Construction of $k$-arc transitive digraphs

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Abstract

A digraph is $k$-arc transitive if it has a group of automorphisms which acts transitively on the set of $k$-arcs. Unlike the undirected case, in which the cycles are the only $k$-arc transitive finite graphs for $k \geq 8$, there are $k$-arc transitive finite digraphs with arbitrary out-degree for every positive integer $k$. We show that every regular finite digraph admits a covering digraph which is $k$-arc transitive. The result provides a technique to construct $k$-arc transitive digraphs from arbitrary regular digraphs. Some examples are given from complete graphs with and without loops. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a positive integer $k$, a $k$-arc of a digraph $\Gamma=(V,E)$ is a sequence $(x_0, \ldots, x_k)$ of $k+1$ vertices of $\Gamma$ such that, for each $0 \leq i < k$, $(x_i, x_{i+1})$ is an arc of the digraph. A digraph $\Gamma$ is $k$-arc transitive if it has an automorphism group $G < \text{Aut} \, \Gamma$ which acts transitively on $k$-arcs.

The corresponding notion for undirected graphs led to remarkable results. A well-known result by Tutte [19] states that finite cubic graphs cannot be $k$-arc transitive for $k > 5$. Weiss [20] proved several years later that the only finite connected $k$-arc transitive graphs with $k \geq 8$ are the cycles.

The situation is different in the directed case. Praeger [14] gave infinite families of $k$-arc transitive digraphs for each positive integer $k$ and each out-degree $v$, and new constructions were given by Conder et al. [4]. Recently, Cameron et al. [3] gave...
constructions of infinite highly transitive digraphs, digraphs which are \( k \)-arc transitive for every positive integer \( k \).

The explicit constructions of \( k \)-arc transitive digraphs given in the above-mentioned references are group theoretically oriented: the graph is constructed from its given automorphism group. Here we use a graph theoretical approach. Given a regular digraph we use it to construct \( k \)-arc transitive digraphs for each positive integer \( k \).

Our main result can be stated in terms of covering digraphs. Let \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) be two regular digraphs. A digraph homomorphism from \( \Gamma_1 \) to \( \Gamma_2 \) is a map \( \sigma : V_1 \to V_2 \) which sends arcs to arcs, that is, \( (x, y) \in E_1 \) implies \( (\sigma(x), \sigma(y)) \in E_2 \). Let \( P \) be a partition of the vertex set \( V \) of a digraph \( \Gamma \). The quotient digraph \( \Gamma/P \) has the sets of the partition as vertices and there is one arc from set \( X \in P \) to set \( Y \in P \) if there is an arc \( (x, y) \in X \times Y \) in \( \Gamma \). Then, the map \( \sigma : V \to V/P \), where \( \sigma(x) \) is the part of \( P \) containing \( x \), is a digraph homomorphism of \( \Gamma \) onto \( \Gamma/P \).

The digraph \( \Gamma_1 \) is a cover of \( \Gamma_2 \), with covering map \( \sigma \), if \( \sigma \) is a surjective digraph homomorphism of \( \Gamma_1 \) onto \( \Gamma_2 \), there is a positive integer \( h \) such that \( |\sigma^{-1}(x)| = h \) for all \( x \in V_2 \), and \( \sigma \) is a local isomorphism, that is, for each vertex \( y \in V_1 \) the sets \( \Gamma_1(y) \) and \( \Gamma_2(\sigma(y)) \) of neighbors of \( y \) and \( \sigma(y) \) in \( \Gamma_1 \) and \( \Gamma_2 \), respectively, have the same size. We write \( \Gamma_1 \twoheadrightarrow \Gamma_2 \), or \( \Gamma_1 \twoheadrightarrow \Gamma_2 \) when the reference to \( \sigma \) is to be made explicit.

The idea of the technique to construct \( k \)-arc transitive digraphs is based on the simple remark that a digraph \( \Gamma \) is \( k \)-arc transitive if and only if its \( k \)-iterated line digraph is vertex transitive, see Section 2. Extending a previous result by Gross [8], Babai [1] proves that a regular digraph is a Schreier coset digraph of a finite group \((\text{the quotient digraph of a Cayley digraph Cay}(G, S) by a subgroup } H \text{ of } G\). In other words, a regular digraph admits a covering digraph which is vertex transitive. The key result is then that suitably chosen vertex transitive covers of \( k \)-iterated line digraphs are \( k \)-iterated line digraphs themselves. Therefore, they provide instances of \( k \)-arc transitive digraphs which turn out to be covering digraphs of the original digraph.

Our main result can be stated in the following way.

**Theorem 1.** Let \( \Gamma \) be a regular digraph. For each positive integer \( k \) there is a cover \( \Gamma_k \) of \( \Gamma \) which is a \( k \)-arc transitive digraph.

The above statement is a generalization of the results by Babai [1], which correspond to the cases \( k = 0, 1 \) of the theorem. Actually, the mentioned results by Babai provide infinite chains of covers by arc transitive digraphs. Here we can also require to have an infinite chain of \( k \)-arc transitive digraphs, in the sense that the digraphs in the chain are not \((k + 1)\)-arc transitive, but the construction presents some technical difficulties. Even if \( \Gamma_k \twoheadrightarrow \Gamma \) and \( \Gamma_k \twoheadrightarrow \Gamma \), for \( k > k' \), it is not always the case that \( \Gamma_k \twoheadrightarrow \Gamma_k \).

The main result is based in a technical theorem on \( k \)-iterated line digraphs which is presented in Section 3. It turns out that \( \Gamma_k \) has an automorphism group which acts regularly on \( k \)-arcs. However, the technique can be extended to provide digraphs with transitive (nonregular) groups. These problems are addressed in [13].
The paper is organized as follows. In Section 2, we give the terminology and preliminary results to prove Theorem 1. Section 3 is devoted to characterizing covers of \( k \)-line digraphs which are \( k \)-line digraphs themselves. The proof of Theorem 1 is given in Section 4. Some explicit constructions of \( k \)-arc transitive digraphs using the proof of Theorem 1 are given in Section 5. Final remarks in Section 6 conclude the paper.

2. Preliminary results

A digraph or directed graph \( \Gamma = (V,E) \) consists of a set \( V \) of vertices and a subset \( E \) of ordered pairs from \( V \), called arcs. If \((x,y) \in E\) is an arc from \( x \) to \( y \) we say that \( x \) is adjacent to \( y \) and also that \( y \) is adjacent from \( x \). A digraph is regular of degree \( r \) or \( r \)-regular when each vertex is adjacent to and from exactly \( r \) vertices. A digraph is strongly connected when for any \( x,y \in V \) there is a directed path from \( x \) to \( y \).

All digraphs are assumed to be finite, regular and (strongly) connected, that is, the in- and out-neighborhoods of each vertex have all the same cardinality and there is a directed path from each given vertex to any other one in the digraph. We allow loops but no multiple arcs. Note that a finite connected \( k \)-arc transitive digraph with \( k \geq 0 \) must be regular. Directed cycles are \( k \)-arc transitive for every positive integer \( k \). Therefore we always consider regular digraphs of degree \( r \geq 1 \).

By the König–Hall theorem, a \( r \)-regular digraph \( \Gamma = (V,E) \) is the sum of permutation digraphs \( F = \{F_1,\ldots,F_r\} \) corresponding to a set \( \{f_1,\ldots,f_r\} \) of permutations [18, Theorem 5.3]. We call such a set \( F \) a 1-factorization. That is, a 1-factorization is a set of 1-regular spanning subdigraphs of \( \Gamma \) whose sets of arcs partition \( E \). Each \( F_i \) is a disjoint union of directed cycles, and we interpret \( f_i \) as the corresponding permutation of \( V \) whose disjoint cycle decomposition is \( F_i \), \( 1 \leq i \leq r \). We still denote by \( F = \{f_1,\ldots,f_r\} \) the set of these permutations. We denote by \( G(\Gamma,F) \) the permutation group on \( V \) generated by the permutations in \( F \) and refer to it as the permutation group of the 1-factorization \( F \). We use the exponential notation for the action of permutations. Therefore, the composition of permutations is read from left to right: \( x^{f_1 f_2} = (x^{f_1})^{f_2} \).

Let \( G \) be a finite group and \( S \subseteq G \). The Cayley digraph of \( G \) defined by \( S, \text{ Cay}(G,S) \), has the elements of \( G \) as vertices and there is an arc \((x,ys)\) for each \( x \in G \) and each \( s \in S \). When \( F \) is a 1-factorization of a digraph \( \Gamma \) and \( G = G(\Gamma,F) \), we say that \( \text{Cay}(G,F) \) is a Cayley cover of \( \Gamma \) and write \( \text{Cay}(G,F) = \tilde{\Gamma}_F \) (or simply \( \tilde{\Gamma} \) if the reference to \( F \) is clear from the context). The following easy lemma justifies this terminology. It is a restatement of [1, Proposition 2.2] and a proof is included for the benefit of the reader.

**Proposition 2.** Let \( F \) be a 1-factorization of a regular connected digraph \( \Gamma = (V,E) \). Then, \( \tilde{\Gamma}_F \) is a cover of \( \Gamma \).

**Proof.** Let \( x_0 \) be a vertex of \( \Gamma \). Let \( G = G(\Gamma,F) \) and \( \pi : G \to V \) be defined as \( \pi(g) = x_0^g \). Since the arc \((g,.gf)\) is sent to the arc \((x_0^g,((x_0^g)^g)^f)\) for each \( g \in G \) and \( f \in F \), then \( \pi \)
is a digraph homomorphism. Since $\Gamma$ is connected, $G$ is transitive on $V$. Therefore, $\pi$ is a surjective map. Both graphs are $r$-regular with $r=|F|$ and $|\pi(gF)|=|x_0^g|=|F|$ for each $g \in G$. Hence $\pi$ is a local isomorphism. Finally, $|\pi^{-1}(x)| = |G_{x_0}|$ for all $x \in V$, where $G_{x_0}$ is the stabilizer of $x_0$ in $G$. □

Note that when $\Gamma$ is a Cayley digraph itself, say $\Gamma = \text{Cay}(G,S)$, then the set $S = \{s_1, \ldots, s_r\}$ gives rise to the 1-factorization $\{S_1, \ldots, S_r\}$ where $S_i$ is the 1-regular spanning subdigraph which contains the arcs of the form $(g, gs_i)$, $g \in G$ and $G = G(\Gamma,S)$ if $\Gamma$ is connected.

Recall that the line digraph $L\Gamma = (V_L, E_L)$ of $\Gamma = (V,E)$ has the arcs of $\Gamma$ as vertices and $((x_1, x_2), (y_1, y_2))$ is an arc in $L\Gamma$ whenever $x_2 = y_1$. If $\Gamma$ is a regular digraph, then the map $\phi : \text{Aut} \Gamma \rightarrow \text{Aut} L\Gamma$ defined as

$$(x, y)^{\phi(g)} = (x^g, y^g), \quad (x, y) \in E$$

for each $g \in \text{Aut} \Gamma$, is a group isomorphism, see for instance [10]. We identify $\phi(g) \in \text{Aut} L\Gamma$ with $g \in \text{Aut} \Gamma$. With this identification, we can write $\text{Aut} L\Gamma = \text{Aut} \Gamma$.

For $k \geq 2$, the $k$-line digraph of $G$ is defined recursively by $L^k \Gamma = L(L^{k-1}\Gamma)$. Thus, the vertices of $L^k \Gamma$ can be thought of as $k$-arcs in $\Gamma$. By the above remark, we have $\text{Aut} L^k \Gamma = \text{Aut} \Gamma$. Therefore, we have

**Remark 3.** A finite connected digraph $\Gamma$ is $k$-arc transitive if and only if its $k$-iterated line digraph is vertex transitive.

For a positive integer $k$ and a vertex $x \in V$, $\Gamma^k(x)$ denotes the set of vertices $y \in V$ for which there is a directed walk of length exactly $k$ from $x$ to $y$. We will use the following characterization of line digraphs given by Heuchene [11].

**Theorem 4.** Let $\Gamma = (V,E)$ be a connected $r$-regular digraph. There is an $r$-regular digraph $\Gamma_0$ (possibly with multiple arcs) such that $\Gamma = L^k \Gamma_0$ if and only if $\{\Gamma^k(x) | x \in V\}$ is a partition of $V$ into sets of cardinality $r^k$. Moreover, if $|\Gamma^{k+1}(x)| = r^{k+1}$ for each $x \in V$, then $\Gamma_0$ has no multiple arcs.

According to the above characterization, $\Gamma$ is isomorphic to the quotient digraph $L\Gamma/P$, where $P$ is the partition of $V$ into sets of vertices with a common neighborhood. More precisely, let $\sigma_1 : V_L \rightarrow V$ be defined as $\sigma_1(x,y) = y$ and $\sigma_k : V(L^k \Gamma) \rightarrow V$ be defined as $\sigma_k(x_0, \ldots, x_k) = x_k$.

**Corollary 5.** $L^k \Gamma \xrightarrow{\sigma_k} \Gamma$ for $k \geq 1$.

We say that $\sigma_1$ is the *left standard covering map* from $L\Gamma$ onto $\Gamma$.

Denote by $\Gamma^{-1}$ the *inverse* digraph of $\Gamma$, that is, the digraph obtained from $\Gamma$ by inverting the direction of each arc. As a direct consequence of Theorem 4 we have the following Corollary.
Corollary 6. Let $\Gamma = (V,E)$ be a $k$-line digraph of some regular digraph $\Gamma_0$ of degree $r$. Then,

(a) for each $x \in V$ we have $|\Gamma^x(x)| = r^s$ for $1 \leq s \leq k$,
(b) for each $x \in V$ we have $|\Gamma^{-1} \Gamma^k(x)| = r^k$ for $1 \leq s \leq k$,
(c) $\{\Gamma^{-1} \Gamma^x(x) \mid x \in V\}$ is a partition of $V$ into sets of cardinality $r^s$ for $1 \leq s \leq k$.

When $\Gamma$ is a Cayley digraph, Heuchene’s characterization can be stated in the following form.

Corollary 7. Let $\Gamma = \text{Cay}(G,S)$ with $|S| = r$. There is an $r$-regular digraph $\Gamma_0$ (possibly with multiple arcs) such that $\Gamma = L^k \Gamma_0$ if and only if, there is a subgroup $H_k$ of $G$ such that $H_k = x S^k$ for some $x \in S^{-k}$, and $|H_k| = r^k$.

Moreover, if $H_k \cap s H_k s^{-1} = \{1\}$ for each $s \in S$, then $\Gamma_0$ has no multiple arcs.

Proof. Suppose first that $\Gamma = L^k \Gamma_0$ for some digraph $\Gamma_0$. Let $H_k = x S^k$ for some fixed $x \in S^{-k}$. For each $y \in S^{-k}$, we have $1 \in x S^k \cap y S^k = \Gamma^k(x) \cap \Gamma^k(y)$. Then Theorem 4 implies

$$H_k = x S^k = y S^k = S^{-k} S^k$$

and $|H_k| = r^k$. (1)

In particular, $H_k = S^{-k} z$ for each $z \in S^k$ as well. Therefore, $H_k^2 = (S^{-k} x^{-1})(x S^k) = H_k$, so that $H_k$ is a subgroup of $G$.

Conversely, suppose that $x S^k$ is a subgroup $H_k$ of size $|S|^k$ for some $x \in S^{-k}$. Then $\{\Gamma^k(g) \mid g \in G\}$ is the partition of $G$ into the left cosets $g x^{-1} H_k$, $g \in G$. By Theorem 4, $\Gamma$ is a $k$-line digraph.

In order to prove the last part of the statement, suppose that $|\Gamma_k^{k+1}(s^{-k})| = |H_k S| < |S|^{k+1}$. Then there are elements $u, v \in S$ such that $H_k u = H_k v$. By (1) $x H_k = x^{-k} x S^k$, which implies $x H_k \cap H_k x = x H_k \cap H_k y \supset x^{-k+1} S^{k-1} y$. Hence, $\Gamma_0$ has no multiple arcs unless $H_k \cap x H_k x^{-1}$ is nontrivial for some $x \in S$. \qed

Remark 8. Note that a $k$-line digraph is a $k'$-line digraph for $1 \leq k' \leq k$ as well. Let $\Gamma^{-1}$ denote the inverse of $\Gamma$, that is, the digraph obtained from $\Gamma$ by reversing the directions of the arcs. Note that $L^k (\Gamma^{-1})_0 = (L^k \Gamma_0)^{-1}$. Therefore, as a consequence of the above Corollary, Cay($G,S$) is a $k$-line digraph if and only if $K_k = f S^{-k}$ is a subgroup of size $|S|^k = |S|^k$.

3. Uniform factorizations

Let $\Gamma = (V,E)$ be a regular connected $k$-line digraph. We want to introduce $1$-factorizations of $\Gamma$ such that the resulting Cayley covers are also $k$-line digraphs. For each $x \in V$, let $\Gamma(x)$ denote the subdigraph of $\Gamma$ induced by the set of vertices $V_x = \Gamma^k(x) \cup \Gamma^{-1} \Gamma^k(x) \cup \cdots \cup \Gamma^{-r} \Gamma^k(x)$. Note that, according to Corollary 6, $|V_x| = (k+1)r^k$, where $r$ is the degree of $\Gamma$. 
A 1-factorization of $\Gamma$ is $k$-uniform if, for each pair of vertices $x,y \in V$, there is a digraph isomorphism $\phi : \Gamma^k_x \rightarrow \Gamma^k_y$ such that $\phi(x) = y$ and $\phi$ preserves the colors given by $F$, that is, $\phi(z^f) = (\phi(z))^f$ for each $z \in V \setminus \Gamma^k(x)$ and each $f \in F$.

**Theorem 9.** Let $F$ be a 1-factorization of a regular connected $k$-line digraph $\Gamma = (V, E)$. Then, $\Gamma_F$ is a $k$-line digraph if and only if $F$ is $k$-uniform.

**Proof.** Suppose first that $\Gamma = \Gamma_F$ is a $k$-line digraph. By Corollary 6, we have $|V(\Gamma^k_F)| = |V| = (k+1)|F|^k$ for each $f \in V(\Gamma) = G(\Gamma, F) \subseteq \text{Sym}(V)$, $x \in V$. We show that, for each $x \in V$, there is a digraph isomorphism $\phi_x : \Gamma^k_1 \rightarrow \Gamma^k_x$ which preserves the colors given by $F$ and is such that $\phi_1(x) = x$, where 1 denotes the identity permutation of $V$. Therefore, for each $x,y \in V$, $\phi_y \phi_x^{-1} : \Gamma^k_x \rightarrow \Gamma^k_y$ is the desired isomorphism and $F$ is $k$-uniform. Indeed, the isomorphism $\phi_x$ is the restriction to $\Gamma^k_1$ of the natural digraph homomorphism $\phi_x(f) = x^f$ of $\Gamma$ onto $\Gamma$, which clearly satisfies the requirements.

Conversely, suppose that $F$ is a $k$-uniform factorization of $\Gamma$. For each pair $x,y \in V$, denote by $\phi_{xy}$ a digraph isomorphism $\phi_{xy} : \Gamma^k_x \rightarrow \Gamma^k_y$ such that $\phi_{xy}(x) = y$ and $\phi_{xy}$ preserves the colors given by $F$.

Let $K_k = fF^{-k}$ for some $f \in F^k$. By Corollary 7 and Remark 8 it suffices to show that $K_k$ is a subgroup of order $|F|^k$. For each $x \in V$, we have $x^{K_k} = \Gamma^{-k}(x^f)$. Since $\Gamma$ is a $k$-line digraph, $\{x^{K_k}, x \in V\}$ is a partition of $V$ into sets of cardinality $|F|^k$. In particular, $|K_k| = |F|^k$.

Since $x \in x^{K_k}$, for each $x \in V$ then $x^{K_k} = x^{K_k}$. Let $g_1, g_2 \in K_k$. By the above remark, there is $h \in K_k$ such that $x^{g_1g_2} = x^h$. For each $y \in V$, we have

$$y^{g_1g_2} = (\phi_{xy}(x))^{g_1g_2} = \phi_{xy}(x^{g_1g_2}) = \phi_{xy}(x^h) = \phi_{xy}(x)^h = y^h.$$ 

Hence, $g_1g_2 = h$ and thus $K_k^2 = K_k$. Therefore $K_k$ is a subgroup. $\square$

We next give a procedure to construct $k$-uniform factorizations of a $k$-line digraph. Let $\Gamma$ be a connected regular digraph of degree $r$ and $\Gamma^* = L^k \Gamma$. Let $F = \{f_1, \ldots, f_r\}$ be a 1-factorization of $\Gamma$. We identify the vertices of $\Gamma^*$, which correspond to $k$-arcs of $\Gamma$, with the elements in $V \times F \times \cdots \times F$ in the following way. The $k$-arc $(x_0, x_1, \ldots, x_k)$ of $\Gamma$ is identified with the element $(x_0, h_1, \ldots, h_k) \in V \times F^k$ such that $x_i^{h_{i+1}} = x_{i+1}$ for $i = 0, \ldots, k-1$. Since $F$ is a 1-factorization, this is a well-defined one-to-one correspondence between the two sets. With this identification, the vertex $(x_0, h_1, \ldots, h_k)$ of $\Gamma^*$ is adjacent to the vertices $(x_0, h_1, \ldots, h_k, f_j)$, $j = 1, \ldots, r$.

Let $(F_0, \cdot)$ be a group of order $|F|$, and let $\phi : F \rightarrow F_0$ be a bijection. Denote by $\oplus$ the binary operation induced on $F$, that is, $f_1 \oplus f_2 = \phi^{-1}(\phi(f_1) \cdot \phi(f_2))$. Then $(F, \oplus)$ is a group. For $j = 1, \ldots, r$, define $f_j^* : V \times F^k \rightarrow V \times F^k$ as follows:

$$(x_0, h_1, \ldots, h_k)^j = (x_0^h, h_2, \ldots, h_k, h_1 \oplus f_j).$$ 

Thus, $f_j^*$ is a permutation of $V \times F^k$ such that each vertex of $L^k \Gamma$ is sent to one of its neighbors. Moreover, the images of a vertex by $f_i^*$ and by $f_j^*$, $i \neq j$, are two distinct
neighbors of that vertex. In other words, the set \( F^* = \{ f_1^*, \ldots, f_r^* \} \) is a 1-factorization of \( L^k \Gamma \). We say that \( F^* \) is the 1-factorization of \( L^k \Gamma \) induced by \((F, \oplus)\), or simply by \( F \) if the group structure is clear from the context.

**Theorem 10.** Let \( F \) be a 1-factorization of a connected \( r \)-regular digraph \( \Gamma = (V, E) \), and let \((F, \oplus)\) be a group defined as described above. Let \( F^* \) be the 1-factorization of \( \Gamma^* = L^k \Gamma \) induced by \( F \). Then, \( F^* \) is a \( k \)-uniform factorization, and so \( \Gamma^* \) is a \( k \)-line digraph.

**Proof.** We must show that, for each pair of vertices \( x, y \in V^* = V(\Gamma^*) \), there is a digraph isomorphism \( \phi_{xy} : (V^*)_x \rightarrow (V^*)_y \) which preserves the colors given by \( F \).

For each \( i = 0, \ldots, k \) and each element \( z \in (\Gamma^*)^{-i}(\Gamma^*) \) \( \forall (x) = x^{(\Gamma^*)^{-i}}(V^*)_x \), there is a permutation \( g \in (F^*)^i(F^*)^{-i} \) such that \( z = x^g \). We claim that it is unique.

For \( f \in F \), let us write \( \ominus f \) for the inverse of \( f \) in \((F, \oplus)\). As an element of \( V \times F^k \), a vertex \( x \in V^* \) can be written as \( x = (x, h_1, \ldots, h_k) \in V \times F^k \) with \( x \in V \) and \( h_j \in F \), \( 1 \leq j \leq k \). Thus, with this notations,

\[
(x, h_1, \ldots, h_k)^{(\Gamma^*)^{-i}} = (x^{(h_1 \ominus f)^{-1}}, h_k \ominus f, h_1, \ldots, h_{k-1}).
\]

For a fixed \( 0 \leq i \leq k \), a permutation \( g \in (F^*)^i(F^*)^{-i} \) can be written as

\[
g = \left( \prod_{j=1}^{k} g_j^i \right) \left( \prod_{j=1}^{i} (g_k^i)^{-1} \right)
\]

for some \( g_1^i, \ldots, g_k^{i+1} \in F^* \).

For \( 1 \leq i \leq k \) we have

\[
x^g = (x, h_1, \ldots, h_k)^g = (x^u, h_1 \ominus g_1, \ldots, h_k \ominus g_k)^\prod_{j=1}^{i} (g_k^i)^{-1}
\]

where \( u = \prod_{j=1}^{k} h_j \) and \( v = \prod_{j=1}^{i} (h_{k+1-j} \ominus g_{k+1-j} \ominus g_{k-j})^{-1} \). Similarly, if \( i = 0 \) then

\[
x^g = (x^u, h_1 \ominus g_1, \ldots, h_k \ominus g_k).
\]

From the above relations, \( x^g = x^{g^i} \) implies \( g_1 = g_1', \ldots, g_i = g_i' \) and, if \( i \geq 1 \), \( g_{k+1-1} \ominus g_{k+i} = g_{k+1-1} \ominus g_{k+i} \), \( g_k \ominus g_{k+1} = g_k \ominus g_{k+1} \). Therefore, we have \( y^{g^i} = y^{g^i} \) for each other vertex \( y \in V^* \). This proves our claim.

We define

\[
\phi_{xy} : (V^*)_x \rightarrow (V^*)_y
\]

\[
\phi_{xy}(x^g) = y^{g^i}
\]

where \( g \) runs over \( \bigcup_{j=0}^{k} (F^*)^i(F^*)^{-i} \). Since each permutation \( g \) defines univocally an element in \( (V^*)_x \) for each vertex \( x \in V^* \), the map is well defined. Since \( \phi_{xy} \) is invertible then it is bijective. Moreover, by definition, if \( x^g \in (V^*)_x \) \( (\Gamma^*)^i(x) \) then \( \phi_{xy}(x^g) = y^{g^i} = \phi_{xy}(x^g)^i \) for each \( f_i^* \in F^* \), so that \( \phi_{xy} \) preserves the colors given by \( F^* \). This concludes the proof. \( \Box \)
4. Proof of Theorem 1

We are now ready to prove our main result. Let \( \Gamma \) be a connected regular digraph which is a line digraph. It can be easily checked that there is a unique digraph \( \Gamma_0 \), up to isomorphism, such that \( \Gamma = L\Gamma_0 \). Therefore we can write \( \Gamma_0 = L^{-1}\Gamma \). Similarly, we can write \( \Gamma_0 = L^{-k}\Gamma \) when \( \Gamma \) is a \( k \)-line digraph of \( \Gamma_0 \). Following the notation of Theorem 9, we next show that the graph \( L^{-k}\Gamma \) is a cover of \( \Gamma_0 \). Moreover, by Remark 3, \( L^{-k}\Gamma \) is a \( k \)-arc transitive digraph since its \( k \)-line digraph \( \Gamma_0 \) is a Cayley digraph.

Proposition 11. Let \( \Gamma, \Gamma' \) be two regular connected digraphs. Then \( \Gamma \) is a cover of \( \Gamma' \) if and only if \( L\Gamma \) is a cover of \( L\Gamma' \).

Proof. Let \( \Gamma \rightarrow \Gamma' \) be a cover with covering map \( \pi \) and define \( \pi_L : L\Gamma \rightarrow L\Gamma' \) as \( \pi_L(x,y) = (\pi(x), \pi(y)) \), \((x,y) \in L\Gamma \). It can be easily checked that \( \pi_L \) is a covering map.

Reciprocally, let \( \pi_L : L\Gamma \rightarrow L\Gamma' \) be a covering map. Let \( \sigma \) and \( \sigma' \) be the left standard covering maps of \( L\Gamma' \) onto \( \Gamma \) and \( L\Gamma' \) onto \( \Gamma' \), respectively. Define \( \pi : \Gamma \rightarrow \Gamma' \) as \( \pi = \sigma' \pi_L \sigma^{-1} \). Let us show that \( \pi \) is well defined. Let \( u_1 \) and \( u_2 \) be two vertices of \( L\Gamma \) such that \( \sigma(u_1) = \sigma(u_2) \). Thus, \( u_1 \) and \( u_2 \) have a common neighborhood in \( L\Gamma \). Therefore, \( \pi_L(u_1) \) and \( \pi_L(u_2) \) have a common neighborhood in \( L\Gamma' \). Hence \( \sigma' \pi_L(u_1) = \sigma' \pi_L(u_2) \). It is straightforward to check that \( |\pi^{-1}x'| = |\pi_L^{-1}(\sigma'^{-1}x')| \) for each \( x' \in V(\Gamma') \). Thus, \( \pi \) is a covering map.

In both cases, the following diagram is commutative:

\[
\begin{array}{ccc}
L\Gamma & \xrightarrow{\pi_L} & L\Gamma' \\
\downarrow \sigma & & \downarrow \sigma' \\
\Gamma & \xrightarrow{\pi} & \Gamma'
\end{array}
\]

By combining Theorem 10 with the above proposition we can prove our main result.

Proof of Theorem 1. Let \( \Gamma \) be an \( r \)-regular digraph. Without loss of generality, we may assume that \( \Gamma \) is connected. Let \( \Gamma \) be a 1-factorization \( F \) and \( F^* \) the induced factorization in \( L^k\Gamma \) for some group operation \( \oplus \) defined in \( F \). Denote by \( \sigma_k \) the left standard covering map of \( L^k\Gamma \) onto \( \Gamma \) and by \( \pi_k \) the covering map of \( \overline{L^k\Gamma} \) onto \( L^k\Gamma \).

By Theorem 10, \( \overline{L^k\Gamma} \) is a \( k \)-line digraph, say \( \Gamma_k = L^{-k}(\overline{L^k\Gamma}) \). By Proposition 11, as \( \overline{L^k\Gamma} \) is a cover of \( L^k\Gamma \), then \( \Gamma_k \) is a cover of \( \Gamma \). We thus have the following commutative diagram:

\[
\begin{array}{ccc}
\overline{L^k\Gamma} & \xrightarrow{\pi_k} & L^k\Gamma \\
\downarrow \sigma_k & & \downarrow \sigma_k \\
\Gamma_k & \xrightarrow{\pi} & \Gamma
\end{array}
\]

This completes the proof.
5. Explicit constructions

We next give some examples of actual constructions of $k$-arc regular graphs using the above results.

5.1. The De Bruijn digraphs as $k$-line digraphs

In our first example we take $\Gamma = K_{r}^{+}$, the complete symmetric digraph with loops. Let $(F, \oplus)$ be a group of order $r$. Then Cay$(F,F)$ is isomorphic to $K_{r}^{+}$ and $F$ can be seen itself as a 1-factorization of the digraph. The $k$-line digraph $L^{k}K_{r}^{+}$ is the De Bruijn digraph $B(r,k+1)$ of degree $r$ and diameter $k+1$, see for instance [7]. Its vertices can be seen as elements of $F^{k+1}$, where $x = (x_{0}, x_{1}, \ldots, x_{k})$ is adjacent to the vertices $(x_{1}, \ldots, x_{k}, x)$ with $x \in F$. Let $\rho$ be the rotation automorphism of $F^{k+1}$, defined as

$$(x_{0}, x_{1}, \ldots, x_{k})^\rho = (x_{1}, \ldots, x_{k}, x_{0})$$

and let $G$ be the semidirect product $F^{k+1} \rtimes \mathbb{Z}_{k+1}$, where $\rho : \mathbb{Z}_{k+1} \rightarrow \text{Aut}(F^{k+1})$ is given by $\rho(i) = \rho^i$. We define the action of $G$ on $F^{k+1}$ as

$$x^{(\rho^i,y)} = (x^{\rho^i} \oplus y),$$

where $x, y \in F^{k+1}$ and the addition $\oplus$ is componentwise.

It is not difficult to check that the set $F^{*} = \{f^{*}_{1}, \ldots, f^{*}_{r}\}$, where

$$f^{*}_{i} = (0, \ldots, 0, f_{i}, 1) \in F^{k+1} \rtimes \mathbb{Z}_{k+1}, \quad i = 1, \ldots, r,$$

and 0 is the identity element of $(F, \oplus)$, defines a $k$-uniform 1-factorization of $L^{k}K_{r}^{+}$ through the action given by (3) and that the elements generate the whole group $G$. Therefore, Cay$(G,F^{*})$ is a $k$-line digraph and a cover of $L^{k}K_{r}^{+}$. The digraph $\Gamma_{0}$ such that $L^{k}\Gamma_{0} = \text{Cay}(G,F^{*})$ is the quotient digraph of Cay$(G,F^{*})$ modulo $\mathbb{K} = (F^{*})^{k}(F^{*})^{-k}$. This time we obtain a Cayley digraph again, as it is not difficult to check: $\Gamma_{0} = \text{Cay}(\mathbb{Z}_{k+1} \times \mathbb{Z}_{r}, S)$, where $S = \{(1, i), 1 \leq i \leq r\}$. This class of $k$-arc regular digraphs were first considered by Praeger [14] as examples of $k$-arc transitive digraphs. They were also considered from a different point of view in [2], where they were given the name of complete generalized cycles. For applications of these digraphs to the design of permutation networks, see [5].

5.2. The Kautz digraphs as $k$-line digraphs

Let us now take $\Gamma = K_{r+1}$, the complete symmetric digraph without loops, when $r+1$ is a prime number. Consider the 1-factorization of $K_{r+1}$ given by the additive Cayley digraph Cay$(\mathbb{Z}_{r+1}, F)$, where $F = \mathbb{Z}_{r+1} \setminus \{0\}$. Then, the product in the field $\mathbb{Z}_{r+1}$ induces a group operation in $F$. The $k$-line digraph $L^{k}K_{r+1}$ is the Kautz digraph $K(r,k+1)$ of degree $r$ and diameter $k+1$, see for instance [7,11]. Let $F^{*} = \{f^{*}_{1}, \ldots, f^{*}_{r}\}$ be the
1-factorization of $L^kK_{r+1}$ induced by $F$. With the notations of Section 3, we have 
\[(x, h_0, h_1, \ldots, h_{k-1})^F_i = (x + h_0, h_1, \ldots, h_0 f_i), \quad 1 \leq i \leq r\]
for each vertex $(x, h_0, h_1, \ldots, h_{k-1}) \in \mathbb{Z}_{r+1} \times F^k$.

We shall show that the resulting $k$-arc regular cover of $K_{r+1}$ is 
$L^{-k}(L^kF^*) = \text{Cay}((\mathbb{Z}_{r+1})^k \rtimes \mathbb{Z}_k, S)$,
where $S = \{f_i = (0, \ldots, 0, f_i, 1) \in (\mathbb{Z}_{r+1})^k \rtimes \mathbb{Z}_k, 1 \leq i \leq r\}$, $\rho$ denotes the rotation automorphism of $(\mathbb{Z}_{r+1})^k$ defined as $(x_0, x_1, \ldots, x_{k-1})^\rho = (x_1, \ldots, x_{k-1}, x_0)$, and, by abuse of language, we still denote by $\rho : \mathbb{Z}_k \rightarrow \text{Aut}((\mathbb{Z}_{r+1})^k)$ the group homomorphism $\rho(i) = \rho^i$. Fig. 1 shows the resulting digraph when $r + 1 = 3$ and $k = 2$.

We first describe the permutation group of the 1-factorization $F^*$. Let $\Omega$ be the semidirect product $\mathbb{Z}_{r+1} \rtimes F$, where the elements in $F$ act as automorphisms of the additive group $\mathbb{Z}_{r+1}$ in the usual way by multiplication. Let $G' = \Omega^k \rtimes \mathbb{Z}_k$, where $\rho$ is the rotation automorphism of $\Omega^k$, $\rho = (\omega_1, \omega_2, \ldots, \omega_k) = (\omega_2, \ldots, \omega_k, \omega_1)$, and, by abuse of notation, we still denote by $\rho : \mathbb{Z}_k \rightarrow \text{Aut}(\Omega^k)$ the group homomorphism $\rho(i) = \rho^i$. Consider the action of $G'$ on $\Omega^k$ given by 
\[x^{(y,i)} = x^\rho \ast y, \quad x, y \in \Omega^k,\]
where $\ast$ is the componentwise product inherited from $\Omega$. More precisely, if $x = ((x_0, u_0), \ldots, (x_{k-1}, v_{k-1})) \in \Omega^k$ and $y = ((y_0, v_0), \ldots, (y_{k-1}, v_{k-1}), i) \in \Omega^k \rtimes \mathbb{Z}_k$, then 
\[x^{(y,i)} = ((x_i + u_i y_0, u_i v_0), \ldots, (x_{i+k-1} + u_{i+k-1} y_{k-1}, u_{i+k-1} v_{k-1})), \quad (4)\]
where the subindices are taken modulo $k$. 
Lemma 12. The group $G = G(L^k K_{r+1}, F^*)$ of the 1-factorization $F^*$ of $L^k K_{r+1}$ induced by $F$ is isomorphic to $G'$.

Proof. Define $\alpha : \Omega^k \to \mathbb{Z}_{r+1} \times F^k$ by

$$\alpha((x_0, u_0), \ldots, (x_{k-1}, v_{k-1})) = \left( \sum_{j=0}^{k-1} x_j, u_0, \ldots, u_{k-1} \right).$$

By the action given by (4), we clearly have $x^{(x,y)} = (x')^{(x',y)}$ whenever $\alpha(x) = \alpha(x')$. Thus, the partition $P = \{ \alpha^{-1}(x, u_0, \ldots, u_{k-1}), (x, u_0, \ldots, u_{k-1}) \in \mathbb{Z}_{r+1} \times F^k \}$ is a complete block system of $G'$. Moreover, the only element which fixes each block of $P$ is the identity. Indeed, if $(y, i) \in \Omega^k \times \mathbb{Z}_{r+1}$ satisfies $\alpha(x^{(y,i)}) = \alpha(x)$ for all $x \in \Omega^k$, then, using the notation in (4), we have $\sum_{j=0}^{k-1} y_j u_{i+j} = 0$ and $v_j u_{i+j} = u_j$ for all choices of $(u_0, \ldots, u_{k-1}) \in F^k$ and $0 \leq j \leq k - 1$. Therefore, we have $y_0 = \cdots = y_{k-1} = 0$, $v_0 = \cdots = v_{k-1} = 1$ and $i = 0$. Finally, if $f_i = ((0,1), \ldots, (0,1), (1, i), 1)$, $1 \leq i \leq r$, then

$$\alpha(x^{(f)}) = (\alpha(x))^{(f)}, \quad 1 \leq i \leq r.$$ 

Therefore, the group $G$ is isomorphic to the subgroup of $G'$ generated by $f_1, \ldots, f_r$, which is easily seen to be the whole of $G'$. □

In particular, the Cayley cover $L^k T_{F^*}$, is isomorphic to Cay($G'$, $\{f_1, \ldots, f_r\}$).

Now, equality (5.2) follows from the fact that this last digraph is also isomorphic to $L^k$ Cay($((\mathbb{Z}_{r+1})^k \rtimes \mathbb{Z}_r, S)$, where $S = \{ \hat{f}_{1}, \ldots, \hat{f}_{r} \}$. Indeed, the map $\phi : G' \to ((\mathbb{Z}_{r+1})^k \rtimes \mathbb{Z}_r) \times S^k$ defined as $\phi(x, i) = ((x_0, \ldots, x_{k-1}, i), \hat{f}_{u_0}, \ldots, \hat{f}_{u_{k-1}})$, is easily checked to be a digraph isomorphism from Cay($G'$, $\{f_1, \ldots, f_r\}$) to $L^k$ Cay($((\mathbb{Z}_{r+1})^k \rtimes \mathbb{Z}_r, S)$).

6. Conclusions

We have introduced a natural technique to construct $k$-arc transitive digraphs from a given regular digraph $\Gamma$, based on Theorem 9. The construction is combinatorial in nature and it produces a cover of the original digraph $\Gamma$. The explicit construction of such covers depends on two steps. First, the actual computation of the permutation group $G$ of a uniform 1-factorization of $L^k \Gamma$. When the 1-factorization $F^*$ of $L^k \Gamma$ is induced by a 1-factorization $F$ of $\Gamma$ and a group structure on $F$, it is proved in [18] that the resulting permutation group $G$ is similar, as a permutation group, to a subgroup of the wreath product $G_0 \rtimes (F^k \rtimes \mathbb{Z}_r)$, where $G_0$ is the permutation group of $F$, $G_0 = G(\Gamma, F)$. The second step consists in the computation of $L^{-1} L^k T_{F^*}$. This last digraph is the Schreier coset digraph of the Cayley cover $L^k T_{F^*}$, modulo the subgroup.
\((F^*)^k(F^*)^{-k}\). However, an explicit description of such Schreier digraphs is not always easy to obtain.

Most of the examples of \(k\)-arc transitive digraphs for large values of \(k\) that we were able to produce, and also the ones which can be found in the literature, turn out to be homomorphic to cycles. Recently, Cameron et al. [3] gave constructions of infinite highly arc transitive digraphs (digraphs which are \(k\)-arc transitive for every positive integer \(k\)), in most cases which turned out to be homomorphic to an infinite path. The authors raised the question of whether there are infinite highly arc transitive digraphs satisfying some local finiteness condition nonhomomorphic to the infinite path (see [3] for details.) The question was answered in the positive by Evans [6] when the digraph has finite out-degree but infinite in-degree, and in the negative by Praeger [15] when the digraph has finite, but unequal, in-degree and out-degree. The regular case for locally finite digraphs has been answered positively very recently by Malnič et al. [12]. Some of the finite examples obtained by Conder et al. [4] are not homomorphic to cycles. Actually, it is not difficult to see that a finite Cayley digraph is homomorphic to a cycle if and only if there is a normal subgroup \(H\) of the base group such that all generators are contained in one coset of \(H\), see for instance [13]. The examples given in [4] are instances of Cayley graphs of the alternating group or the symmetric group, so that it can be easily checked when they are homomorphic to cycles.

The construction of \(k\)-arc transitive digraphs as quotient digraphs of \(k\)-line digraphs we have given is in some sense universal. If the digraph \(\Gamma\) admits an automorphism group \(G\) which acts regularly on \(k\)-arcs, then \(\Gamma\) is a quotient digraph of the Cayley digraph \(L^k \Gamma\). When \(G\) acts transitively but not regularly on \(k\)-arcs, then \(L^k \Gamma\) is vertex transitive but not a Cayley digraph for \(G\). However, by a theorem by Sabidussi [17], \(L^k \Gamma\) is then a quotient digraph of a Cayley digraph so that \(\Gamma\) is still a quotient digraph of a Cayley digraph which is a \(k\)-line digraph.

A last remark concerns multiple arcs. The main results of the paper can be re-stated for the case where the original digraph has multiple arcs, including multiple loops. We have chosen not to consider this general case for the sake of simplicity. We note that the existence of multiple arcs is only natural as they easily appear in quotient digraphs.

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References