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Integral sum graphs from identification[1]

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Abstract

The idea of integral sum graphs was introduced by Harary (1994). A graph G is said to be an *integral sum graph* if its nodes can be given a labeling f with distinct integers, so that for any two distinct nodes u and v of G, uv is an edge of G if and only if $f(u)+f(v)=f(w)$ for some node w in G. A tree is said to be a *generalized star* if it can be obtained from a star by extending each edge to a path. A node of a tree T is said to be a *fork* of T if its degree is not equal to two. In this paper, we first introduce some methods of identification on constructing new connected integral sum graphs from given integral sum graphs. Applying the methods of identification, we then prove that the generalized stars and the trees with all forks at least distance 4 apart are integral sum graphs.

1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of [3] unless otherwise specified.

The idea of integral sum graphs was introduced by Harary [5]. A graph G is said to be an *integral sum graph* if its nodes can be given a labeling f with distinct integers, so that for any two distinct nodes u and v of G, uv is an edge of G if and only if $f(u)+f(v)=f(w)$ for some node w in G. (Furthermore, such a labeling f is then called an *integral sum labeling* of G.) If there is an integral sum labeling f of G with $f(x)>0$ for all nodes x in G, then G is said to be a *sum graph*. In fact, the concept of sum graphs was introduced earlier in Harary [4], and much work has been devoted to the sum graphs (see e.g., [2, 5–7]).

It is easily seen that a non-trivial graph G (i.e., G has more than one node) is disconnected if G is a sum graph. However, many integral sum graphs are connected. Harary [5] found that all paths and stars are integral sum graphs and conjectured that every integral sum tree is a caterpillar. This conjecture was recently disproved

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in [1], where an infinite number of integral sum trees which are not caterpillars were presented. In the same paper [1], we proved another conjecture of Harary [5] that gives the integral sum number of a complete graph K_n (i.e., the smallest non-negative integer m such that $K_n \cup mK_1$ is an integral sum graph). In the present paper, we shall present some methods on constructing new connected integral sum graphs from given integral sum graphs by identification. Applying the methods of identification, we then prove that the generalized stars and the trees with all forks at least distance 4 apart are integral sum graphs. (A *fork* of a tree T is a node of T with degree not equal to two. A *generalized star* is a tree which can be obtained from a star by extending each edge to a path. An equivalent definition for a generalized star is given in the next section.)

2. Preliminaries

Let G_1 and G_2 be two graphs. Suppose $r_1 \in V(G_1)$ is a fixed node of G_1, called the *root* of G_1, and $r_2 \in V(G_2)$ is the root of G_2. We let $(G,r) \equiv (G_1,r_1) \bowtie (G_2,r_2)$ denote the graph G with *root* r, which is obtained from G_1 and G_2 by identifying r_1 and r_2 as one node r. When we do not consider the node r as the root of the obtained graph, we simply denote the graph as $G=(G_1,r_1) \bowtie (G_2,r_2)$. It is clear that $V(G)=(V(G_1)-\{r_1\})\cup(V(G_2)-\{r_2\})\cup\{r\}$ and $E(G)=E(G_1)\cup E(G_2)$. For the sake of convenience, we may consider G_1 and G_2 as subgraphs of G and consider r, r_1 and r_2 as the same node. It is also clear that the operation of identification $\bowtie$ is commutative and associative.

Now we may give an equivalent definition for generalized stars. A graph G is said to be a *generalized star* with *root* r if $(G,r)=(G_1,r_1) \bowtie (G_2,r_2) \bowtie \cdots \bowtie (G_k,r_k)$ where $k \geqslant 1$ and each G_i is a path whose root r_i is one of its end-nodes. In the special case where each G_i is a path with one edge, the generalized star is the usual k-star, that is, the complete bipartite graph $K_{1,k}$.

Let $G=(V(G),E(G))$ be a graph with node set $V(G)$ and edge set $E(G)$. Let $\overline{G}$ denote the complement of G. Assume that f is a labeling of $V(G)$ with distinct integers. An edge $uv \in E(G)\cup E(\overline{G})$ is said to be *f-proper* if $f(u)+f(v)=f(w)$ for some $w \in V(G)$. Then we immediately have the following fact.

Fact 1. *The labeling f is an integral sum labeling of G if and only if all edges of G are f-proper and all edges of $\overline{G}$ are not f-proper.*

For an integral sum labeling f of G, the following facts can also be easily seen:

Fact 2. *For any nonzero integer m, $m\cdot(f(x))$ also gives an integral sum labeling of G. (We will denote this labeling as mf.)*

Fact 3. *Suppose that G is a nontrivial graph. Then $f(x)\neq 0$ for every node x of G if and only if the maximum degree $\Delta(G)<|V(G)|-1$.*

The following definitions will be needed in Section 3.

A path in a tree T is said to be *forkless* in T if each inner-node of the path is not a fork of T. A forkless path P in a tree T is said to be a *maximal forkless path* in T if P is not a subgraph of a longer forkless path in T. (That is, both end nodes of a maximal forkless path P in T are forks of T.)

Recall that a *caterpillar* T is a tree in which the removal of all end-nodes results in a path (called the *spine* of T). As special examples, the paths and the stars are caterpillars. Let T be a caterpillar with root r. Then r is said to be the *neck* of T if r is an end-node of the spine of T. And r is said to be the *head* of T if r is an end-node of T which is adjacent to an end-node of the spine of T.

3. Main results

Theorem 1. *Let (G_1, r_1) be a caterpillar whose root r_1 is its neck or head. Let (G_2, r_2) be a connected graph with root r_2 which has an integral sum labeling φ such that*

(i) $\varphi(x) \neq 0$ *for every node* $x \in V(G_2)$, *and*

(ii) $|\varphi(r_2)| > |\varphi(x)|$ *for any* $x \in V(G_2) - \{r_2\}$.

Then $G = (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph.

It should be noted that when G_2 is a nontrivial graph, condition (i) is equivalent to the structural condition: $\Delta(G_2) < |V(G_2)| - 1$ (see Fact 3).

To prove Theorem 1, we need the following

Lemma 1. *Let (G_1, r_1) be a star whose root r_1 is its center (i.e., r_1 is adjacent to all other nodes of G_1), and let (G_2, r_2) be the same as in Theorem 1. Then $G = (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph.*

Proof. Clearly we may assume that G_2 is nontrivial. Since G_2 is connected, it is easy to see that $|V(G_2)| > 3$ from condition (i) and Fact 3.

Without loss of generality, we may assume that $\varphi(r_2) > 0$. (Otherwise, by Fact 2 we may consider a new integral sum labeling $(-1)\varphi$ instead.) Let $v \in V(G_2)$ such that $\varphi(v) < \varphi(x)$ for any other $x \in V(G_2)$. Then we must have $v \neq r_2$ and $\varphi(v) < 0$. (Otherwise, we have $\varphi(v) > 0$. Then $\varphi(x) > 0$ for all $x \in V(G_2)$. That is, φ is a sum labeling of G_2, which contradicts the fact that any nontrivial connected graph is not a sum graph.)

Let $V(G_1) = \{r_1\} \cup \{a_1, a_2, \ldots, a_q\}$. We extend the labeling φ of G_2 to a labeling f of G as

$$f(x) = \varphi(x) \quad \text{for } x \in V(G_2),$$

and

$$f(a_i) = \varphi(v) - i\varphi(r_2) \quad \text{for } i = 1, 2, \ldots, q.$$

It is clear that f is a labeling of $V(G)$ with distinct integers. So, by Fact 1, we only need to show that every edge in $E(G)$ is f-proper and any edge in $E(\overline{G})$ is not f-proper.

Note that $E(G)=E(G_1)\cup E(G_2)$. It is easily seen that every $e\in E(G_2)$ is f-proper since $f|_{V(G_2)}=\varphi$ is an integral sum labeling of G_2. For $e\in E(G_1)$, we let $e=r_1a_i$ where $1\leqslant i\leqslant q$. Then

$$\begin{aligned} f(r_1)+f(a_i) &= \varphi(r_2)+(\varphi(v)-i\varphi(r_2)) \\ &= \varphi(v)-(i-1)\varphi(r_2) \\ &= \begin{cases} f(v) & \text{when } i=1 \\ f(a_{i-1}) & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, we have shown that every edge in $E(G)$ is f-proper.

Now we shall show that any $e\in E(\overline{G})$ is not f-proper by contradiction. Otherwise, suppose that $ab\in E(\overline{G})$ is f-proper, i.e., $f(a)+f(b)=f(c)$ for some $c\in V(G)$. Without loss of generality, we may distinguish the following three cases.

Case 1: $\{a,b\}\subset V(G_1)-\{r_1\}$. Then there are distinct $i,j\in\{1,2,\ldots,q\}$ such that $f(a)=\varphi(v)-i\varphi(r_2)$ and $f(b)=\varphi(v)-j\varphi(r_2)$.

If $c\in V(G_1)$, then $c=r_1$ or a_k for some $1\leqslant k\leqslant q$. So we have

$$\begin{aligned} (\varphi(v)-i\varphi(r_2))+(\varphi(v)-j\varphi(r_2)) &= f(a)+f(b)=f(c) \\ &= \varphi(r_2) \text{ or } \varphi(v)-k\varphi(r_2). \end{aligned}$$

It follows that $\varphi(v)=((i+j+1)/2)\varphi(r_2)$ or $(i+j-k)\varphi(r_2)$. Note that $\varphi(v)\neq 0$. Then we see that $|\varphi(v)|\geqslant|\varphi(r_2)|$. It contradicts the given condition (ii) for G_2.

If $c\in V(G)-V(G_1)$, then $c\in V(G_2)-\{r_2\}$, and

$$(\varphi(v)-i\varphi(r_2))+(\varphi(v)-j\varphi(r_2))=\varphi(c).$$

So, $|\varphi(c)|=|2\varphi(v)-(i+j)\varphi(r_2)|\geqslant|(i+j)\varphi(r_2)|-2|\varphi(v)|>|\varphi(r_2)|$. It also contradicts the given condition (ii) for G_2.

Case 2: $\{a,b\}\subset V(G_2)$. Then $ab\in E(\overline{G_2})$. So, ab is not φ-proper since φ is an integral sum labeling of G_2. By definition, $f|_{V(G_2)}=\varphi$, then we must have $c\notin V(G_2)$, i.e., $c=a_i$ for some $1\leqslant i\leqslant q$. It follows that

$$\varphi(a)+\varphi(b)=f(a)+f(b)=f(c)=f(a_i)=\varphi(v)-i\varphi(r_2).$$

Since $\varphi(a)+\varphi(b)>2\varphi(v)$, we have

$$\varphi(v)-i\varphi(r_2)>2\varphi(v), \qquad -i\varphi(r_2)>\varphi(v), \qquad 0<i\varphi(r_2)<-\varphi(v),$$

$$|\varphi(r_2)|\leqslant|i\varphi(r_2)|<|\varphi(v)|.$$

It contradicts the given condition (ii) for G_2.

Case 3: $a\in V(G_1)-\{r_1\}$ and $b\in V(G_2)$. Then $f(a)=\varphi(v)-i\varphi(r_2)$ for some $1\leqslant i\leqslant q$.

If $c \in V(G_2)$, then

$$\varphi(c) = f(c) = f(a) + f(b) = (\varphi(v) - i\varphi(r_2)) + \varphi(b)$$
$$< \varphi(v) - i\varphi(r_2) + \varphi(r_2) = \varphi(v) - (i-1)\varphi(r_2) \leqslant \varphi(v).$$

It contradicts the assumption that $\varphi(v) < \varphi(x)$ for any other $x \in V(G_2)$. Thus we have $c \in V(G) - V(G_2)$, i.e., $c \in V(G_1) - \{r_1\}$. So, $f(c) = \varphi(v) - j\varphi(r_2)$ for some $j \neq i$, $1 \leqslant j \leqslant q$. It follows that $\varphi(b) = f(b) = f(c) - f(a) = (i-j)\varphi(r_2)$, which implies that $|\varphi(b)| \geqslant |\varphi(r_2)|$, contradicting the given condition (ii) for G_2.

This completes the proof for Lemma 1. □

Remark. The labeling f given for $G = (G_1, r_1) \bowtie (G_2, r_2)$ in Lemma 1 satisfies the condition that $|f(a_q)| > |f(x)|$ for any $x \in V(G) - \{a_q\}$. Taking a_q as the root of G, we may do the identification indicated in Lemma 1 for any star and (G, a_q). It is easily seen that this process can be repeated again and again.

Note that a path with n edges can be obtained from n stars $K_{1,1}$ by doing the identification one by one. Then we immediately have the following

Corollary 1. *Let G_1 be a path and let r_1 and r_1' be the end-nodes of G_1. Let (G_2, r_2) be the same as in Lemma 1. Then $G = (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph. Furthermore, G has an integral sum labeling f such that $|f(r_1')| > |f(x)|$ for any $x \in V(G) - \{r_1'\}$.*

Now the proof for Theorem 1 goes as follows.

Proof of Theorem 1. Note that the edge from head to neck in G_1 can be adjoined to G_2 by Corollary 1. So we only need to consider the case when r_1 is the neck of G_1. If G_1 is a star, the result is already given in Lemma 1. Thus, we may assume that G_1 is not a star, and write its spine as $a_1 a_2 \cdots a_n$ with $n \geqslant 2$ and $a_1 = r_1$. It is not difficult to see that G_1 can be obtained from n stars $S_1, S_2, \ldots, S_n$ (with a_i as the center of S_i), by identifying one end-node of S_i with the center a_{i+1} of S_{i+1} for $i = 1, 2, \ldots, n-1$. Then, the result immediately follows from Lemma 1 and the Remark right after the proof of Lemma 1.

This completes the proof for Theorem 1. □

For any given path, an integral sum labeling was already given in Harary [5]. Now we shall give new integral sum labelings for a path with more than three edges in the following two corollaries, which will be needed later. For convenience, we always denote a path with length n as $P_n = a_0 a_1 a_2 \cdots a_n$, where the nodes are listed in the natural order, i.e., a_i is adjacent to a_{i+1} for $i = 0, 1, \ldots, n-1$.

Corollary 2. *Let* $f(a_0)=1$, $f(a_1)=t$, $f(a_2)=1+t$, $f(a_3)=-t$, *and* $f(a_k)=f(a_{k-2})-f(a_{k-1})$ *for all* $k\geqslant 4$, *where* t *is an integer greater than* 1. *Then*

(i) $|f(a_n)|>|f(a_{n-1})|$ *and* $f(a_n)\cdot f(a_{n-1})<0$ *for any* $n\geqslant 4$;

(ii) f *gives an integral sum labeling for* $P_n(n\geqslant 4)$.

Proof. By induction on n.

The proof for (i) is straightforward. For (ii), we note that $P_{n+1}=(P_n,a_n)\bowtie(a_na_{n+1},a_n)$ where a_na_{n+1} is a star ($K_{1,1}$). Then, the proof is easily completed by using (i) and Lemma 1 with the labeling defined in its proof. □

Corollary 3. *Let* $f(a_0)=1-t$, $f(a_1)=t$, $f(a_2)=1$, $f(a_3)=-t$, *and* $f(a_k)=f(a_{k-2})-f(a_{k-1})$ *for all* $k\geqslant 4$, *where* t *is an integer greater than* 2. *Then* f *gives an integral sum labeling for* $P_n(n\geqslant 4)$.

The proof for Corollary 3 is omitted here since it is almost the same as the proof for Corollary 2.

Theorem 2. *Let* $(G_1,r_1)=a_0a_1a_2\cdots a_n$ *be a path with length* $n\geqslant 4$ *and root* $r_1=a_0$. *Let* (G_2,r_2) *be a connected graph with root* r_2, *which satisfies the following*:

(1) *the maximum degree* $\Delta(G_2)<|V(G_2)|-1$, *and*

(2) *there is an integral sum labeling* ψ *of* G_2 *such that*

$$\psi(x)\neq-\psi(r_2)\quad \text{for any } x\in V(G_2)-\{r_2\}.$$

Then $G=(G_1,r_1)\bowtie(G_2,r_2)$ *is an integral sum graph.*

It should be noted that the conditions (1) and (2) for (G_2,r_2) do not really restrict much. For example, all the rooted trees depicted in Figs. 1–4 (where the black nodes are the roots) satisfy these conditions.

In order to prove Theorem 2, we first give the following lemma.

Lemma 2. *Let* (G_i,r_i) *be a graph with root* r_i *and* φ_i *be its integral sum labeling*, $i=1,2$. *Suppose that*

(i) $\varphi_i(x)\neq 0$, *for any* $x\in V(G_i)-\{r_i\}$, $i=1,2$;

(ii) $\varphi_1(x)=\varphi_2(y)$ *if and only if* $x=r_1$ *and* $y=r_2$;

(iii) $\varphi_1(a)\pm\varphi_1(b)\neq\varphi_2(x)$ *for all distinct* $a,b\in V(G_1)$ *and* $x\in V(G_2)-\{r_2\}$; *and*

(iv) $\varphi_2(x)\pm\varphi_2(y)\neq\varphi_1(a)$ *for all distinct* $x,y\in V(G_2)$ *and* $a\in V(G_1)-\{r_1\}$.

Then $G=(G_1,r_1)\bowtie(G_2,r_2)$ *is an integral sum graph.*

Proof. We define a labeling of G as follows:

$$f(x)=\begin{cases}\varphi_1(x) & \text{if } x\in V(G_1),\\ \varphi_2(x) & \text{if } x\in V(G_2).\end{cases}$$

It is clearly seen from (ii) that f is a labeling of $V(G)$ with distinct integers. So, by Fact 1, we only need to show that every edge in $E(G)$ is f-proper and any edge in $E(\overline{G})$ is not f-proper.

Since $E(G)=E(G_1)\cup E(G_2)$, $f|_{V(G_1)}=\varphi_1$ and $f|_{V(G_2)}=\varphi_2$, we immediately see that every $e\in E(G)$ is f-proper.

Now we shall show that any $e\in E(\overline{G})$ is not f-proper by contradiction. Otherwise, suppose that $uv\in E(\overline{G})$ is f-proper, i.e., $f(u)+f(v)=f(w)$ for some $w\in V(G)$. Without loss of generality, we may distinguish the following three cases.

Case 1: $\{u,v\}\subset V(G_1)$.

Then $uv\in E(\overline{G_1})$. So, uv is not φ_1-proper. It follows that $w\notin V(G_1)$, i.e., $w\in V(G_2)-\{r_2\}$. Thus we have $\varphi_1(u)+\varphi_1(v)=\varphi_2(w)$. It contradicts (iii).

Case 2: $\{u,v\}\subset V(G_2)$.

As in Case 1, we have $\varphi_2(u)+\varphi_2(v)=\varphi_1(w)$ with $w\in V(G_1)-\{r_1\}$. It contradicts (iv).

Case 3: $u\in V(G_1)-\{r_1\}$ and $v\in V(G_2)-\{r_2\}$.

If $w\in V(G_1)$, then $\varphi_1(u)+\varphi_2(v)=\varphi_1(w)$, i.e., $\varphi_1(w)-\varphi_1(u)=\varphi_2(v)$. It contradicts (iii), since we can easily see $u\neq w$ from the given condition (i).

If $w\in V(G_2)$, we can get a contradiction similarly.

Therefore, Lemma 2 is proved by contradiction. □

Now we may give a proof for Theorem 2 as follows.

Proof for Theorem 2. By Corollary 2, there is an integral sum labeling φ of G_1 such that $\varphi(a_0)=1$, $\varphi(a_1)=t$, $\varphi(a_2)=1+t$, $\varphi(a_3)=-t$, and $\varphi(a_k)=\varphi(a_{k-2})-\varphi(a_{k-1})$ for all $k\geqslant 4$, where we assume that $t>2\cdot\max\{|\psi(x)|\colon x\in V(G_2)\}$.

The given condition (1) implies that G_2 is nontrivial and $\psi(x)\neq 0$ for all $x\in V(G_2)$. Now, by Fact 2 in Section 2, we can define an integral sum labeling of G_1 as $\varphi_1=m\varphi$, where $m=\psi(r_2)$. For the sake of convenience, we also use φ_2 to denote ψ. Then, to show G is an integral sum graph, we only need to show that φ_1 and φ_2 satisfy the conditions (i)–(iv) in Lemma 2. Clearly, (i) is satisfied. Note that $\varphi_1(r_1)=m\varphi(a_0)=m=\varphi_2(r_2)$. It is then easily seen that (ii) is satisfied, since Corollary 2(i) implies that $|\varphi_1(x)|\geqslant t>|\varphi_2(y)|$ for any $x\in V(G_1)-\{r_1\}$ and $y\in V(G_2)$. It is also obvious that (iv) is satisfied, since $|\varphi_2(x)\pm\varphi_2(y)|<t\leqslant|\varphi_1(a)|$ for all distinct $x,y\in V(G_2)$ and $a\in V(G_1)-\{r_1\}$. So we only need to consider (iii). By direct verification, we can see that

$$\varphi_1(a)\pm\varphi_1(b)=0,\ \pm m,$$

or

$$|\varphi_1(a)\pm\varphi_1(b)|\geqslant|m|(t-1)>|\varphi_2(x)|,$$

for all distinct $a,b\in V(G_1)$ and $x\in V(G_2)-\{r_2\}$.

By the given condition (2), we have $\varphi_2(x)\neq 0,\pm m$, for any $x\in V(G_2)-\{r_2\}$. Then we immediately see that (iii) is satisfied.

This completes the proof for Theorem 2. □

Applying Theorem 2, we can get the following

Theorem 3. *Any tree T with all forks at least distance 4 apart is an integral sum graph.*

Proof. Let T be a tree with all forks at least distance 4 apart, and let $P_1, P_2, \ldots, P_n$ be the maximal forkless paths of T. It is easily seen that T can be obtained from these paths by identifying them one by one as follows:

$$T_1 = (P_1, a_1), \qquad T_2 = (P_2, a_2) \bowtie (T_1, x_1),$$

$$T_3 = (P_3, a_3) \bowtie (T_2, x_2), \quad \ldots, \quad T_n = (P_n, a_n) \bowtie (T_{n-1}, x_{n-1}),$$

where $T_n = T$, and all a_i's and x_j's are in the set S of forks of T.

To prove T is an integral sum graph, we use the mathematical induction on n to show a stronger claim as follows.

Claim. For each $i = 1, 2, \ldots, n$, there is an integral sum labeling f_i such that for any node $b \in V(T_i) \cap S$, $f_i(x) \neq -f_i(b)$ for any $x \in V(T_i) - \{b\}$.

When $n = 1$, $T = T_1$ is a path of length at least 4. So, the claim is true by Corollary 2. Now we assume that $n > 1$ and that the claim is true for $n - 1$. By the induction hypothesis, for each $i = 1, 2, \ldots, n - 1$, there is an integral sum labeling f_i such that for any node $b \in V(T_i) \cap S$, $f_i(x) \neq -f_i(b)$ for any $x \in V(T_i) - \{b\}$. Then it is clear that (T_{n-1}, x_{n-1}) satisfies the conditions (1) and (2) in Theorem 2. So we may use Theorem 2 for $T_n = (P_n, a_n) \bowtie (T_{n-1}, x_{n-1})$. Then, by the construction of the labeling f given in the proof for Lemma 2 (with φ_1 and φ_2 given in the proof for Theorem 2), we get the desired labeling f_n for T_n. This completes the induction.

Therefore, T is an integral sum graph. □

Recall that the subdivision $S(G)$ of a graph G is a graph obtained from G by replacing each edge uv of G by a path of length 2 from u to v. In other words, $S(G)$ is obtained from G by inserting a new node on each edge of G so that each inserted new node has degree 2 in $S(G)$. We call $S(S(G))$ as the *double subdivision* of G. Then, by Theorem 3, we immediately have the following

Corollary 4. *The double subdivision of any tree is an integral sum graph.*

Theorem 4. *Every generalized star is an integral sum graph.*

In order to prove Theorem 4, we need more lemmas.

Lemma 3. *Let $P = a_0 a_1 a_2 \cdots a_n$ be a path with length $n \geqslant 4$. Then for any given $0 \leqslant i \leqslant n$, there is an integral sum labeling ψ of P such that $\psi(x) \neq -\psi(a_i), \pm 2\psi(a_i)$ for any $x \in V(P)$.*

Proof. For $i \neq 1,3$, we can get the desired labeling ψ by Corollary 2. So, we only need to prove it for $i=1$ or 3.

We first consider $n \geqslant 5$.

When $n=5$, we define the labeling ψ as $\psi(a_0)= -7$, $\psi(a_1)=5$, $\psi(a_2)= -2$, $\psi(a_3)=3$, $\psi(a_4)=1$, $\psi(a_5)=4$;

When $n \geqslant 6$, we define the labeling ψ as $\psi(a_0)= -9$, $\psi(a_1)= -4$, and $\psi(a_k)= \psi(a_{k-2}) - \psi(a_{k-1})$ for all $k \geqslant 2$.

It is not difficult to show that the defined ψ is the desired labeling for $i=1$ or 3, since a direct verification works when $n=5,6$ and we may use Corollary 1 when $n>6$.

Now, let us consider $n=4$. We need to define ψ differently for the cases $i=1$ and $i=3$.

For $i=1$, we define ψ as $\psi(a_0)= -4$, $\psi(a_1)=3$, $\psi(a_2)= -1$, $\psi(a_3)=2$, $\psi(a_4)=1$.

For $i=3$, we define ψ as $\psi(a_0)=1$, $\psi(a_1)=2$, $\psi(a_2)= -1$, $\psi(a_3)=3$, $\psi(a_4)= -4$.

It is easy to verify that ψ is the desired labeling in each case.

This completes the proof for Lemma 3. □

Lemma 4. *Let $(G_1,r_1)=a_0a_1a_2\cdots a_n$ be a path with length $n \geqslant 4$ and root $r_1=a_n$. Let $(G_2,r_2)=b_0b_1b_2\cdots b_m$ be a path with length $m \geqslant 1$, whose root r_2 is any node of G_2. Then there is an integral sum labeling ψ of $(G,r)=(G_1,r_1)\bowtie(G_2,r_2)$ such that $\psi(x)\neq -\psi(r), \pm 2\psi(r)$ for any $x \in V(G)$.*

Proof. If $m \geqslant 4$, we see from Lemma 3 that (G_2,r_2) satisfies the conditions (1) and (2) in Theorem 2. Then, by the construction of the labeling f given in the proof for Lemma 2 (with φ_1 and φ_2 defined as in the proof for Theorem 2), we can get the desired integral sum labeling of G.

Note that when $r_2=b_0$ or b_m, G is a path and the result then follows from Lemma 3 immediately. So, we only need to consider $m=2$ or 3, and we may assume that $r_2=b_1$ without loss of generality.

When $m=2$, (G_2,r_2) is a star whose root r_2 is its center. With the labeling given in Corollary 2, (G_1,r_1) satisfies the conditions (i) and (ii) in Theorem 1. Then we can use Lemma 1 for $(G,r)=(G_2,r_2)\bowtie(G_1,r_1)$. As the labeling given in the proof of Lemma 1, we get the desired integral sum labeling ψ of G. It is easy to verify that $\psi(x)\neq -\psi(r), \pm 2\psi(r)$ for any $x \in V(G)$.

When $m=3$, we first define a labeling f_1 of G_1 as $f_1(a_i)=f(a_{n-i})$ for all $i=0,1,\ldots,n$, where f is the labeling given in Corollary 2 (with $t>4$), and define f_2 of G_2 as $f_2(b_0)=1$, $f_2(b_1)=2$, $f_2(b_2)= -1$, and $f_2(b_3)=3$. Then we define the labeling ψ of G as

$$\psi(x)=\begin{cases} 2f_1(x) & \text{if } x \in V(G_1), \\ f_2(x) & \text{if } x \in V(G_2). \end{cases}$$

It is not difficult to verify that ψ is the desired integral sum labeling of G.

This completes the proof for Lemma 4. □

Lemma 5. *Let $G_1 = a_0a_1a_2 \cdots a_n$ be a path with $4 \leqslant n \leqslant 6$ and root $r_1 = a_{\lfloor n/2 \rfloor}$. Let (G_2, r_2) be a connected graph with root r_2, which satisfies the following:*
(1) *the maximum degree $\Delta(G_2) < |V(G_2)| - 1$, and*
(2) *there is an integral sum labeling ψ of G_2 such that*

$$\psi(x) \neq -\psi(r_2), \pm 2\psi(r_2) \text{ for any } x \in V(G_2) - \{r_2\}.$$

Then there is an integral sum labeling f of $(G, r) = (G_1, r_1) \bowtie (G_2, r_2)$ such that $f(x) \neq -f(r), \pm 2f(r)$ for any $x \in V(G)$.

Proof. We first define a labeling φ of G_1 as follows. For $n = 4$ or 5, φ is defined the same as in Corollary 3; for $n = 6$, φ is defined as $\varphi(a_0) = -1 + 2t$, $\varphi(a_1) = 1 - t$, $\varphi(a_2) = t$, $\varphi(a_3) = 1$, $\varphi(a_4) = -t$, $\varphi(a_5) = 1 + t$, and $\varphi(a_6) = -1 - 2t$. In all the cases, we assume that $t > 1 + 2 \cdot max\{|\psi(x)| : x \in V(G_2)\}$.

Now, as in the proof of Theorem 2, we let $\varphi_1 = m\varphi$ where $m = \psi(r_2)$, and $\varphi_2 = \psi$. Then we define a labeling f of G as in the proof of Lemma 2.

It is easily verified that $f(x) \neq -f(r), \pm 2f(r)$ for any $x \in V(G)$. Then we only need to show that f is an integral sum labeling. It suffices to show that φ_1 and φ_2 satisfy the conditions (i)–(iv) in Lemma 2.

Obviously, (i) and (ii) are satisfied. Note that we have either $\varphi_1(a) \pm \varphi_1(b) = 0$, $\pm m, \pm 2m$, or $|\varphi_1(a) \pm \varphi_1(b)| \geqslant t - 1 > |\varphi_2(x)|$ for all distinct $a, b \in V(G_1)$ and $x \in V(G_2) - \{r_2\}$. Then, by the given condition (2), we see that (iii) is satisfied. Finally, it is also easy to see that (iv) is satisfied, since $|\varphi_2(x) \pm \varphi_2(y)| < t - 1 < |\varphi_1(a)|$ for all distinct $x, y \in V(G_2)$ and $a \in V(G_1) - \{r_1\}$.

This completes the proof for Lemma 5. □

Lemma 6. *Let $(G, r) = (G_1, r_1) \bowtie (G_2, r_2)$, where $G_1 = a_0a_1a_2 \cdots a_n$ is a path with $n \geqslant 4$ and root $r_1 = a_0$, and G_2 is a connected graph with root r_2. If*
(i) *$\Delta(G_2) < |V(G_2)| - 1$, and*
(ii) *there is an integral sum labeling ψ of G_2 such that*

$$\psi(x) \neq -\psi(r_2), \pm 2\psi(r_2) \text{ for any } x \in V(G_2) - \{r_2\},$$

then there is an integral sum labeling f of G such that

$$f(x) \neq -f(r), \pm 2f(r) \text{ for any } x \in V(G).$$

Proof. By Theorem 2, G is an integral sum graph. It is easy to verify that the labeling f of G given in the proof for Lemma 2 (with φ_1 and φ_2 given in the proof for Theorem 2) is the desired f. □

In the proof for Theorem 4, for the sake of simplicity, we shall use the following notation. For a given generalized star $(G, r) = (H_1, r_1) \bowtie (H_2, r_2) \bowtie \cdots \bowtie (H_k, r_k)$ where $k \geqslant 1$ and each H_i is a path with root r_i being one of its end-nodes, we shall denote it as $(G, r) = (G(n_1, n_2, \ldots, n_t), r)$, where n_i is the number of the paths with length i in the set $\{H_1, H_2, \ldots, H_k\}$, and t is the largest i such that $n_i > 0$.

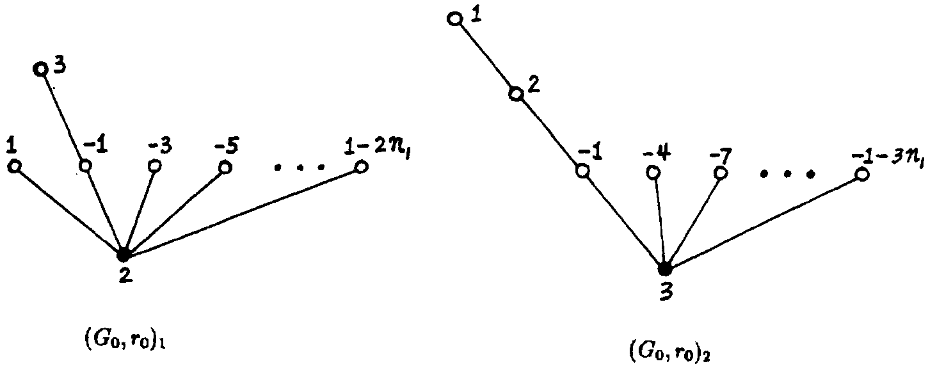

Fig. 1.

Now the proof for Theorem 4 goes as follows.

Proof for Theorem 4. For a generalized star $(G,r)=(G(n_1,n_2,\ldots,n_t),r)$, let

$$N=\sum_{i\geqslant 4} n_i.$$

We show the following stronger claim by induction on $N\geqslant 0$.

Claim. *G is a path, a star with center r, or a graph with an integral sum labeling ψ of G such that $\psi(x)\neq -\psi(r),\pm 2\psi(r)$ for any $x\in V(G)-\{r\}$.*

For $N=0$, we may distinguish the following cases.

Case 1: $\sum n_i\leqslant 2$. Then clearly G is a path and the claim is true.

Case 2: $\sum n_i\geqslant 3$. Then we distinguish the following subcases.

Subcase 1: $n_2+n_3=0$. Then clearly G is a star and the claim is true.

Subcase 2: $n_1>0$ and $n_2+n_3=2k+1(k\geqslant 0)$. We show the claim by induction on k. To do so, we denote the graph (G,r) as (G_k,r_k).

When $k=0$, $n_1\geqslant 2$. Then (G_k,r_k) is one of the two graphs depicted in Fig. 1.

It is not difficult to verify that for each graph in Fig. 1 (where the black node is the root of the graph), the labeling given there is the desired ψ.

When $k\geqslant 1$, it is easy to see that $(G_k,r_k)=(G_{k-1},r_{k-1})\bowtie(P,r)$ where $P=a_0a_1\cdots a_n$ is a path with $4\leqslant n\leqslant 6$ and $r=a_{\lfloor n/2\rfloor}$. So the claim directly follows from the induction hypothesis and Lemma 5.

Subcase 3: $n_1>0$ and $n_2+n_3=2k(k\geqslant 1)$. We also denote the graph (G,r) as (G_k,r_k) and use induction on k.

When $k=1$, $n_1\geqslant 1$ and the graph is one of the three graphs depicted in Fig. 2.

Note that $(G_1,r_1)_1$ can be obtained from a graph of the type $(G_0,r_0)_1$ (in Fig. 1) and $K_{1,1}$ by identification. Then, by Lemma 1, we see that the labeling given in Fig. 2 for $(G_1,r_1)_1$ is the desired integral sum labeling. Similarly, since $(G_1,r_1)_2$ can be

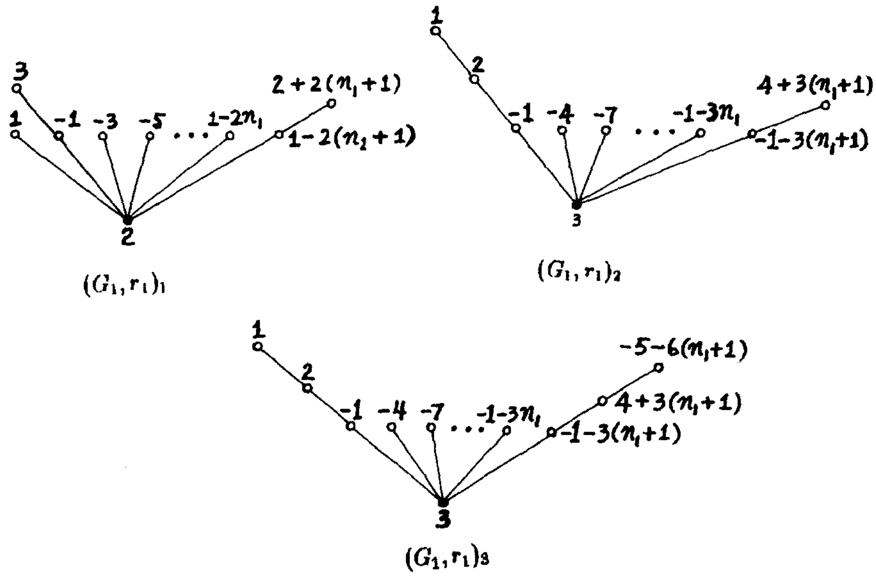

Fig. 2.

obtained from a graph of the type $(G_0,r_0)_2$ (in Fig. 1) and $K_{1,1}$ by identification, and $(G_1,r_1)_3$ can be obtained from $(G_1,r_1)_2$ and $K_{1,1}$ by identification, then we easily see that the labelings of $(G_1,r_1)_2$ and $(G_1,r_1)_3$ given in Fig. 2 are the desired integral sum labelings.

When $k\geqslant 2$, it is easy to see that $(G_k,r_k)=(G_{k-1},r_{k-1}) \bowtie (P,r)$ where $P=a_0a_1\cdots a_n$ is a path with $4\leqslant n\leqslant 6$ and $r=a_{\lfloor n/2\rfloor}$. So the claim directly follows from the induction hypothesis and Lemma 5.

Subcase 4: $n_1=0$ and $n_2+n_3=2k+1(k\geqslant 0)$. In fact, we must have $k\geqslant 1$ since $\sum n_i\geqslant 3$.

When $k=1$, the graph is one of the four graphs depicted in Fig. 3. It is easy to see that the given labeling for each in Fig. 3 is the desired.

Then, as in the above subcases, we can use induction on k to show the claim for all $k\geqslant 1$.

Subcase 5: $n_1=0$ and $n_2+n_3=2k(k\geqslant 1)$. Similar to the above subcases, we only need to show that when $k=1$ the graph has a labeling as described in the Claim.

When $k=1$, the graph is one of the three graphs depicted in Fig. 4. It is easy to see that the given labeling for each in Fig. 4 is the desired.

In the above we have shown the claim is true for $N=0$.

Now we consider $N\geqslant 1$. Note that

$(G,r)=(G(n_1,\ldots,n_t),r)=(G(n_1,\ldots,n_t-1),r)\bowtie(P_t,r)$ where P_t is a path with length $t\geqslant 4$ and r is an end node of P_t. By the induction hypothesis, $(G(n_1,\ldots,n_t-1),r)$ is a star with center r, a path, or a graph with an integral sum labeling ψ of G such that $\psi(x)\neq-\psi(r)$, $\pm 2\psi(r)$ for any $x\in V(G)-\{r\}$. For the latter two cases, we can

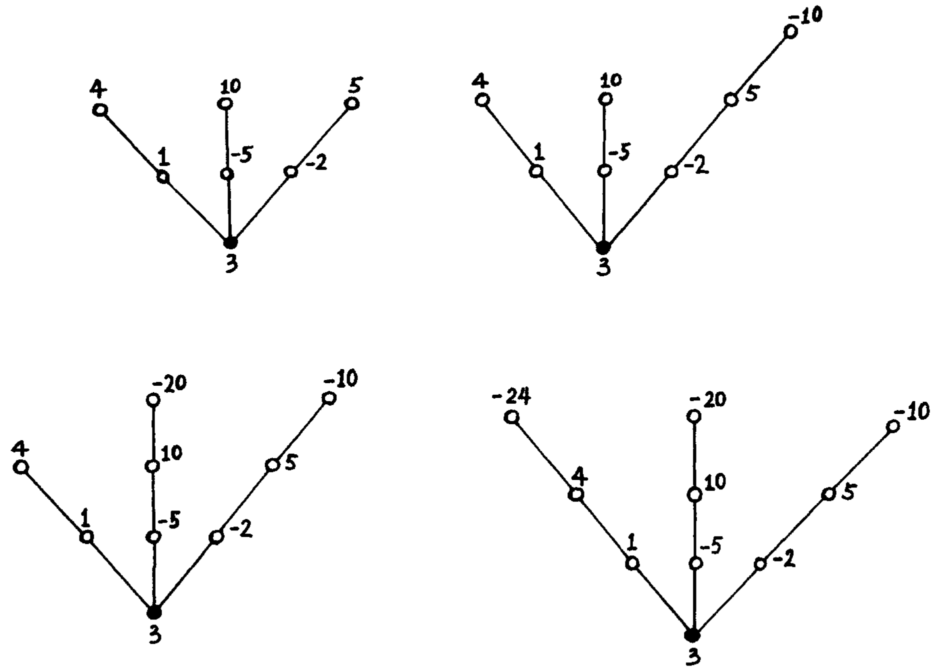

Fig. 3.

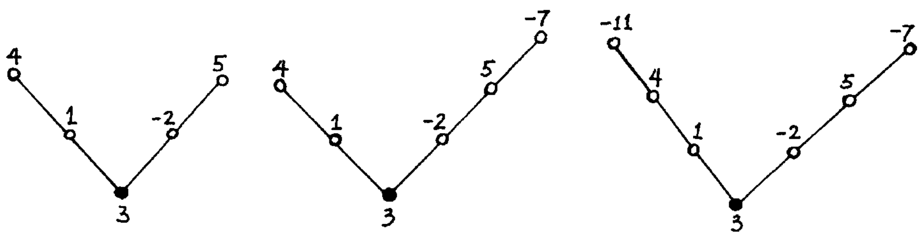

Fig. 4.

easily see the claim is true directly from Lemmas 4 and 6. For the remaining case, $(G(n_1,\ldots,n_t-1),r)$ is a star with center r. Given an integral sum labeling for P_t as in Corollary 2, we immediately see that G is an integral sum graph by Lemma 1. Now let ψ be the integral sum labeling of G defined just as the labeling f given in the proof for Lemma 1. Then it is easy to verify that $\psi(x) \neq -\psi(r), \pm 2\psi(r)$ for any $x \in V(G) - \{r\}$.

This finishes the induction for the claim.

Therefore, any generalized star is an integral sum graph. □

4. Concluding remarks

1. Our main purpose here is to introduce the 'identification' methodology into the study of integral sum graphs. We believe that this methodology may have applications elsewhere with similar problems, such as sum graphs, graceful graphs and other graph labeling problems, etc.

2. As applications of the 'identification' methodology, we have shown that generalized stars and trees with all forks at least distance 4 apart are integral sum graphs. Informed by a referee, all caterpillars and trees on 10 nodes or less are also known to be integral sum graphs. It seems reasonable to raise the following:

Conjecture. Every tree is an integral sum graph.

3. It should be mentioned that the 'identification' methodology is not for studying trees only. It can also be used to obtain many classes of integral sum graphs which are not trees. For example, let (G_1, r_1) be a caterpillar whose root r_1 is its neck or head, and let (G_2, r_2) be a cycle of length 5 whose root r_2 is any one of its nodes, then $(G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph by Theorem 1, since we may give an integral sum labeling φ of G_2 as '$3, -2, 1, 2, -1$' with $\varphi(r_2) = 3$.

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References

[1] Z. Chen, Harary's conjectures on integral sum graphs, Discrete Math. 160 (1996) 241–244.
[2] T. Hao, On sum graphs, J. Combin. Math. Combin. Computing 6 (1989) 207–212.
[3] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[4] F. Harary, Sum graphs and difference graphs, Congr. Numer. 72 (1990) 101–108.
[5] F. Harary, Sum graphs over all the integers, Discrete Math. 124 (1994) 99–105.
[6] N. Hartsfield, W.F. Smyth, A family of sparse graphs of large sum number, Discrete Math., to appear.
[7] W.F. Smyth, Sum graphs of small sum number, Colloquia Math. Soc. Janos Bolyai 60 (1991) 669–678.