colon \alpha_{\mathcal{U}}(O) < \omega_1\bigr\},$$
$$t(\mathcal{U}) = \bigl\{O \times \bigl\{d_\beta\colon \alpha_{\mathcal{U}}(O) \leqslant \beta \leqslant \omega_1\bigr\}\colon O \in s(\mathcal{U})\bigr\}.$$

For $x \in \mathbb{R}$, denote

$$h_{1,\mathcal{U}}(x) = \liminf_{y \to x} \bigl\{\alpha_{\mathcal{U}}(O)\colon y \in O \in \mathcal{O}\bigr\}$$

(where $y \to x$ is understood with respect to the topology $\mathcal{E}$; note that this inf is actually min),

$$h_{2,\mathcal{U}}(x) = \min\bigl\{\alpha\colon (\exists U \in \mathcal{U}) \text{ such that } \{x\} \times \{d_\beta\colon \alpha \leqslant \beta \leqslant \omega_1\} \subset U\bigr\},$$
$$h_{\mathcal{U}}(x) = \max\bigl\{h_{1,\mathcal{U}}(x), h_{2,\mathcal{U}}(x)\bigr\},$$
$$H_{\mathcal{U}}(x) = \{x\} \times \bigl\{d_\beta\colon h_{\mathcal{U}}(x) \leqslant \beta \leqslant \omega_1\bigr\}.$$

Put $I(\mathcal{U}) = \mathbb{R} \setminus \bigcup s(\mathcal{U})$, $k(\mathcal{U}) = \{H_{\mathcal{U}}(x)\colon x \in I(\mathcal{U})\}$,

$$r(\mathcal{U}) = s(\mathcal{U}) \cup \bigl\{\{x\}\colon x \in I(\mathcal{U})\bigr\},$$
$$\alpha^*(\mathcal{U}) = \max\bigl\{\sup\bigl\{\alpha_{\mathcal{U}}(O)\colon O \in s(\mathcal{U})\bigr\}, \sup\bigl\{h_{2,\mathcal{U}}(x)\colon x \in I(\mathcal{U})\bigr\}\bigr\} + 1.$$

Note that if $\alpha \geqslant \alpha^*(\mathcal{U})$, then $\mathbb{R} \times \{d_\alpha\} \subset \bigcup(t(\mathcal{U}) \cup k(\mathcal{U}))$. For $\alpha < \alpha^*(\mathcal{U})$, put

$$\mathcal{U}_\alpha = \big(\mathcal{U} \mid (\mathbb{R} \times \{d_\alpha\})\big) \wedge \big\{V \times \{d_\alpha\}\colon\ V \in r(\mathcal{U})\big\},$$
$$s_\alpha(\mathcal{U}) = \big\{O \times \{d_\alpha\}\colon\ O \in \mathcal{O} \text{ and } (\exists U \in \mathcal{U}_\alpha) \text{ such that } O \times \{d_\alpha\} \subset U\big\},$$
$$i_\alpha(\mathcal{U}) = \Big\{\big\{(x, d_\alpha)\big\}\colon\ (x, d_\alpha) \in \big(\mathbb{R} \times \{d_\alpha\}\big) \setminus \bigcup s_\alpha(\mathcal{U})\Big\},$$
$$r_\alpha(\mathcal{U}) = s_\alpha(\mathcal{U}) \cup i_\alpha(\mathcal{U}).$$

Finally, put $R(\mathcal{U}) = t(\mathcal{U}) \cup k(\mathcal{U}) \cup \bigcup\{r_\alpha(\mathcal{U})\colon\ \alpha \in A(\mathcal{U})\}$. Then $R(\mathcal{U})$ is a countable open refinement of $\mathcal{U}$ covering $\mathbb{R} \times L$. To check monotonicity of R, let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of $(\mathbb{R}, \mathcal{E}_B) \times L$, and suppose $\mathcal{V}$ refines $\mathcal{U}$. Let $W \in R(\mathcal{V})$. We have to find $W' \in R(\mathcal{U})$ such that $W' \supset W$. There are three possibilities.

Case 1. $W \in t(\mathcal{V})$. The existence of W' follows from monotonicity of s and t.

Case 2. $W \in k(\mathcal{V})$. Then $W = H_\mathcal{V}(x)$ for some $x \in I(\mathcal{V})$. Obviously, $h_{1,\mathcal{U}}(x) \leqslant h_{1,\mathcal{V}}(x)$, $h_{2,\mathcal{U}}(x) \leqslant h_{2,\mathcal{V}}(x)$, and thus $h_\mathcal{U}(x) \leqslant h_\mathcal{V}(x)$. Therefore $H_\mathcal{U}(x) \supset H_\mathcal{V}(x)$. So if $x \in I(\mathcal{U})$, then $H_\mathcal{U}(x) \in k(\mathcal{U}) \subset R(\mathcal{U})$ and we can take $W' = H_\mathcal{U}(x)$.

Otherwise, if $x \notin I(\mathcal{U})$, we have $x \in \bigcup s(\mathcal{U})$, so $x \in O^*$ for some $O^* \in s(\mathcal{U})$. Then $\alpha_\mathcal{U}(O^*) \leqslant h_{1,\mathcal{U}}(x) \leqslant h_{1,\mathcal{V}}(x) \leqslant h_\mathcal{V}(x)$. So for $W' = O^* \times \{d_\beta\colon\ \alpha_\mathcal{U}(O^*) \leqslant \beta \leqslant \omega_1\}$ we have $W' \supset H_\mathcal{V}(x)$, and $W \in t(\mathcal{U}) \subset R(\mathcal{U})$.

Case 3. $W \in r_\alpha(\mathcal{V})$ for some $\alpha < \alpha^*(\mathcal{V})$. If $\alpha < \alpha^*(\mathcal{U})$, then the existence of W' follows from the fact that $\mathcal{V}_\alpha$ refines $\mathcal{U}_\alpha$ and monotonicity of s_α and r_α.

Suppose $\alpha \geqslant \alpha^*(\mathcal{U})$. Since $W \in r_\alpha(W)$, we have either (a) $W \in s_\alpha(W)$, or (b) $W \in i_\alpha(W)$. In the case (a), $W = O \times \{d_\alpha\}$ for some $O \in \mathcal{O}$, such that there is $V \in \mathcal{V}_\alpha$ with $O \times \{d_\alpha\} \subset V$. But $\mathcal{V}_\alpha$ refines $\mathcal{U}_\alpha$, so there is $U \in \mathcal{U}_\alpha$ such that $U \supset V \supset O \times \{d_\alpha\}$. So $W \in s_\alpha(\mathcal{U})$, and we can set $W' = W$.

In the case (b), W is a one point set, so the existence of W' follows from the fact that $R(\mathcal{U})$ is a cover. □

Taking into account Propositions 11 and 12 one may wonder if there is a first countable space X with uncountably many isolated points such that the product $X \times (\omega + 1)$ is mL. The answer is affirmative. Let Z be the lexicographic product $\mathbb{R} \times 3$. It follows from a result in [2] that Z is mL. (Alternatively, it is enough to note that Z concentrates on $\mathbb{R} \times (\{0, 2\}) \subset Z$.) Furthermore, Z is first countable, compact, and $c(Z) = \mathfrak{c}$.

Proposition 14. *The* (*Cartesian*) *product* $Z \times (\omega + 1)$ *is mL.*

Proof. For $p, q \in \mathbb{Q}$, $p < q$, and $n \in \omega$, put

$$O_{p,q,n} = (p, q) \times 3 \times [n, \omega].$$

For $p \in \mathbb{Q}$, $x \in \mathbb{R}$, $p < x$, and $n \in \omega$, put

$$R_{p,x,n} = \big(\big((p, x) \times 3\big) \cup \big(\{x\} \times \{0\}\big)\big) \times [n, \omega].$$

For $x \in \mathbb{R}$, $q \in \mathbb{Q}$, $x < q$, and $n \in \omega$, put

$$L_{x,q,n} = \big(\big((x, q) \times 3\big) \cup \big(\{x\} \times \{2\}\big)\big) \times [n, \omega].$$

Let $\mathcal{U}$ be an open cover of $Z \times (\omega 1)$. Put

$$s_O(\mathcal{U}) = \{O_{p,q,n}\colon\ p, q \in \mathbb{Q},\ p < q,\ n \in \omega,\ O_{p,q,n} \prec \mathcal{U}\},$$
$$s_R(\mathcal{U}) = \big\{R_{p,x,n}\colon\ p \in \mathbb{Q},\ x \in \mathbb{R},\ p < x,\ n \in \omega,\ R_{p,x,n} \prec \mathcal{U},\ R_{p,x,n} \not\prec s_O(\mathcal{U})\big\},$$
$$s_L(\mathcal{U}) = \big\{L_{x,q,n}\colon\ x \in \mathbb{R},\ q \in \mathbb{Q},\ x < q,\ n \in \omega,\ L_{x,q,n} \prec \mathcal{U},\ L_{x,q,n} \not\prec s_O(\mathcal{U})\big\},$$
$$s(\mathcal{U}) = s_O(\mathcal{U}) \cup s_R(\mathcal{U}) \cup s_L(\mathcal{U}).$$

Then it is easy to see that $s(\mathcal{U})$ is countable, $s(\mathcal{U}) \prec \mathcal{U}$, s is monotonic with respect to $\mathcal{U}$, and $\bigcup s(\mathcal{U}) \supset \mathbb{R} \times \{0, 2\} \times \{\omega\}$. Put

$$I(\mathcal{U}) = \big(\mathbb{R} \times \{1\} \times \{\omega\}\big) \setminus \bigcup s(\mathcal{U}).$$

Then $I(\mathcal{U})$ is at most countable. Put

$$i(\mathcal{U}) = \big\{y \in \mathbb{R}\colon\ \langle y, 1, \omega\rangle \in I(\mathcal{U})\big\}.$$

For $y \in i(\mathcal{U})$, put

$$n_R(y,\mathcal{U}) = \min\{n\colon (\exists p < y) R_{p,y,n} \prec \mathcal{U}\},$$
$$n_L(y,\mathcal{U}) = \min\{n\colon (\exists q > y) L_{y,q,n} \prec \mathcal{U}\},$$
$$n_i(y,\mathcal{U}) = \min\{n\colon \{y\} \times \{1\} \times [n,\omega] \prec \mathcal{U}\},$$
$$n(y,\mathcal{U}) = \max\{n_R(y,\mathcal{U}), n_L(y,\mathcal{U}), n_i(y,\mathcal{U})\}.$$

Put

$$N(\mathcal{U}) = \{\{y\} \times \{1\} \times [n(y,\mathcal{U}),\omega]\colon y \in i(\mathcal{U})\},$$
$$t(\mathcal{U}) = s(\mathcal{U}) \cup N(\mathcal{U}).$$

Then $t(\mathcal{U})$ is countable, $t(\mathcal{U}) \prec \mathcal{U}$, t is monotonic with respect to $\mathcal{U}$, and $\bigcup t(\mathcal{U}) \supset Z \times \{\omega\}$.

Let r be a mL operator for Z (as was noted before the proposition, such an operator exists). For an open cover $\mathcal{U}$ of $Z \times (\omega + 1)$, and $n \in \omega$, put

$$\mathcal{U}_n = \{O\colon O \text{ is open in } Z,\ O \times \{n\} \prec \mathcal{U}\}.$$

Then $\mathcal{U}_n$ is an open cover of Z. Put

$$r_n(\mathcal{U}) = \{V \times \{n\}\colon V \in r(\mathcal{U}_n)\}.$$

Then $r_n(\mathcal{U})$ is countable, $r_n(\mathcal{U}) \prec \mathcal{U}$, r_n is monotonic with respect to $\mathcal{U}$, and $\bigcup r_n(\mathcal{U}) \supset Z \times \{n\}$. Put

$$R(\mathcal{U}) = t(\mathcal{U}) \cup \bigcup\{r_n(\mathcal{U})\colon n \in \omega\}.$$

Then R is a mL operator for $Z \times (\omega + 1)$. □

So, in Proposition 12, “containing a weaker metrizable topology” cannot be generalized to “first countable”.

It is clear that monotone Lindelöfness is hereditary with respect to closed subspaces, the square of $(\mathbb{R}, \mathcal{E}_B)$ is Lindelöf and contains a closed subspace homeomorphic to $(\mathbb{R}, \mathcal{E}_B) \times (\omega + 1)$, so Proposition 11 implies

Corollary 15. *There is a mL space the square of which is Lindelöf but not mL.*

Question 16. For which $n > 1$ is there X such that X^n is mL while X^{n+1} is Lindelöf but not mL?

Question 17. Is there a mL space Y such that the product of Y with the one point Lindelöfication of the discrete space of cardinality ω_1 is not mL?

More generally:

Question 18. Let Y be mL and X a mL P-space. Must the product $Y \times X$ be mL?

3. Powers of subspaces of the Sorgenfrey line

Let $(\mathbb{R}, \mathcal{S})$ be the Sorgenfrey line. Even if the square of $(\mathbb{R}, \mathcal{S})$ is not normal, under CH the powers of some uncountable subspaces of $(\mathbb{R}, \mathcal{S})$ are Lindelöf.

Proposition 19. *(See [10].)* (CH) *For every n there exists $X \subset (\mathbb{R}, \mathcal{S})$ such that X^n is Lindelöf but X^{n+1} is not normal.*

Proposition 20. *(See [3].)* (CH) *For every n and every uncountable $Y \subset (\mathbb{R}, \mathcal{S})$ there is an uncountable $X \subset Y$ such that X^n is Lindelöf.*

A set $A \subset (\mathbb{R}, \mathcal{S})^n$ is called a *discrete surface* if for all distinct $x = \langle x_1, \dots, x_n\rangle$, $y = \langle y_1, \dots, y_n\rangle \in A$ there are $i, j \in n$ such that $x_i < y_i$ and $x_j > y_j$ [3].

Proposition 21. *(See [3].) Let $X \subset (\mathbb{R}, \mathcal{S})$. X^n is Lindelöf iff it does not contain an uncountable discrete surface.*

Proposition 22. *Let $X \subset (\mathbb{R}, \mathcal{S})$, and let $n \in \mathbb{N}$. If X^n is Lindelöf, then it is mL.*

Proof. We consider points of X^n as functions from n to X. For $a, b \in \mathbb{R}^n$, we write $a < b$ when $a(i) < b(i)$ for $0 \leqslant i < n$, $(a, b) = \{x \in \mathbb{R}^n\colon a < x < b\}$ etc. Pick a dense countable subspace $D \subset X$.

Let X^n be Lindelöf, $\mathcal{U}$ an open cover of X^n, and let $d \in D^n$. Put

$$s_d(\mathcal{U}) = \{[c, d) \cap X^n\colon c \in D^n \,\&\, c < d \,\&\, [c, d) \cap X^n \prec \mathcal{U}\},$$
$$S_d(\mathcal{U}) = \bigcup s_d(\mathcal{U}),$$
$$T_d(\mathcal{U}) = \{y \in X^n \setminus S_d(\mathcal{U})\colon y < d \,\&\, [y, d) \cap X^n \prec \mathcal{U}\}.$$

Let $\emptyset \neq A \subset n$. Say that a point $t \in T_d(\mathcal{U})$ is of $\mathcal{U}$-type A if there is $z \in T_d(\mathcal{U})$ such that $z(i) < t(i)$ for all $i \in A$, and $z(i) = t(i)$ for all $i \in n \setminus A$. Let $I(t) = \{A \subset n\colon t \text{ is of } \mathcal{U}\text{-type } A\}$, and $MI(t) = \{A \in I(t)\colon \nexists B \in I(t) \text{ such that } B \supset A$ and $B \neq A\}$. Let

$$T_{d,A}(\mathcal{U}) = \{t \in T_d(\mathcal{U})\colon A \in MI(t)\},$$
$$\mathcal{A}_d(\mathcal{U}) = \{A\colon \emptyset \neq A \subset n \,\&\, T_{d,A}(\mathcal{U}) \neq \emptyset\},$$
$$T_{d,\emptyset}(\mathcal{U}) = \{t \in T_d(\mathcal{U})\colon t \text{ is not of } \mathcal{U}\text{-type } A \text{ for any } A \text{ such that } \emptyset \neq A \subset n\}.$$

Then $T_d(\mathcal{U}) = T_{d,\emptyset}(\mathcal{U}) \cup \bigcup\{T_{d,A}(\mathcal{U})\colon A \in \mathcal{A}_d(\mathcal{U})\}$.

It is clear that $n \notin MI(t)$ for any t. For $A \subset n$, let $A' = n \setminus A$. Let $A \in \mathcal{A}_d(\mathcal{U})$. We claim that $|\pi_{A'}(T_{d,A}(\mathcal{U}))| \leqslant \omega$.

Suppose the contrary. Note that for every $p \in \pi_{A'}(T_{d,A}(\mathcal{U}))$ there are $q_1, q_2 \in D^A \cap \pi_A(T_{d,A}(\mathcal{U}))$ such that $q_1(i) < q_2(i)$ for all $i \in A$. There is an uncountable $K \subset \pi_{A'}(T_{d,A}(\mathcal{U}))$ such that for all $p \in K$, q_1 and q_2 are the same. Since K is uncountable, it is not a discrete surface, and thus it contains points p_1, p_2 such that $p_1(i) < p_2(i)$ for all $i \in A'$. Consider points $t_1, t_2 \in T_{d,A}(\mathcal{U})$ defined by

$$t_j(i) = \begin{cases} q_j(i) & \text{if } i \in A, \\ p_j(i) & \text{if } i \in A', \quad j = 1, 2. \end{cases}$$

Then $t_1 < t_2 < d$. Pick $c \in D^n$ so that $t_1 < c < t_2 < d$. Then $t_2 \in [c, d)$, a contradiction.

Suppose $A \in \mathcal{A}_d(\mathcal{U})$ and $p \in \pi_{A'}(T_{d,A}(\mathcal{U}))$. Let

$$P_{d,A,p}(\mathcal{U}) = \{x \in T_{d,A}(\mathcal{U})\colon x < d \,\&\, \pi_{A'}(x) = p \,\&\, \pi_{\{i\}}(x) \in D \text{ for all } i \in A\}.$$

Then $|P_{d,A,p}(\mathcal{U})| \leqslant \omega$. Put

$$r_{d,A,p}(\mathcal{U}) = \{[x, d)\colon x \in P_{d,A,p}(\mathcal{U})\},$$
$$r_{d,A}(\mathcal{U}) = \bigcup\{r_{d,A,p}(\mathcal{U})\colon p \in \pi_{A'}(T_{d,A}(\mathcal{U}))\},$$
$$r_d(\mathcal{U}) = \bigcup\{r_{d,A}(\mathcal{U})\colon A \in \mathcal{A}_d(\mathcal{U})\} \cup s_d(\mathcal{U}).$$

Then $T_d(\mathcal{U}) \setminus \bigcup r_d(\mathcal{U}) = T_{d,\emptyset}(\mathcal{U})$. It is clear that $T_{d,\emptyset}(\mathcal{U})$ is a discrete surface, and thus it is countable. Put

$$\tilde{r}_d(\mathcal{U}) = r_d(\mathcal{U}) \cup \{[t, d)\colon t \in T_{d,\emptyset}(\mathcal{U})\},$$
$$r(\mathcal{U}) = \bigcup\{\tilde{r}_d(\mathcal{U})\colon d \in D^n\}.$$

Then r is a mL operator for X^n. This follows from the following observations: if $\mathcal{U}$ and $\mathcal{V}$ are two open covers of X^n, and $\mathcal{U} \prec \mathcal{V}$, then:

(1) $S_d(\mathcal{U}) \subset S_d(\mathcal{V})$,
(2) $S_d(\mathcal{U}) \cup T_d(\mathcal{U}) \subset S_d(\mathcal{V}) \cup T_d(\mathcal{V})$,
(3) if $x \in S_d(\mathcal{V}) \cup T_d(\mathcal{V})$, then there is $y \in S_d(\mathcal{V}) \cup T_d(\mathcal{V})$ such that $y \leqslant x$ and $[y, d) \in \tilde{r}_d(\mathcal{V})$. □

Corollary 23. (CH) *For every n and every uncountable $Y \subset (\mathbb{R}, \mathcal{S})$ there is an uncountable $X \subset Y$ such that X^n is mL.*

4. Some L spaces are mL

We start with a proposition that helps to show that two known examples of L-spaces (constructed assuming CH) are mL. Perhaps it can be applied to some other Lindelöf spaces with point countable bases.

Proposition 24. *Let $X = \{x_\alpha\colon \alpha < \omega_1\}$ be a hereditarily Lindelöf space, and $\{B_{n,\alpha}\colon n \in \omega, \alpha < \omega_1\}$ a family of open sets in X such that*

(1) *$\{B_{n,\alpha}\colon n \in \omega, B_{n,\alpha} \neq \emptyset\}$ is a base of neighborhoods of x_α, and*
(2) *if $\alpha < \beta < \omega_1$, and $x_\beta \in B_{n,\alpha}$, then $B_{n,\beta} \subset B_{n,\alpha}$.*

Then X is mL.

Proof. Let $\mathcal{U}$ be an open cover of X. We define families $s_\alpha(\mathcal{U})$ inductively for $\alpha < \omega_1$. Let $\alpha < \omega_1$ and suppose $s_\gamma(\mathcal{U})$ has been defined for each $\gamma < \alpha$. Put $s_\alpha(\mathcal{U}) = \{B_{n,\alpha}\colon$ (*a*) $B_{n,\alpha} \prec \mathcal{U}$, and (b) there are no $\gamma < \alpha$ and $B_{m,\gamma} \in s_\gamma(\mathcal{U})$ such that $B_{n,\alpha} \subset B_{m,\gamma}\}$. Finally, set $r(\mathcal{U}) = \bigcup\{s_\alpha(\mathcal{U})\colon \alpha < \omega_1\}$. Then $r(\mathcal{U})$ is an open cover of X that refines $\mathcal{U}$. The proof of monotone Lindelöfness of X is now concluded by these two claims:

Claim 1. *$s_\alpha(\mathcal{U})$ are eventually empty and thus $r(\mathcal{U})$ is countable.*

Proof. If $r(\mathcal{U})$ is uncountable, then for some n so is $r_n(\mathcal{U}) = \{B_{m,\alpha} \in r(\mathcal{U})\colon m = n\}$. Denote $X_n = \{x_\alpha\colon B_{n,\alpha} \in r_n(\mathcal{U})\}$, and $A_n = \{\alpha\colon B_{n,\alpha} \in r_n(\mathcal{U})\}$. The open cover $\{B_{n,\alpha}\colon \alpha \in A_n\}$ of X_n has a countable subcover, say $\{B_{n,\alpha}\colon \alpha \in A\}$. Pick $\beta \in A_n$ with $\beta > \sup A$. Then $x_\beta \in B_{n,\alpha}$ for some $\alpha \in A$, and thus by (2) and (b), $B_{n,\beta}$ cannot be in $s_\beta(\mathcal{U})$. A contradiction. □

Claim 2. *r is monotonic.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X such that $\mathcal{V}$ refines $\mathcal{U}$, and let $B_{n,\alpha} \in s_\alpha(\mathcal{V}) \subset r(\mathcal{V})$. Then there is a $V \in \mathcal{V}$ such that $B_{n,\alpha} \subset V$, and thus a $U \in \mathcal{U}$ such that $B_{n,\alpha} \subset V \subset U$. Then either $B_{n,\alpha} \in r(\mathcal{U})$, or there is a $B_{m,\gamma} \in r(\mathcal{U})$ with $\gamma < \alpha$ and $B_{m,\gamma} \supset B_{n,\alpha}$. □

Remarks. (1) The condition $|X| = \omega_1$ is not very restrictive: a first countable Lindelöf space has cardinality $\leqslant \mathfrak{c}$. Under CH this is ω_1, and most applications of this proposition are supposed to be under the assumption of CH.

(2) Adding condition (2′) $x_\alpha \notin B_{n,\beta}$ whenever $\alpha < \beta$ makes X an L-space with a point-countable base. This condition holds in both applications below.

Now we apply Proposition 24 to L-spaces from [5] which are subspaces of $\mathcal{P}(\omega)$ with the Vietoris topology. The following three facts are from [5]:

Proposition 25. *(See [5].)*

(A) *In the Vietoris topology a neighborhood base for $x \in \mathcal{P}(\omega)$ consists of all sets of the form $[f, x]$ ($f \in [x]^{<\omega}$) where $[f, x] = \{s\colon f \subseteq s \subseteq x\}$. In other words, for the discrete space ω, the Vietoris topology on $\mathcal{P}(\omega)$ coincides with the Pixley–Roy topology* [4].
(B) *The following axiom is a consequence of* CH
(DOWN) *There is a sequence $\langle x_\alpha\colon \alpha < \omega_1\rangle$ in $\mathcal{P}(\omega)$ such that*
 (1) *if $\alpha < \beta < \omega_1$, then $x_\alpha \not\subseteq x_\beta$; and*
 (2) *if $I \subseteq \omega_1$ is uncountable, then there are distinct $\alpha, \beta \in I$ with $x_\beta \subseteq x_\alpha$.*
(C) *If x_α are as in* (DOWN), *then $X = \{x_\alpha\colon \alpha < \omega_1\} \subset \mathcal{P}(\omega_1)$ is an L-space.*

Using this we get.

Proposition 26. *If x_α are as in* (DOWN), *then $X = \{x_\alpha\colon \alpha < \omega_1\} \subset \mathcal{P}(\omega)$ is mL.*

Proof. To apply Proposition 24, enumerate $[\omega]^{<\omega}$ as $\{f_n\colon n \in \omega\}$, and put $B_{n,\alpha} = [f_n, x_\alpha]$. □

Now we apply Proposition 24 to an L space from [6]. This space, denoted by $\mathcal{L}$ is a subspace of the space $\mathcal{K}$ of all nonempty compact nowhere dense subsets of $\mathbb{R}$ equipped with the Pixley–Roy topology, that is, a basic neighborhood of $K \in \mathcal{K}$ is of the form $(*)$ $[K,U] = \{S \in \mathcal{K}\colon K \subset S \subset U\}$ where U is a neighborhood of K in $\mathbb{R}$. The following two facts are from [6]:

Proposition 27. *(See [6].)*

(A) *$\mathcal{K}$ is a CCC Baire space in which no nonempty open set is separable.*
(B) (CH) *By Proposition* 7*, $\mathcal{K}$ contains a dense Lusin subspace $\mathcal{L}$*; *$\mathcal{L}$ is an L space.*

Proposition 28. (CH) *The space $\mathcal{L}$ from Proposition* 27(B) *is mL.*

Proof. To apply Proposition 24, enumerate $\mathcal{L}$ as $\{L_\alpha\colon \alpha < \omega_1\}$, and all unions of finite families of open intervals in $\mathbb{R}$ with rational endpoints as $\{U_n\colon n \in \omega\}$ and put $B_{n,\alpha} = [L_\alpha, U_n]$. □

Another well-known example of an L space is the Souslin line. Monotone Lindelöfness of Souslin lines is discussed in [2]; some questions remain open.

5. Subspaces of infinite products

A space X is *monotonically Lindelöf at $p \in X$* if there is an operator r_p that assigns to every nonempty family $\mathcal{U}$ of neighborhoods of p a nonempty countable family $r_p(\mathcal{U})$ of neighborhoods of p so that $r_p(\mathcal{U})$ refines $\mathcal{U}$, and $r_p(\mathcal{U})$ refines $r_p(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$ [8]. Clearly, a mL space must be mL at each point. By a subbase of neighborhoods of $p \in X$ we mean a family $\mathcal{A}$ of neighborhoods of p such that finite intersections of elements of $\mathcal{A}$ form a base of neighborhoods of p. The next proposition is a slight generalization of a proposition from [8].

Proposition 29. *Let X be a space, $p \in X$, κ and τ infinite cardinals* ($\kappa < \tau$)*, and $\mathcal{B}$ a subbase of neighborhoods of p of the form $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha\colon \alpha < \tau\}$. Suppose that*

(1) *for every neighborhood U of p, $|\{\alpha < \tau\colon \exists B \in \mathcal{B}_\alpha U \subset B\}| < \kappa$, and*
(2) *every subfamily of $\mathcal{B}$ which is still a subbase at p contains elements from more than κ many $\mathcal{B}_\alpha$s.*

Then X is not mL at p.

Proof. Suppose there were an operator r_p like in the definition of monotone Lindelöfness at p. By induction, we define a decreasing sequence $\{\mathcal{U}_\gamma\colon \gamma \leqslant \kappa\}$ of families of neighborhoods of p, and an increasing sequence $\{T_\gamma\colon \gamma \leqslant k\}$ of subsets of τ. Set $\mathcal{U}_0 = \mathcal{B}$, and $T_0 = \emptyset$. Now, suppose $0 < \gamma \leqslant \kappa$, and $\mathcal{U}_\beta$ and T_β have been defined for all $\beta < \gamma$. Put $T_\gamma = \{\alpha < \tau\colon \exists \beta < \gamma,\ \exists U \in r_p(\mathcal{U}_\beta),\ \exists B \in \mathcal{B}_\alpha$ such that $U \subset B\}$ and $\mathcal{U}_\beta = \bigcup\{\mathcal{B}_\alpha\colon \alpha \notin T_\beta\}$. By (1), $|T_\gamma| \leqslant \kappa$, and thus $\mathcal{U}_\gamma \neq \emptyset$. At step $\gamma = \kappa$ we get a contradiction with (2). □

Corollary 30. *The one point compactification of a discrete space of uncountable cardinality is not mL.*

Proof. Let $X = \{p\} \cup D$, where $|D| > \omega$, points of D are isolated in X, and a basic neighborhood of p contains p and all but finitely many points of D. To apply Proposition 29, set $\kappa = \omega$, $\tau = |D|$, enumerate $D = \{d_\alpha\colon \alpha < \tau\}$, and put $\mathcal{B}_\alpha = \{X \setminus \{d_\alpha\}\}$. □

Corollary 31. *The one point Lindelöfication of the discrete space of cardinality $\geqslant \omega_2$ is not mL.*

Proof. Let $X = \{p\} \cup D$, where $|D| > \omega_1$, points of D are isolated in X, and a basic neighborhood of p contains p and all but countably many points of D. To apply Proposition 29, set $k = \omega_1$, $\tau = |D|$, enumerate $D = \{d_\alpha\colon \alpha < \tau\}$, and put $\mathcal{B}_\alpha = \{X \setminus \{d_\alpha\}\}$. □

Corollary 32. *If X is a dense subset of the product of at least ω_1 many factors each of which consist of more than one point, then X is not mL at any point.*

Proof. Put $\kappa = \omega$. Without loss of generality, assume that $X \subset P = \prod\{P_\alpha\colon \alpha < \tau\}$ where $\tau \geqslant \omega_1$ and all P_α consist of more than one point. Denote π_α the projection of P onto the αth factor, $\mathcal{C}_\alpha$ the family of all open sets in P_α that contain $\pi_\alpha(p)$ but not the entire $\pi_\alpha(X)$, and $\mathcal{B}_\alpha = \{X \cap (\pi_\alpha)^{-1}(B)\colon B \in \mathcal{C}_\alpha\}$. Then $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha\colon \alpha < \tau\}$ is a subbase at p like in the previous proposition. □

Recall that $C_p(X)$, the space of continuous functions on X with the pointwise convergence topology, is Lindelöf in many nontrivial cases. In contrast with this we get:

Corollary 33. *The following conditions are equivalent*:

(1) *$C_p(X)$ is mL,*
(2) *$C_p(X)$ is mL at any point,*
(3) *X is countable.*

6. $\beta\omega$ is not mL

Proposition 34. *$\beta\omega \setminus \omega$ is not mL.*

Proof. Let $\mathcal{A}$ be an uncountable independent family of subsets of ω. For $A \in \mathcal{A}$, put $A^0 = \mathrm{cl}_{\beta\omega}(A) \cap (\beta\omega \setminus \omega)$ and $A^1 = \mathrm{cl}_{\beta\omega}(\omega \setminus A) \cap (\beta\omega \setminus \omega)$. For $\mathcal{C} \subset \mathcal{A}$, put $\tilde{\mathcal{C}} = \{A^i\colon A \in \mathcal{C}$ and $i \in 2\}$. If $\mathcal{C} \neq \emptyset$, then $\tilde{\mathcal{C}}$ is an open cover of $\beta\omega \setminus \omega$. Suppose there is a mL operator r for $\beta\omega \setminus \omega$.

We inductively define nonempty subfamilies $\mathcal{U}_\alpha \subset \mathcal{A}$ for $0 \leqslant \alpha \leqslant \omega$ so that $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$ whenever $\alpha < \beta$. Put $\mathcal{U}_0 = \mathcal{A}$. Now let $0 \leqslant \alpha < \omega$, and $\mathcal{U}_\alpha$ has been defined. For every $R \in r(\tilde{\mathcal{U}_\alpha})$ pick $A_R \in \mathcal{U}_\alpha$ so that $R \subset (A_R)^i$ for some $i \in 2$. Put $\mathcal{U}_{\alpha+1} = \mathcal{U}_\alpha \setminus \{A_R\colon R \in r(\tilde{\mathcal{U}_\alpha})\}$. Then for every $\alpha < \omega$, $\mathcal{U}_\alpha$ is a co-countable subfamily of $\mathcal{A}$, and hence so is $\mathcal{U}_\omega = \bigcap\{\mathcal{U}_\alpha\colon \alpha < \omega\}$.

Notice that for every $R \in r(\tilde{\mathcal{U}_\alpha})$, there is $A_R \in \mathcal{U}_\alpha \setminus \mathcal{U}_{\alpha+1}$ and $i \in 2$ such that $R \subset (A_R)^i$. Then by monotonicity,

(∗) if $0 \leqslant \alpha < \beta \leqslant \omega$, then for every $R \in r(\tilde{\mathcal{U}_\beta})$, there is $A \in \mathcal{U}_\alpha \setminus \mathcal{U}_{\alpha+1}$ and $i \in 2$ such that $R \subset (A)^i$.

By compactness, there is a finite subcover $\mathcal{D} \subset r(\tilde{\mathcal{U}_\omega})$. Let $|\mathcal{D}| = M$. Pick m so that $2^m > M$. For each $D \in \mathcal{D}$ and each $\alpha \in \{0, \ldots, m-1\}$ by (∗) one can pick $A_{D,\alpha} \in \mathcal{U}_\alpha \setminus \mathcal{U}_{\alpha+1}$ so that $D \subset (A_{D,\alpha})^i$ for some $i \in 2$, then $A_{D,\alpha} \neq A_{D,\beta}$ whenever $\alpha \neq \beta$. Put $\mathcal{K} = \{A_{D,\alpha}\colon D \in \mathcal{D},\ 0 \leqslant \alpha < m\}$. Then $|\mathcal{K}| \geqslant m$. Now we note that $\bigwedge\{\tilde{A}\colon A \in \mathcal{K}\}$ is a partition of $\beta\omega \setminus \omega$ into $2^{|\mathcal{K}|}$ nonempty clopen sets, call them blocks, and each $D \in \mathcal{D}$ intersects at most $2^{|\mathcal{K}|-n}$ blocks. So elements of $\mathcal{D}$, together, cannot intersect all blocks, a contradiction. □

Since monotone Lindelöfness is preserved by closed subspaces, we get

Corollary 35. *$\beta\omega$ is not mL.*

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