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Note

Regular factors in regular graphs

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Abstract

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Let G be a k-regular, $(k-1)$-edge-connected graph with an even number of vertices, and let m be an integer such that $1 \leqslant m \leqslant k-1$. Then the graph obtained by removing any $k-m$ edges of G, has an m-factor.

All graphs considered are finite. We shall allow graphs to contain multiple edges and we refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. We say that G has a k-factor, if there exists a k-regular spanning subgraph of G. If $S, T \subseteq V(G)$, then $e_G(S, T)$ denotes the number and $E_G(S, T)$ the set of edges having one end-vertex in S and the other in set T. If $S \subseteq V(G)$ then $\omega(G-S)$ denotes the number of components of the graph $G-S$.

Given an ordered pair (D, S) of disjoint subsets of $V(G)$ and a component C of $(G-D)-S$, put $r_G(D, S; C) = e_G(V(C), S) + k\,|V(C)|$. We say that C is odd or even component of $(G-D)-S$ according to whether $r_G(D, S; C)$ is odd or even. The number of odd components of $(G-D)-S$ is denoted by $q_G(D, S; k)$.

Tutte [5] proved the following theorem.

Tutte's k-factor theorem. *A graph G has a k-factor if and only if*

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G-D}(x)) \leqslant k\,|D| \tag{1}$$

for all $D, S \subseteq V(G)$, $D \cap S = \emptyset$.

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He also noted that for any graph G and any positive integer k,

$$q_G(D,S;k)+\sum_{x\in S}(k-d_{G-D}(x))-k\,|D|\equiv k\,|V(G)|\,(\mathrm{mod}\,2). \tag{2}$$

The first results on factors in graphs were obtained by Petersen [2].

Petersen's decomposition theorem. *A graph G can be decomposed into d 2-factors if and only if G is $2d$-regular.*

Petersen's 1-factor theorem. *Every 3-regular graph without cut-edges has a 1-factor.*

Petersen's 1-factor theorem can be generalised in the following way ([1, p. 80, ex.5.3.2]).

Theorem 1. *If G is k-regular, $(k-1)$-edge-connected with an even number of vertices, then G has a 1-factor.*

Plesnik [3] obtained the following stronger result.

Theorem 2. *Let G be k-regular, $(k-1)$-edge-connected and with an even number of vertices. Then the graph, obtained by removing any $k-1$ edges of G, has a 1-factor.*

Some related results can also be found in another paper of Plesnik [4].
We shall prove the following generalization of Theorem 2.

Theorem 3. *Let G be a k-regular, $(k-1)$-edge-connected graph with an even number of vertices, and let m be an iteger such that $1\leq m\leq k-1$. Then the graph, obtained by removing any $k-m$ edges of G, has an m-factor.*

Lemma 4. *Let G be a $(k-1)$-edge-connected graph and let D, S be two disjoint subsets of $V(G)$. Remove $k-m$ edges of G and let G_1 be the remaining graph. Then*:

(i) $$\sum_{x\in S}d_{G_1-D}(x)\geq\sum_{x\in S}d_G(x)-\sum_{x\in D}d_G(x)-2(k-m),$$

(ii) $$2\sum_{x\in S}d_{G_1-D}(x)\geq(k-1)(\omega(G_1-D)-S)+\sum_{x\in S}d_G(x)-\sum_{x\in D}d_G(x)-2(k-m).$$

Proof. (i) $\sum_{x\in D}d_G(x)\geq e_G(D,S)=\sum_{x\in S}d_G(x)-\sum_{x\in S}d_{G-D}(x)$. But $\sum_{x\in S}d_{G-D}(x)\leq\sum_{x\in S}d_{G_1-D}(x)+2(k-m)$ because every edge that we delete, contributes at most two, to the summation $\sum_{x\in S} d_{G-D}(x)$. So

$$\sum_{x\in D} d_G(x) \geq \sum_{x\in S} d_G(x) - \sum_{x\in S} d_{G_1-D}(x) - 2(k-m).$$

(Note that we did not use the hypothesis that G is $(k-1)$-edge-connected.)

(ii) Define $X = E(G)\setminus E(G_1)$ and $H = (G_1 - D) - S$. Let $C_1, C_2, \ldots, C_z$ be the components of H and $a_i(b_i)$ the number of edges of X joining C_i to $D \cup S$ (to the other C_j's). Clearly

$$e_G(D, S) = \sum_{x\in S} d_G(x) - \sum_{x\in S} d_{G-D}(x). \tag{3}$$

But

$$\sum_{x\in S} d_{G-D}(x) = \sum_{x\in S} d_{G_1-D}(x) + 2\,|E_G(S, S)\cap X| + |E_G(S, V(H))\cap X| \tag{4}$$

and

$$|E_G(S, V(H))\cap X| \leq \sum_{i=1}^{z} a_i. \tag{5}$$

Further, since $|X| = k-m$, $\sum_{i=1}^{z} a_i + (\frac{1}{2})\sum_{i=1}^{z} b_i + |E_G(S,S)\cap X| \leq k-m$. So

$$2\,|E_G(S, S)\cap X| \leq 2k - 2m - 2\sum_{i=1}^{z} a_i - \sum_{i=1}^{z} b_i \tag{6}$$

Substituting (4), (5) and (6) in (3), we have

$$e_G(D, S) \geq \sum_{x\in S} d_G(x) - \sum_{x\in S} d_{G_1-D}(x) - 2k + 2m + \sum_{i=1}^{z} a_i + \sum_{i=1}^{z} b_i. \tag{7}$$

Also

$$\sum_{x\in D} d_G(x) \geq e_G(D, S) + \sum_{i=1}^{z} e_{G_1}(V(C_i), D). \tag{8}$$

Since G is $(k-1)$-edge-connected we have $\sum_{i=1}^{z} e_G(V(C_i), V(G - V(C_i))) \geq (k-1)z$. But

$$\sum_{i=1}^{z} e_G(V(C_i), V(G - V(C_i)))$$
$$\leq \sum_{i=1}^{z} a_i + \sum_{i=1}^{z} b_i + \sum_{x\in S} d_{G_1-D}(x) + \sum_{i=1}^{z} e_{G_1}(V(C_i), D).$$

So

$$\sum_{i=1}^{z} a_i + \sum_{i=1}^{z} b_i + \sum_{x\in S} d_{G_1-D}(x) + \sum_{i=1}^{z} e_{G_1}(V(C_i), D) \geq (k-1)z.$$

Thus $\sum_{i=1}^{z} e_{G_1}(V(C_i), D) \geq (k-1)z - \sum_{i=1}^{z} a_i - \sum_{i=1}^{z} b_i - \sum_{x\in S} d_{G_1-D}(x)$ and so (8) implies,

$$\sum_{x\in D} d_G(x) - (k-1)z + \sum_{i=1}^{z} a_i + \sum_{i=1}^{z} b_i + \sum_{x\in S} d_{G_1-D}(x) \geq e_G(D, S). \tag{9}$$

From (7) and (9),

$$\sum_{x\in D} d_G(x) - (k-1)z + 2\sum_{x\in S} d_{G_1-D}(x) \geq \sum_{x\in S} d_G(x) - 2(k-m).$$

Thus

$$2\sum_{x\in S} d_{G_1-D}(x)$$
$$\geq (k-1)(\omega((G_1-D)-S) + \sum_{x\in S} d_G(x) - \sum_{x\in D} d_G(x) - 2(k-m). \quad \square$$

Lemma 5. *Let G be a graph having a 1-factor and m be an integer. If there exists $D, S \subseteq V(G)$, $D \cap S = \emptyset$, such that*

$$q_G(D, S; m) + \sum_{x\in S} (m - d_{G-D}(x)) > m\,|D|, \tag{10}$$

then $|S| \geq |D| + 1$.

Proof. Suppose that m is even and let C be a component of $(G-D)-S$. Then by definition, if C is an odd component, $e(V(C), S)$ is odd. Thus $e(V(C), S) \geq 1$ and so $q_G(D, S; m) \leq \sum_{x\in S} d_{G-D}(x)$. Therefore using (10) we have $|S| \geq |D| + 1$.

Now suppose that m is odd. From (10)

$$q_G(D, S; m) + \sum_{x\in S} (1 - d_{G-D}(x)) > |D| + (m-1)(|D| - |S|) \tag{11}$$

By hypothesis G has a 1-factor. Thus by Tutte's theorem,

$$q_G(D, S; 1) + \sum_{x\in S} (1 - d_{G-D}(x)) \leq |D| \tag{12}$$

Since m is odd, $q_G(D, S; m) = q_G(D, S; 1)$. Thus (11) and (12) imply that $|D| < |S|$. So Lemma 5 holds. $\square$

Proof of Theorem 3. Let X be the set of edges of G that we delete and define $G_1 = G - X$. If $m = k-1$ the theorem holds because Theorem 2 implies that G possesses a 1-factor containing any given edge and this in turn implies that G possesses a $(k-1)$-factor avoiding a given edge. Hence we may assume that $m \leq k-2$.

Suppose that G_1 does not have an m-factor. Then by Tutte's theorem and (2), there exist $D, S \subseteq V(G)$, $D \cap S = \emptyset$ such that

$$q_{G_1}(D, S; m) + \sum_{x\in S} (m - d_{G_1-D}(x)) \geq m\,|D| + 2. \tag{13}$$

So

$$\omega((G_1-D)-S) + m(|S| - |D|) - 2 \geq \sum_{x\in S} d_{G_1-D}(x). \tag{14}$$

Thus using Lemma 4(i) and since G is k-regular, (14) implies

$$k\,|D| \geq k\,|S| - \omega((G_1 - D) - S) - m(|S| - |D|) + 2 - 2(k - m).$$

Thus

$$\omega((G_1 - D) - S) \geq (k - m)(|S| - |D| - 2) + 2. \tag{15}$$

Also using Lemma 4(ii), (14) implies

$$2(k - m) - 4 \geq (k - 2m)(|S| - |D|) + (k - 3)\omega((G_1 - D) - S). \tag{16}$$

By Theorem 2, G_1 has a 1-factor and since (13) holds using Lemma 5, $|S| \geq |D| + 1$. Also since $1 \leq m \leq k - 2$ we have $k \geq 3$.

Suppose that $|S| = |D| + 1$. Then clearly $\omega((G_1 - D) - S) \geq 1$ since $|V(G)|$ is even. So (16) implies $2(k - m) - 4 \geq (k - 2m) + (k - 3)$ which is a contradiction.

If $|S| \geq |D| + 2$, (15) implies

$$\omega((G_1 - D) - S) \geq (k - m - 1)(|S| - |D| - 2) + (|S| - |D| - 2) + 2$$

and so $\omega((G_1 - D) - S) \geq |S| - |D|$. Hence from (16) we have $2(k - m) - 4 \geq (k - 2m + k - 3)(|S| - |D|)$ which implies that $-4 \geq -2$ since $m \leq k - 2$ and $|S| \geq |D| + 2$. This contradiction completes the proof of the theorem. □

Petersen's decomposition theorem has the following corollary which is similar to Theorem 3.

Corollary 6. *Let G be a k-regular graph where k is an even integer. Then for every subset M of $E(G)$ of cardinality $k/2 - 1$, the graph $G - M$ has a 2-factor.*

Proof. By Petersen's decomposition theorem G can be decomposed into $k/2$ 2-factors. So there exists at least one 2-factor, say H, such that $E(H) \cap M = \emptyset$. Hence the corollary follows. □

We next examine if Theorem 3 is best possible. The condition that $|V(G)|$ must be even is necessary, because there exists a graph G on an odd number of vertices which is k-regular, k-edge-connected and has a subset M of $E(G)$, where $|M| = k/2$, such that $G - M$ does not have a 2-factor. In other words we cannot get a better result than the one which follows from Corollary 6.

We form G as follows. We start from a complete bipartite graph T with bipartition (X, Y), where $X = \{u_1, \ldots, u_k\}$ and $Y = \{v_1, \ldots, v_k\}$. Remove the edges $(u_1, v_1), \ldots, (u_{k/2}, v_{k/2})$ from T and add a new vertex w. Join w to $u_1, u_2, \ldots, u_{k/2}, v_1, \ldots, v_{k/2}$. Clearly the new graph G is k-regular and k-edge-connected. Now define $R = E(w, Y)$, and let $G_1 = G - R$, $D = X$ and $S = Y \cup \{w\}$. Then

$$q_{G_1}(D, S; 2) + \sum_{x \in S} (2 - d_{G_1 - D}(x)) = 2\,|D| + 2,$$

since $q_{G_1}(D, S; 2) = 0$, $\sum_{x \in S} (2 - d_{G_1 - D}(x)) = 2(k + 1)$, and $|D| = k$. Thus by Tutte's theorem, G_1 does not have a 2-factor.

The condition that the graph must be $(k-1)$-edge-connected is also necessary. Let k be an even integer and m be an integer such that $m \leq k-1$. We will describe a graph G which has an even number of vertices, is k-regular, $(k-2)$-edge-connected and has a subset M of $E(G)$, where $|M| = k-m$, such that $G-M$ does not have an m-factor. We form G as follows. We start again from a complete bipartite graph T with bipartition (X, Y) where $X = \{u_1, \ldots, u_k\}$, $Y = \{v_1, \ldots, v_k\}$. Remove the edges (u_1, v_1), $(u_2, v_2), \ldots, (u_{k-m}, v_{k-m})$ from T and add a vertex w and a graph H with odd number of vertices which is $(k-1)$-edge-connected having $k-2$ vertices of degree $k-1$ and all the other vertices are of degree k. Let $r_1, r_2, \ldots, r_{k-2}$ be the vertices of H which have degree $k-1$. Join w, to $v_1, \ldots, v_{k-m}$ to $r_1, \ldots, r_{m-1}$ and to u_1. Also join $r_m, \ldots, r_{k-2}$ to $u_2, \ldots, u_{k-m}$ respectively. The resulting graph G is k-regular, $(k-2)$-edge-connected and has an even number of vertices. Let $X = D$, $S = Y \cup \{w\}$, $M = E_G(w, Y)$ and $G_1 = G - M$. Now $m\,|V(H)| + e_{G_1}(V(H), S)$ is an odd number since $|V(H)|$ is odd and $e_{G_1}(V(H), S) = m-1$. Thus $q_{G_1}(D, S; m) = 1$. Also $|D| = k$ and

$$\sum_{x \in S} (m - d_{G_1 - D}(x)) = mk + 1.$$

So

$$q_{G_1}(D, S; m) + \sum_{x \in S} (m - d_{G_1 - D}(x)) > m\,|D|.$$

Thus G_1 does not have an m-factor.

We should also point out that Theorem 3 has the following corollary.

Corollary 7. *Let G be a k-regular, $(k-1)$-edge-connected graph with an even number of vertices, and let m be an integer such that $1 \leq m \leq k-1$. Then any m edges of G are contained in an m-factor of G.*

Proof. Let M be a set of m edges of G. We define $G_1 = G - M$. Then by Theorem 3, G_1 has a $(k-m)$-factor, say F. Clearly $G - E(F)$ is an m-regular graph which contains all elements of M and so the corollary holds. □

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