Topological uniform descent and Weyl type theorem

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Abstract

The generalized Weyl’s theorem holds for a Banach space operator $T$ if and only if $T$ or $T^*$ has the single valued extension property in the complement of the Weyl spectrum (or B-Weyl spectrum) and $T$ has topological uniform descent at all $\lambda$ which are isolated eigenvalues of $T$. Also, we show that the generalized Weyl’s theorem holds for analytically paranormal operators.

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1. Introduction

Let $B(X)$ denote the algebra of bounded linear operators on a Banach space $X$. An operator $T \in B(X)$ is said to be Fredholm if $R(T)$ is closed and both the deficiency induces $n(T) = \dim N(T)$ and $d(T) = \dim X/R(T)$ are finite, and then the index of $T$, $\text{ind}(T)$, is defined to be $\text{ind}(T) = n(T) - d(T)$. The ascent of $T$, $\text{asc}(T)$, is the least non-negative integer $n$ such that $N(T^n) = N(T^{n+1})$ and the descent, $\text{des}(T)$, is the least non-negative integer $n$ such that $R(T^n) = R(T^{n+1})$. The operator $T$ is Weyl if it is Fredholm of index zero, and $T$ is said to be Browder if it is Fredholm “of finite ascent and descent”. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$ are defined by

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\[ \sigma_c(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \}, \]
\[ \sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl} \}, \]
\[ \sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \}. \]

Let \( \rho(T) \) denote the resolvent set of the operator \( T \) and \( \sigma(T) = \mathbb{C} \setminus \rho(T) \) denote the usual spectrum of \( T \). We use \( \pi_{00}(T) \) denote the set of isolated eigenvalues \( \lambda \) of \( T \) for which \( \dim N(T - \lambda I) < \infty \). Also let \( \pi^a_{00}(T) \) be the set of \( \lambda \in \mathbb{C} \) such that \( \lambda \) is an isolated point of \( \sigma_a(T) \) and \( 0 < \dim N(T - \lambda I) < \infty \), where \( \sigma_a(T) \) denotes the approximate point set of the operator \( T \in B(X) \). We say that the Browder’s theorem holds for \( T \) if
\[ \sigma_w(T) = \sigma_b(T), \]
the Weyl’s theorem holds for \( T \) if
\[ \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \]
and the a-Weyl’s theorem holds for \( T \) if
\[ \sigma_a(T) \setminus \sigma_{ea}(T) = \pi^a_{00}(T), \]
where \( \sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \text{SF}_+^\circ(X) \} \) and \( \text{SF}_+^\circ(X) = \{ T \in B(X), T \text{ is upper semi-Fredholm of } \text{ind}(T) \leq 0 \} \). The concept of a-Weyl’s theorem was introduced by Rakočević: a-Weyl’s theorem for \( T \iff \text{Weyl’s theorem for } T \), but the converse is generally false [13].

For a bounded linear operator \( T \) and a nonnegative integer \( n \) define \( T_{[n]} \) to be the restriction of \( T \) to \( R(T^n) \) viewed as a map from \( R(T^n) \) into \( R(T^n) \) (in particular \( T_{[0]} = T \)). If for some integer \( n \) the range space \( R(T^n) \) is closed and \( T_{[n]} \) is an upper (resp. a lower) semi-Fredholm operator, then \( T \) is called an upper (resp. a lower) semi-B-Fredholm operator. We call \( T \) B-Weyl if for some integer \( n \) the range space \( R(T^n) \) is closed and \( T_{[n]} \) is Weyl. Let \( \sigma_{BW}(T) \) be the B-Weyl spectrum. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

Let \( T \in B(X) \) and let (see [10])
\[ \Lambda(T) = \{ n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow [R(T^n) \cap N(T)] \subseteq [R(T^m) \cap N(T)] \}. \]
Then the degree of stable iteration \( \text{dis}(T) \) of \( T \) is defined as \( \text{dis}(T) = \inf \Lambda(T) \).

Let \( T \) be a semi-B-Fredholm operator and let \( d \) be the degree of the stable iteration of \( T \). It follows from Proposition 2.1 in [2] that if \( T_{[d]} \) is a semi-Fredholm operator, and \( \text{ind}(T_{[m]}) = \text{ind}(T_{[d]}) \) for each \( m \geq d \). This enables us to define the index of a semi-B-Fredholm operator \( T \) as the index of the semi-Fredholm operator \( T_{[d]} \).

In the case of a normal operator \( T \) acting on a Hilbert space, Berkani [3, Theorem 4.5] showed that
\[ \sigma_{BW}(T) = \sigma(T) \setminus E(T), \]
\( E(T) \) is the set of all eigenvalues of \( T \) which are isolated in the spectrum of \( T \). This result gives a generalization of the classical Weyl’s theorem. We say \( T \) obeys generalized Weyl’s theorem if
\[ \sigma_{BW}(T) = \sigma(T) \setminus E(T) \] [4, Definition 2.13].

Similarly, let \( \text{SBF}_+^\circ(X) \) be the class of all upper semi-B-Fredholm operators, and \( \text{SBF}_+^\circ(X) \) the class of all \( T \in \text{SBF}_+^\circ(X) \) such that \( \text{ind}(T) \leq 0 \). Also let
\[ \sigma_{\text{SBF}_+^\circ}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not in } \text{SBF}_+^\circ(X) \}. \]
We call $T$ obeys generalized a-Weyl’s theorem if
$$\sigma_{SBF+}(T) = \sigma_a(T) \setminus E^a(T),$$
where $E^a(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma_a(T)$ [4, Definition 2.13]. From Theorem 3.11 in [4], we say that $T$ satisfying generalized a-Weyl’s theorem satisfies a-Weyl’s theorem, but the converse is not true (see Example 3.12 in [4]).

Sufficient conditions for an operator $T \in B(X)$ to satisfy Weyl’s theorem have been considered by a number of authors in the recent past ([1,9], etc.). The plan of this paper is as follows. In Section 2, we prove our main result and give the necessary and sufficiently conditions for $T$ which the generalized Weyl’s theorem holds. In Section 3, we show the generalized Weyl’s theorem for analytically paranormal operators.

2. Generalized Weyl type theorem for operator $T$

If $T \in B(X)$, for each nonnegative integer $n$, $T$ induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. We will let $k_n(T)$ be the dimension of the null space of the induced map and let $k(T) = \sum_{n=0}^{\infty} k_n(T)$. The following definition describes the classes of operators we will study. These definitions were introduced by Grabiner in [8].

**Definition 2.1.** If there is a nonnegative integer $d$ for which $k_n(T) = 0$ for $n \geq d$ (i.e., if the induced maps are isomorphisms for $n \geq d$), we say that $T$ has uniform descent for $n \geq d$.

**Definition 2.2.** Suppose there is a nonnegative integer $d$ for which $T$ has uniform descent for $n \geq d$. If $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then we say that $T$ has topological uniform descent.

It can be shown that if $T$ is upper semi-B-Fredholm, $T$ has topological uniform descent. For an operator $T$ which has the topological uniform descent, there is the property [8, Corollary 4.9].

**Lemma 2.1.** Suppose that $T \in B(X)$ and $\lambda$ belongs to the boundary of the spectrum of $T$. If $T - \lambda I$ has topological uniform descent, then $\lambda$ is a pole of $T$.

An operator $T \in B(X)$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at $\lambda_0 \in \mathbb{C}$ for short, if for every open disc $D_{\lambda_0}$ centered at $\lambda_0$ the only analytic function $f: D_{\lambda_0} \to X$, which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. Trivially, every operator $T$ has SVEP at every point of the resolvent $\mathbb{C} \setminus \sigma(T)$; also $T$ has the SVEP at $\lambda \in \text{iso } \sigma(T)$. We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. Let $\Pi(T)$ be the set of pole points in spectrum set, clearly, $\Pi(T) \subseteq E(T)$.

**Proposition 2.1.** If $T \in B(X)$, then $T - \lambda I$ has topological uniform descent for all $\lambda \in E(T)$ if and only if $E(T) = \Pi(T)$.

**Proof.** Suppose $E(T) = \Pi(T)$. Then for each $\lambda \in E(T)$ there corresponds an integer $p \geq 1$ such that $X = N[(T - \lambda I)^p] \oplus R[(T - \lambda I)^p]$. Thus for each $n \geq p$, $R[(T - \lambda I)^n]$ is closed. Since $N(T - \lambda I) \cap R[(T - \lambda I)^n] = \{0\}$ for every $n \geq p$, it follows that $k_n(T - \lambda I) = 0$ for $n \geq p$. Then $T - \lambda I$ has uniform descent for $n \geq p$. Therefore from Definition 2.2, we know that $T - \lambda I$ has topological uniform descent.
Conversely, from Lemma 2.1, we can see the result is true. □

If $T$ has SVEP, then $T$ satisfies Browder’s theorem, thus we can prove that $\sigma (T) \setminus \sigma_{BW}(T) \subseteq E(T)$. In fact, let $\lambda_0 \in \sigma (T) \setminus \sigma_{BW}(T)$, from the Remark in [5] and the proof in Lemma 4.2 in [8], there exists $\epsilon > 0$ such that $T - \lambda I$ is Weyl and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Since Browder’s theorem holds for $T$, it follows that $T - \lambda I$ is Browder. Then $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$ [14, Lemma 3.4]. This implies $\lambda_0 \in \text{iso} \sigma (T)$. We claim that $N(T - \lambda_0 I) \neq \{0\}$. If not, $T - \lambda_0 I$ must be bounded from below, and therefore $T - \lambda_0 I$ has topological uniform descent. From Lemma 2.1, $T - \lambda_0 I$ Browder, therefore $T - \lambda_0 I$ is invertible because $N(T - \lambda_0 I) = \{0\}$. It is a contradiction. In additional, if $T$ has topological uniform descent for $\lambda \in E(T)$, then $T$ satisfies generalized Weyl’s theorem. The following theorem shows that the SVEP hypothesis on $T$ can be weakened to SVEP at all points in the complement of $\sigma_w(T)$.

**Theorem 2.1.** $T \in B(X)$ satisfies generalized Weyl’s theorem if and only if

(a) $T$ or $T^*$ has SVEP at all $\lambda \in \sigma (T) \setminus \sigma_w(T)$;
(b) $T$ has topological uniform descent at all $\lambda \in E(T)$.

**Proof.** Suppose $T$ satisfies generalized Weyl’s theorem. Since generalized Weyl’s theorem implies Weyl’s theorem, it follows that $\sigma (T) \setminus \sigma_w(T) = \pi_0(T)$. Then for every $\lambda \in \sigma (T) \setminus \sigma_w(T)$, we have that $\lambda \in \text{iso} \sigma (T)$. Therefore both $T$ and $T^*$ has SVEP at all $\lambda \in \sigma (T) \setminus \sigma_w(T)$. The fact that the Weyl’s theorem holds for $T$ implies $E(T) = \Pi(T)$. Then from Proposition 2.1, $T$ has topological uniform descent at all $\lambda \in E(T)$.

For the converse, let $\lambda \in E(T)$. By hypothesis (b), $\lambda \in \Pi(T)$. Thus $T - \lambda I$ is B-Weyl, which means that $\lambda \in \sigma (T) \setminus \sigma_{BW}(T)$. For the reverse inclusion, let $\lambda \in \sigma (T) \setminus \sigma_{BW}(T)$. Then there exists $\epsilon > 0$ such that $T - \mu I$ is Weyl and $N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n]$ if $0 < |\mu - \lambda| < \epsilon$. If $T$ or $T^*$ has SVEP at $\mu$, then $T - \mu I$ has finite ascent and finite descent [1, Corollary 2.10]. Then $N(T - \mu I) = N(T - \mu I) \cap \bigcap_{n=1}^{\infty} R[(T - \mu I)^n] = \{0\}$ [14, Lemma 3.4]. Thus $\lambda \in \text{iso} \sigma (T)$. From Proposition 2.1, $\lambda \in \Pi(T)$. Then $\lambda \in E(T)$. From the preceding proof, we know that the generalized Weyl’s theorem holds for $T$.

**Remark 2.1.** An operator $T$ is said to be semi-regular if $R(T)$ is closed and $N(T) \subseteq \bigcap_{n=1}^{\infty} R(T^n)$. The Kato spectrum $\sigma_k(T)$ of $T$ is defined by $\sigma_k(T) = \{ \lambda \in C \mid T - \lambda I \text{ is not semi-regular} \}$. Condition (a) in Theorem 2.1 can be instead by $(a')\sigma(T) = \sigma_w(T) \cup \sigma_k(T)$ or $\sigma(T^*) = \sigma_w(T^*) \cup \sigma_k(T^*)$. In the following, we will prove that condition (a) is equivalent to condition $(a')$. Suppose $T$ or $T^*$ has SVEP at $\lambda \in \sigma (T) \setminus \sigma_w(T)$, let $\lambda_0 \notin [\sigma_w(T) \cup \sigma_k(T)]$. Then $T - \lambda_0 I$ is Weyl and $N(T - \lambda_0 I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda_0 I)^n]$. Since $T$ or $T^*$ has SVEP, then $T - \lambda_0 I$ has finite ascent. Then $N(T - \lambda_0 I) = N(T - \lambda_0 I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda_0 I)^n] = \{0\}$ [14, Lemma 3.4], which means that $T - \lambda_0 I$ is invertible. Then $\sigma (T) = \sigma_w(T) \cup \sigma_k(T)$. Conversely, suppose $\sigma (T) = \sigma_w(T) \cup \sigma_k(T)$. Let $\lambda_0 \in \sigma (T) \setminus \sigma_w(T)$, then there exists $\epsilon > 0$ such that $\lambda \notin [\sigma_w(T) \cup \sigma_k(T)]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Then $\lambda_0 \in \text{iso} \sigma (T)$, which means that $T$ or $T^*$ has SVEP at $\lambda_0$. Also, Condition (a) in Theorem 2.1 can be instead by $(a'')$: $T - \lambda I$ or $T^* - \lambda I$ has finite ascent for $\lambda \in \sigma (T) \setminus \sigma_w(T)$.

Suppose $T$ has topological uniform descent at all $\lambda \in \text{iso} \sigma (T)$, then $\text{iso} \sigma (T) = E(T) = \Pi(T)$ (Lemma 2.1 and Proposition 2.1). Since $\text{iso} \sigma (T) = \text{iso} \sigma (T^*)$ and $\Pi(T) = \Pi(T^*)$, we have that $\text{iso} \sigma (T^*) = \Pi(T^*)$. But since $\Pi(T^*) \subseteq E(T^*) \subseteq \text{iso} \sigma (T^*)$, it follows that $\text{iso} \sigma (T^*) = \text{iso} \sigma (T^*) = \Pi(T^*)$. Therefore, $T$ satisfies generalized Weyl’s theorem.
\( E(T^*) = \Pi(T^*) \). Using Proposition 2.1 we know \( T^* \) has topological uniform descent at all \( \sigma(T) \). Then we have the generalized Weyl’s theorem for \( T^* \) as follows.

**Corollary 2.1.** If \( T \) or \( T^* \) has SVEP at all \( \lambda \in \sigma(T) \setminus \sigma_w(T) \) and \( T \) has topological uniform descent at all \( \sigma(T) \), then the generalized Weyl’s theorem holds for \( T^* \).

**Proof.** Using Theorem 2.1 and the statements before Corollary 2.1, we only need to prove that \( T \) or \( T^* \) has SVEP at all \( \lambda \in \sigma(T) \setminus \sigma_w(T) \). Since \( \sigma(T) = \sigma(T^*) \) and \( \sigma_w(T) = \sigma_w(T^*) \), it follows that \( T^* \) or \( T \) has SVEP at all \( \lambda \in \sigma(T^*) \setminus \sigma_w(T^*) \). Then Theorem 2.1 tells us that the generalized Weyl’s theorem holds for \( T^* \). \( \square \)

For the generalized a-Weyl’s theorem, we have:

**Theorem 2.2**

(1) Suppose \( T \) has topological uniform descent at each \( \lambda \in E(T) \). If \( T^* \) has SVEP, then the generalized a-Weyl’s theorem holds for \( T \).

(2) Suppose \( T \) has topological uniform descent at each \( \lambda \in \sigma(T) \). If \( T \) has SVEP, then the generalized a-Weyl’s theorem holds for \( T^* \).

**Proof**

(1) The hypothesis \( T^* \) has SVEP implies \( \sigma(T) = \sigma_a(T) [11, p. 35] \), and hence \( E(T) = E^a(T) \). We claim that \( \sigma_{SBF^+}(T) = \sigma_{BW}(T) \). In fact, the inclusion \( \sigma_{SBF^+}(T) \subseteq \sigma_{BW}(T) \) is clear. For the inverse inclusion, let \( T - \lambda_0 I \) be upper semi-B-Fredholm of \( \text{ind}(T - \lambda_0 I) \leq 0 \), then there exists \( \epsilon > 0 \) such that \( T - \lambda I \) is upper semi-Fredholm of \( \text{ind}(T - \lambda I) = \text{ind}(T - \lambda_0 I) \leq 0 \) if \( 0 < |\lambda - \lambda_0| < \epsilon \) [2, Corollary 3.2]. Therefore \( T^* - \lambda I \) is lower semi-Fredholm of \( \text{ind}(T^* - \lambda I) \geq 0 \). Since \( T^* \) has SVEP, it induces that \( \text{ind}(T^* - \lambda I) \leq 0 \). Then \( T^* - \lambda I \) is Weyl and hence \( T - \lambda I \) is Weyl if \( 0 < |\lambda - \lambda_0| < \epsilon \). This proves that \( \text{ind}(T - \lambda_0 I) = \text{ind}(T - \lambda I) = 0 \), which means that \( \lambda_0 \notin \sigma_{BW}(T) \). Since the generalized Weyl’s theorem holds for \( T \), it follows that the generalized a-Weyl’s theorem holds for \( T \).

(2) If \( T \) has SVEP, then \( \sigma(T^*) = \sigma_a(T^*) \) and \( E^a(T^*) = E(T^*) \). Similar to the preceding proof, we know \( \sigma_{SBF^+}(T^*) = \sigma_{BW}(T^*) \). From Corollary 2.1, the generalized Weyl’s theorem holds for \( T^* \), then the generalized a-Weyl’s theorem holds for \( T^* \). \( \square \)

In the following, let \( H(T) \) be the class of all complex-valued functions which are analytic on \( \sigma(T) \) and are not constant on any component of \( \sigma(T) \).

**Theorem 2.3.** Suppose \( T \) has topological uniform descent at each \( \lambda \in \sigma(T) \).

(1) If \( T \) or \( T^* \) has SVEP, then the generalized Weyl’s theorem holds for \( f(T) \) for each \( f \in H(T) \).

(2) If \( T^* \) has SVEP, then the generalized a-Weyl’s theorem holds for \( f(T) \) for each \( f \in H(T) \).

(3) If \( T \) has SVEP, then the generalized a-Weyl’s theorem holds for \( f(T^*) \) for each \( f \in H(T) \).

**Proof**

(1) Let \( \mu_0 \in [\sigma(f(T)) \setminus \sigma_{BW}(f(T))] \), then there exists \( \epsilon > 0 \) such that \( f(T) - \mu I \) is Weyl and \( N[f(T) - \mu I] \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - \mu I)^{n}] \) if \( 0 < |\mu - \mu_0| < \epsilon \). For each such \( \mu \), suppose
f(T) − μI = (T − λ₁I)n₁(T − λ₂I)n₂ · · · (T − λₖI)nₖ g(T), where λᵢ ≠ λⱼ and g(T) is invertible. Then T − λᵢI is Fredholm and ∑ₖₙ₌₁ ind([(T − λᵢI)nᵢ]) = 0. Since T or T* has SVEP, it follows that ind(T − λᵢI) ≤ 0 for all λᵢ or ind(T − λᵢI) ≥ 0 for all λᵢ. Then ind(T − λᵢI) = 0, which means that T − λᵢI is Weyl. From Theorem 2.1, we know that the generalized Weyl’s theorem holds for T, then T − λᵢI is Browder and hence f(T) − μI is Browder. Thus N[f(T) − μI] = N[f(T) − μI] ∩ ∩ₙ₌₁ R[(f(T) − μI)n] = {0}. It induces f(T) − μI is invertible and hence μ₀ ∈ σ(f(T)). Lemma 2.1 tells us that μ₀ is a pole of f(T), then μ₀ ∈ E(f(T)). For the converse, let μ₀ ∈ E(f(T)) and let f(T) − μ₀I = (T − λ₁I)n₁(T − λ₂I)n₂ · · · (T − λₖI)nₖ g(T), where λᵢ ≠ λⱼ and g(T) is invertible. Without loss of generality, we suppose λᵢ ∈ σ(T). Then λ ∈ iso σ(T). Since T has topological uniform descent at each λᵢ, using Lemma 2.1, we know λᵢ is pole of T. Then T − λᵢI is B-Weyl and then f(T) − μ₀I is B-Weyl [5, Corollary 3.3]. Now we get that σ(f(T)) \ σ_BW(f(T)) = E(f(T)), which means that the generalized Weyl’s theorem holds for f(T).

(2) If T* has SVEP, and let f ∈ H(T), then f(T*) = f(T)* has SVEP [11, Theorem 3.3.9], which implies that σ(f(T)) = σₑ(f(T)). Arguing as in the proof of Theorem 2.2 it is seen that σₑ(f(T)) = σₑ_SBW(f(T)). Since the generalized Weyl’s theorem holds for f(T), it follows that f(T) satisfies generalized a-Weyl’s theorem.

(3) Suppose T has SVEP. Let μ₀ ∈ σₑ(f(T*))) \ σₑ_SBW(f(T*)), then there exists δ > 0 such that f(T*) − μI ∈ SF⁺∞(X) and N[f(T*) − μI] ⊆ ∩ₙ₌₁ R[(f(T*) − μI)n] if 0 < |μ − μ₀| < δ. For each such μ, suppose f(T*) − μI = (T* − λ₁I)n₁(T* − λ₂I)n₂ · · · (T* − λₖI)nₖ g(T*), where λᵢ ≠ λⱼ and g(T*) is invertible. Then T* − λᵢI is upper semi-Fredholm and ∑ₙ₌₁ ind([(T* − λᵢI)nᵢ]) ≤ 0. Since T has SVEP, it follows that ind(T − λᵢI) ≤ 0 for all λᵢ. Then ind(T* − λᵢI) = 0, which means that T* − λᵢI is Weyl. From Corollary 2.1, we know that the generalized Weyl’s theorem holds for T*, then T* − λᵢI is Browder and hence f(T*) − μI is Browder. Thus N[f(T*) − μI] = N[f(T*) − μI] ∩ ∩ₙ₌₁ R[(f(T*) − μI)n] = {0}. It induces f(T*) − μI is invertible and hence μ₀ ∈ iso σ(f(T*)). Lemma 2.1 tells us that μ₀ is a pole of f(T*), then μ₀ ∈ E⁺∞(f(T*)). For the converse, let μ₀ ∈ E⁺∞(f(T*)) and let f(T*) − μ₀I = (T* − λ₁I)n₁(T* − λ₂I)n₂ · · · (T* − λₖI)nₖ g(T*), where λᵢ ≠ λⱼ and g(T*) is invertible. Without loss of generality, we suppose λᵢ ∈ σ(T*). Since T has SVEP, it induces σₑ(T*) = σ(T*). Then λ ∈ iso σ(T). Since T has topological uniform descent at each λᵢ, using Lemma 2.1, we know λᵢ is pole of T. Thus λᵢ is pole of T*. Then T* − λᵢI is B-Weyl and then f(T*) − μ₀I is B-Weyl [5, Corollary 3.3], which means that μ₀ ∈ σₑ(f(T*)) \ σₑ_SBW(f(T*)). This proves that the generalized a-Weyl’s theorem holds for f(T*) for every f ∈ H(T).

Corollary 2.2. Suppose T has topological uniform descent at each λ ∈ iso σ(T), then the generalized Weyl’s theorem holds for f(T) for each f ∈ H(T) if and only if

(1) T or T* has SVEP at λ ∈ σ(T) \ σₑ(T);
(2) f(σₑ(T)) = σₑ(f(T)) for each f ∈ H(T).

Proof. We only prove that if the generalized Weyl’s theorem holds for f(T), then f(σₑ(T)) = σₑ(f(T)) for each f ∈ H(T). σₑ(f(T)) ⊆ f(σₑ(T)) is clear. For the converse, let μ₀ ∉ σₑ(f(T)) and suppose f(T) − μ₀I = (T − λ₁I)n₁(T − λ₂I)n₂ · · · (T − λₖI)nₖ g(T), where λᵢ ≠ λⱼ and g(T) is invertible. Then T − λᵢI is Fredholm. Since the generalized Weyl’s theorem holds for f(T), it follows that f(T) − μ₀I is Browder. It is well known that σₑ(f(T)) = f(σₑ(T))
for each \( f \in H(T) \). Then \( T - \lambda_i I \) is Browder, which means that \( \lambda_i \notin \sigma_w(T) \). This implies that \( \mu_0 \notin f(\sigma_w(T)) \). Then \( f(\sigma_w(T)) = \sigma_w(f(T)) \) for each \( f \in H(T) \). \( \Box \)

**Remark 2.2.** In Corollary 2.2, condition (2) can be instead by the condition “\( \text{ind}(T - \lambda)\text{ind}(T - \mu I) \geq 0 \) for each pair \( \lambda, \mu \in \mathbb{C}\setminus\sigma_c(T) \)” [9, Theorem 5].

3. Generalized Weyl type theorem for analytically paranormal operator

Recall that \( T \in B(X) \) is said to be paranormal if

\[
\|Tx\|^2 \leq \|T^2x\| \|x\| \quad \text{for all } x \in X.
\]

An operator \( T \in B(X) \) is algebraically paranormal if there exists \( f \in H(T) \) such that \( f(T) \) is paranormal. Analytically paranormality is preserved under restriction to invariant subspaces.

We say that \( T \) is algebraically paranormal if there exists a nonconstant complex polynomial \( p \) such that \( p(T) \) is paranormal. For algebraically paranormal operator \( T \), it is shown in [7] that Weyl’s theorem holds for \( f(T) \) for any \( f \in H(T) \). In this section, we will show that for analytically paranormal operator \( T \), the generalized Weyl’s theorem holds for \( f(T) \) for any \( f \in H(T) \).

Let us give a preliminary Lemma [7, Lemma 2.1].

**Lemma 3.1.** Let \( T \) be a paranormal operator, and assume that \( \sigma(T) = \{\lambda\} \). Then \( T = \lambda I \).

The main result in this section is:

**Theorem 3.1.** Let \( T \in B(X) \) be an analytically paranormal operator, then

(a) the generalized Weyl’s theorem holds for \( f(T) \) for every \( f \in H(T) \);
(b) the generalized a-Weyl’s theorem holds for \( f(T^*) \) for every \( f \in H(T) \).

**Proof.** (a) Using Corollary 2.2, we need to prove (1) \( T \) has topological uniform descent at \( \lambda \in \text{iso} \sigma(T) \); and (2) \( T \) or \( T^* \) has SVEP.

(1) Let \( \lambda_0 \in \text{iso} \sigma(T) \). Using the spectral projection, we can represent \( T \) as the direct sum

\[
T = T_1 + T_2, \quad \text{where } \sigma(T_1) = \{\lambda_0\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda_0\}.
\]

Let \( f(T) \) be paranormal for some \( f \in H(T) \), then \( f(T_1) \) is paranormal. Since \( \sigma(f(T_1)) = f(\sigma(T_1)) = f(\lambda_0) \), from Lemma 3.1, we know that \( f(T_1) - f(\lambda_0)I = 0 \). Let \( f(T_1) - f(\lambda_0)I = (T_1 - \lambda_0 I)^{n_0}(T_1 - \lambda_1 I)^{n_1} \cdots (T_1 - \lambda_k I)^{n_k}g(T_1) \), where \( \lambda_i \neq \lambda_j \) and \( g(T_1) \) is invertible. Then \( (T_1 - \lambda_0 I)^{n_0}(T_1 - \lambda_1 I)^{n_1} \cdots (T_1 - \lambda_k I)^{n_k}g(T_1) = 0 \). The fact that \( T_1 - \lambda_i I \) is invertible for every \( \lambda_1, \lambda_2, \ldots, \lambda_k \) tells us that \( (T_1 - \lambda_0 I)^{n_0} = 0 \), which means that \( T_1 - \lambda_0 I \) is nilpotent. Then \( T_1 - \lambda_0 I \) has finite ascent and descent. Since \( T_2 - \lambda_0 I \) is invertible, it follows that \( T - \lambda_0 I \) has finite ascent and descent. This fact means that \( \lambda_0 \) is a pole of \( T \). Let \( \text{asc}(T - \lambda_0 I) = \text{des}(T - \lambda_0 I) = p \), then \( X = N[(T - \lambda_0 I)^p] \oplus R[(T - \lambda_0 I)^p] \). Using the definition of topological uniform descent, we know that \( T \) has topological uniform descent at \( \lambda_0 \).

(2) We claim that \( T \) has SVEP. Since \( f(T) \) is paranormal for some \( f \in H(T) \), it follows that \( f(T) \) has SVEP [6, Corollary 2.10]. Hence by Theorem 3.3.9 in [12], \( T \) has SVEP.

(b) Using Theorem 2.3, we know that the generalized a-Weyl’s theorem holds for \( f(T) \) for any \( f \in H(T) \). \( \Box \)
From Corollary 2.2 and Theorem 3.1, we obtain the following useful consequence.

**Corollary 3.1.** Let \( T \) be an analytically paranormal operator. Then
\[
\sigma_w(f(T)) = f(\sigma_w(T)) \quad \text{for every } f \in H(T).
\]

If \( T^* \) is an analytically paranormal operator, we claim that \( T \) has topological uniform descent at \( \lambda \in \text{iso } \sigma(T) \). In fact, let \( \lambda \in \text{iso } \sigma(T) \), then \( \lambda \in \text{iso } \sigma(T^*) \). From the proof of Theorem 3.1, we can see that \( \lambda \) is a pole of \( T^* \). Thus \( \lambda \) is a pole of \( T \), which means that \( T \) has topological uniform descent at \( \lambda \). Using Theorem 2.3, we can induce the following result.

**Theorem 3.2.** Suppose that \( T^* \in B(X) \) be an analytically paranormal operator, then the generalized a-Weyl’s theorem holds for \( f(T) \) for every \( f \in H(T) \).

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**References**