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Weak Compactness in Spaces of Bochner Integrable Functions and Applications

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Dedicated to the Memory of Norman Levinson

1. Introduction

In this paper we shall give the necessary and sufficient conditions for weak compactness of sets in the space of Bochner integrable functions $L^1_\mu$. Roughly speaking, the necessary and sufficient conditions for bounded sets are: (a) uniform $\sigma$ additivity of the indefinite integrals; (b) the action of the indefinite integrals on each measurable set is a weakly compact subset in $E$, the range space of the functions; (c) weak uniform convergence of $f_\sigma$ to $f$, for $f$ belonging to the subset of $L^1_\mu$ in question, where $f_\sigma$ is the conditional expectation of $f$ with respect to a partition $\sigma$ of measurable sets. The precise statements are given in Theorem 1, part I. In part II of the same theorem, criteria of weak compactness for subsets of $L^p_\mu$, $1 < p < \infty$, are given. In the following sections, additional results are presented, including: weak compactness and weak sequential completeness in $L^p_\mu$, $1 < p < \infty$; weak compactness in the space of finitely additive vector measures; applications to the structure of weakly compact operators between certain function spaces.

The task of establishing weak compactness criteria in Lebesgue spaces has been a long one. This is not surprising, since not only is the weak topology of fundamental importance in the study of Banach spaces, but in classical measure theory the study of weak compactness is equivalent to the study of setwise convergence. In 1909 Lebesgue essentially gave criteria of weak convergence of indefinite integrals in terms of setwise convergence of the integrals. The Vitali–Hahn–Saks and Nikodym theorems are results in weak compactness

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theory, since setwise convergence of a sequence of measures implies that the sequence is a weakly compact set in the space of measures, hence the sequence is uniformly σ additive. The Schur theorem, which states that weak and strong sequential convergence in \( L^1 \) are equivalent, is another weak compactness result, since weak convergence in \( L^1 \) implies setwise convergence on subsets of the integers; as a result, the uniform σ additivity of the measures forces the tail end of the absolute series to tend uniformly to zero, that is, the sequence converges strongly in \( L^1 \). Dunford \([12]\) and Dunford and Pettis \([13]\) gave criteria of weak compactness in \( L^1(\mu) \). The relationship between uniform σ additivity, uniform absolute continuity, and weak compactness was discovered by Dubrovinii \([11]\).*

Criteria for weak compactness in the space of scalar measures is due to Bartle Dunford, and Schwartz \([1]\) and Grothendieck \([15]\) (see also \([8]\)). Leader \([16]\) and Porcelli \([18]\) treated the scalar finitely additive case. Chatterji \([7]\) showed that if \( E \) is reflexive, then condition (a) above is equivalent to weak compactness in \( L^1(\mu) \). Batt and Berg \([2]\) and Brooks \([3]\) extended the result to the space of vector measures. Brooks and Dinculeanu \([5]\) have established sufficient conditions for weak compactness in the space of scalar functions which are integrable with respect to a vector measure (this integration theory was developed by Bartle, Dunford, and Schwartz \([1, 14]\) in order to represent operators \( T : C(S) \to E \)). Brooks and Dinculeanu \([4]\) proved that weak compactness in the space of vector measures is equivalent to (a) and (b) if \( E \) and \( E^* \) have the Radon–Nikodym property (a Banach space \( X \) has property \( R-N \) if every \( X \)-valued measure with bounded variation is an indefinite Bochner integral; reflexive spaces and separable dual spaces have this property). Recently there have been a number of papers relating this important property to the geometry of Banach spaces. Now it appears that condition (c) is the missing condition that allows one to avoid the hypothesis of \( E \) and \( E^* \) having property \( R-N \) in order to obtain sufficient conditions for weak compactness. We still need this property to show that (c) is a necessary condition for weak compactness.

We take this opportunity to express our thanks to D.R. Lewis for his help with Lemma 6 in Section 4.

2. Preliminaries

Let \((X, \Sigma, \mu)\) be an arbitrary measure space, where \( \Sigma \) is a σ-algebra of subsets of \( X \) and \( \mu \) is a positive, possibly infinite, σ additive measure. For \( 1 \leq p \leq \infty \), \( L^p_{\mu}(\mu) \) denotes the usual Banach spaces of functions \( f : S \to E \), where \( E \) is a Banach space and the norms are defined in the usual manner (see \([10, 14]\)). The standard Banach spaces of \( E \)-valued sequences are denoted by \( l^p_E \). Let \( \Sigma_f \) be the class of sets \( A \in \Sigma \) of finite measure. We shall consider \( \mu \) partitions of the form \( \pi = (A_i)_{i \in I} \), where the \( A_i \in \Sigma_f \) are disjoint and \( 0 < \mu(A_i) \). If \( \pi = (B_j)_{j \in J} \) is a similar partition, we write \( \pi \preceq \pi' \) if every \( B_j \) is contained in some \( A_i \) or
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disjoint from all the $A_i$, and if $\bigcup_i A_i \subset \bigcup_j B_j$. If the cardinality of $\pi$ is finite, we say $\pi$ is a finite $\mu$ partition. Let $\pi = (A_i)_{i \in I}$ be a $\mu$ partition and let $f \in L_E^p(\mu)$, $1 \leq p \leq \infty$. We define $U_\pi(f) = f_\pi = \sum_{i \in I} \left( (\mu(A_i))^{-1} \int_{A_i} f \, d\mu \right) \varphi_{A_i}$, where $\varphi_A$ is the characteristic function of $A$. The integrals are well defined since for $A \in \Sigma_I$, we have $\varphi_A \in L_\mu(\mu)$, where $(1/p) + (1/q) = 1$. Since the sets $A_i$ are disjoint, the sum defining $f_\pi$ reduces, for each point $t \in X$, to a single term. If $p < \infty$, then the set $\{i \in I : \int_{A_i} |f| \, d\mu \neq 0\}$ is at most countable. We have $f_\pi \in L_E^q(\mu)$ and $\|f_\pi\|_p \leq \|\int f\|_p$. Another property is, if $g = \sum_{j=1}^K \varphi_{B_j} \cdot x_j^*$ is a step function with $B_j \in \Sigma_I$ and $x_j^* \in E^*$, the dual space of $E$, then if $\pi \geq (B_1, \ldots, B_k)$, one can show that $\int \langle f_\pi, g \rangle \, d\mu = \int \langle f, g \rangle \, d\mu$.

A set $K$ in a topological vector space $S$ will be called relatively compact if its closure is compact, and conditionally compact if every sequence from $K$ contains a Cauchy subsequence. If $S$ is a Banach space endowed with the weak topology, by the Eberlein-Smulian theorem, a set $K$ is relatively weakly compact if and only if every sequence from $K$ contains a subsequence converging weakly to some element of $S$. Hence, relative weak compactness implies conditional weak compactness if $S$ is weakly sequentially complete, and weak compactness is equivalent to conditional weak compactness in this case.

A set $K$ of bounded measures is said to be uniformly $\sigma$ additive if $\lambda(E_i) \rightarrow 0$ uniformly for $\lambda \in K$, whenever $E_i \searrow \emptyset$.

A Banach space $E$ has the Radon–Nikodym property (property $R-N$) if every $\sigma$-additive $E$-valued measure $m$ defined on a $\sigma$ ring $R$, which is absolutely continuous with respect to a positive, finite, $\sigma$-additive measure $\lambda$ on $R$ ($m$ must be of bounded variation) can be expressed as an indefinite integral $m(A) = \int_A g \, d\lambda$, for $A \in R$, where $g \in L_E^1(\lambda)$. Recall that $E$ has property $R-N$ if it is reflexive [17] or if it is a separable dual space [13]. It is well known [10] that if $E^*$, the dual space of $E$, has property $R-N$, then $(L_E^p(\mu))^* = L_E^q(\mu)$, $1 \leq p < \infty$, $(1/p) + (1/q) = 1$, when $\mu$ is $\sigma$-finite.

3. THE MAIN THEOREM

We shall employ the following convention. If $A$ is a statement containing a condition "$C$ (respectively $D$)" then the theorem "If $A$ then $E$ (respectively $F$)" means $C$ implies $E$ and $D$ implies $F$.

**Theorem 1.** Let $K \subset L_E^p(\mu)$, $1 \leq p < \infty$ and consider the following conditions:

1. $|K| = \int \{f \in K \, d\mu : f \in K\}$ is uniformly $\sigma$ additive on $\Sigma$, when $p = 1$.
2. $K(A) = \{\int_A f \, d\mu : f \in K\}$ is conditionally (respectively relatively) weakly compact in $E$, for each $A \in \Sigma$ with $\mu(A) < \infty$.
3. For every countable subset $K_0 \subset K$ there exists a sequence $(\pi_n)$ of $\mu$.
partitions such that $f_{n} \rightarrow f$ weakly (respectively strongly) in $L_{\mu}(\mu)$, uniformly for $f \in K_0$.

(4) $f_{n} \rightarrow f$ weakly in $L_{\mu}(\mu)$, uniformly for $f \in K$, where $(\pi)$ is the net of all finite $\mu$ partitions over any algebra generating $\Sigma$.

I. Let $K \subseteq L_{\mu}(\mu)$.

(a) If $K$ satisfies conditions (1), (2), and (3), then $K$ is conditionally (respectively relatively) weakly compact (if in condition (3) all partitions are finite, then condition (1) is superfluous).

(b) If $K$ is bounded and $K$ satisfies conditions (1), (2) and if $E^*$ has property $R-N$ (respectively $E$ and $E^*$ have property $R-N$), then $K$ is conditionally (respectively relatively) weakly compact.

Conversely:

(c) If $K$ is conditionally (respectively relatively) weakly compact, then $K$ is bounded and satisfies conditions (1) and (2).

(d) If $K$ is conditionally weakly compact and $E^*$ has property $R-N$, then condition (3) (with weak convergence) and condition (4) are satisfied.

II. Let $K \subseteq L_{p}(\mu)$, $1 < p < \infty$.

(a) If $K$ is bounded and satisfies conditions (2) and (3), then $K$ is conditionally (respectively relatively) weakly compact.

(b) If $K$ is bounded and satisfies condition (2) and if $E^*$ has property $R-N$ (respectively $E$ and $E^*$ have property $R-N$), then $K$ is conditionally (respectively relatively) weakly compact.

Conversely:

(c) If $K$ is conditionally (respectively relatively) weakly compact, then $K$ is bounded and satisfies condition (2).

(d) If $K$ is bounded and $E^*$ has property $R-N$, then condition (3) (with weak convergence) and condition (4) are satisfied.

Remarks. (1°) When $p = 1$, condition (1) is equivalent to the following two conditions: (1') $\int_{\Sigma} |f| \, d\mu \leq \mu$ uniformly for $f \in K$; (1r) for every $\epsilon > 0$, there exists a set $X_\epsilon \subseteq \Sigma$ with $\mu(X_\epsilon) < \infty$ such that $\int_{X - X_\epsilon} |f| \, d\mu < \epsilon$ for all $f \in K$. If $\mu(X) < \infty$, condition (1') is superfluous. (Obviously (1') and (1r) imply (1). Conversely, if (1) holds, then by [4, Theorem 2.1] (1') holds. If (1') does not hold, one can find an $\epsilon > 0$, a sequence of disjoint sets $A_n$ of finite measure, and functions $f_n \in K$ such that $\int_{A_n} |f_n| \, d\mu > \epsilon$, for all $n$, which can be shown to contradict condition (1).

(2°) If $E^*$ has property $R-N$, and $K$ is bounded, then conditions (1) and (2) imply conditions (3) (with weak convergence) and condition (4). This follows from I(b) and I(d), and from II(b) and II(d).
If $E$ is the space of scalars and $K \subseteq L^1(\mu)$ satisfies conditions (1), (2), and (3) (with strong convergence), then $K$ is relatively norm compact. In fact, it will be shown in step (F) in the proof of the Theorem that $\theta(K_\varepsilon)$ is conditionally weakly compact in $l^1$, which is equivalent, by the Schur theorem, to being relatively norm compact; hence $K$ is relatively norm compact in $L^1(\mu)$. Note again that if the partitions $\pi$ are finite, then condition (1) is superfluous. This result complements the Phillips criteria for compactness in $L^1(\mu)$ [14].

Proof of the Theorem. We shall divide the proof into several stages. In view of the Eberlein–Smulian theorem we shall be working with sequences from $K$; also each function $f$ has almost separable range and vanishes a.e. $\mu$ outside a sequence of sets of finite measure. As a result, we can assume that $E$ is separable, that $\mu$ is totally $\sigma$ finite and that $\Sigma$ is generated by a countable ring $R$ of sets of finite measure.

I(b), II(b). First we shall prove statements I(b) and II(b). Assume $K \subseteq L^p_{\Sigma}(\mu), 1 \leq p < \infty$ is bounded, satisfies condition (2), and when $p = 1$ also assume that $K$ satisfies condition (1). Set $M = \sup\{\|f\|_p : f \in K\}$.

(A) Every sequence $(f_n)$ from $K$ contains a subsequence $(h_n)$ such that $(\int_A h_n \, d\mu)$ is a weak Cauchy sequence in $E$, for every $A \in R$. In fact, using condition (2), for each set $A \in R$ we can find a subsequence $(f_{n'}^\prime)$ such that $(\int_A f_{n'}^\prime \, d\mu)$ is a weak Cauchy sequence in $E$. Arranging the sets of $R$ into a sequence and using the diagonal process, we can find a subsequence $(h_n)$ such that for each $A \in R$, the sequence $(\int_A h_n \, d\mu)$ is weak Cauchy in $E$.

(B) If $(h_n)$ is the sequence obtained above, then $(\int h_n, g \, d\mu)$ is a Cauchy sequence of scalars for every function $g \in L_{\Sigma}(\mu), (1/p) - (1/q) = 1$. To see this note first that every step function $g \in S_{\Sigma}(R)$ of the form $\sum_{i=1}^n \varphi_i x_i^* \mu$ with $A_i \in R$ and $x_i^* \in E^*$, the sequence of scalars $(\int h_n, g \, d\mu)$ is convergent. Consider first the case $1 < p < \infty$ and let $g \in L_{\Sigma}(\mu)$. Let $\varepsilon > 0$ and let $g_\varepsilon \in S_{\Sigma}(R)$ be such that $\|g - g_\varepsilon\|_q < \varepsilon$. Then

$$\left| \int \langle h_n, g \rangle \, d\mu - \int \langle h_m, g \rangle \, d\mu \right| \leq \left| \int \langle h_n, g_\varepsilon \rangle - \langle h_m, g_\varepsilon \rangle \, d\mu \right|$$

$$+ \left| \int \langle h_n, g - g_\varepsilon \rangle \, d\mu \right|$$

$$+ \left| \int \langle h_m, g - g_\varepsilon \rangle \, d\mu \right|.$$ 

If $n_\varepsilon$ is an integer such that for $n, m \geq n_\varepsilon$, we have

$$\left| \int \langle h_n, g_\varepsilon \rangle \, d\mu - \int \langle h_m, g_\varepsilon \rangle \, d\mu \right| < \varepsilon,$$
then for $n, m \geq n_e$ we have also

$$\left| \int \langle h_n, g \rangle \, d\mu - \int \langle h_m, g \rangle \, d\mu \right| < \epsilon + 2M\epsilon,$$

which proves the assertion for $1 < p < \infty$.

Consider now the case $p = 1$ and let $g \in L^1_\mathcal{E}(\mu)$; there exists a sequence $(g_n)$ from $S_\mathcal{E}(\mu)$ such that $g_n \leq |g|$ and $g_n \to g$ a.e. $\mu$, since $g$ is $\mu$ measurable. From condition (1) we deduce that there exists a bounded control measure $\lambda$ on $\Sigma$ (see [4, Theorem 2.3]) such that $\int |h_n| \, d\mu \leq \lambda$ uniformly and $\lambda(A) = 0$ if and only if $\int_A |h_n| \, d\mu = 0$ for all $n$. Since $\mu(A) = 0$ implies $\lambda(A) = 0$, it follows that $g_n$ and $g$ are also $\lambda$ measurable.

Let $\epsilon > 0$; let $\delta > 0$ be such that $A \in \Sigma$ and $\lambda(A) < \delta$ implies $\int_A |h_n| \, d\mu < \epsilon$ for all $n$. If we apply Egorov's theorem, we can find a subset $X_0 \subset \Sigma$ such that $\lambda(X - X_0) < \delta$ and $g_n \to g$ uniformly on $X_0$. Then $\int_{X - X_0} |h_n| \, d\mu < \epsilon$ for all $n$. Let $g_\epsilon$ be a function from the sequence $(g_n)$ such that $|g - g_\epsilon| < \epsilon$ on $X_0$.

Then

$$\left| \int \langle h_n, g - g_\epsilon \rangle \, d\mu \right| \leq \left| \int_{X_0} \langle h_n, g - g_\epsilon \rangle \, d\mu \right| + \left| \int_{X - X_0} \langle h_n, g - g_\epsilon \rangle \, d\mu \right| \leq \epsilon \int_{X_0} |h_n| \, d\mu + \|g - g_\epsilon\|_\infty \int_{X - X_0} |h_n| \, d\mu$$

$$\leq \epsilon M + 2\|g\|_\infty \epsilon.$$

Since $(\int \langle h_n, g \rangle \, d\mu)$ is convergent, let $n_\epsilon$ be such that for $n, m \geq n_\epsilon$ we have

$$\left| \int \langle h_n, g_\epsilon \rangle \, d\mu - \int \langle h_m, g_\epsilon \rangle \, d\mu \right| < \epsilon.$$

It follows that for $n, m \geq n_\epsilon$ we have

$$\left| \int \langle h_n, g \rangle \, d\mu - \int \langle h_m, g \rangle \, d\mu \right| \leq \left| \int \langle h_n, g_\epsilon \rangle \, d\mu - \int \langle h_m, g_\epsilon \rangle \, d\mu \right| + \left| \int \langle h_n, g - g_\epsilon \rangle \, d\mu \right|$$

$$+ \left| \int \langle h_m, g - g_\epsilon \rangle \, d\mu \right| \leq \epsilon + 2\epsilon M + 4\|g\|_\infty \epsilon,$$

which proves the assertion for $p = 1$.

(B') If $(h_n)$ is a sequence from $K$ and $h \in L^1_\mathcal{E}(\mu)$ is such that for each $A \in R$ we have $\int_A h_n \, d\mu \to \int_A h \, d\mu$ weakly in $E$, then for every $g \in L^1_\mathcal{E}(\mu)$ we have...
\[ \int \langle h_n, g \rangle \, d\mu \rightarrow \int \langle h, g \rangle \, d\mu. \] In fact, this follows immediately for step functions \( g \in S_E^*(R) \) and then for every function \( g \in L_{E^*}^p(\mu) \), as in the proof of step (B), replacing \( h_m \) by \( h \).

(C) From steps (A) and (B) we deduce that if \( L_{E^*}^p(\mu)^* = L_{E^*}^q(\mu) \), then \( K \) is conditionally weakly compact. In particular, this is true if \( E^* \) has property \( R-N \), or if every point has strictly positive measure, for example \( I_{E^*} \).

In this way the first parts of statements I(b) and II(b) are proved.

(D) Assume now that in condition (2) the sets \( K(A) \) are relatively weakly compact, and that \( E \) and \( E^* \) have property \( R-N \) (or that \( L_{E^*}^p(\mu) = I_{E^*} \)). To see that \( K \) is relatively weakly compact, let \( (f_n) \) be a sequence from \( K \). By step (C) the set \( K \) is conditionally weakly compact, therefore \( (f_n) \) contains a weak Cauchy subsequence \( (h_n) \). Since for each \( A \in \Sigma_f = \{ B \in \Sigma : \mu(B) < \infty \} \), the set \( K(A) \) is relatively weakly compact in \( E \), the sequence \( (\int_A h_n \, d\mu) \) converges weakly to an element \( \sigma(A) \in E^* \):

\[ \langle \sigma(A), x^* \rangle = \lim_{n \to \infty} \int_A \langle h_n, x^* \rangle \, d\mu, \quad \text{for} \quad x^* \in E^*. \]

The set function \( \sigma : \Sigma_f \to E \) is finitely additive, and weakly \( \sigma \) additive by a theorem of Nikodym, since \( \langle \sigma(\cdot), x^* \rangle \) is the limit of the sequence of \( \sigma \) additive measures \( \int_A \langle h_n, x^* \rangle \, d\mu \). By Pettis' theorem [14], \( \sigma \) is strongly \( \sigma \) additive on \( \Sigma_f \). The measure \( \sigma \) has finite variation on every set \( A \in \Sigma_f \). In fact, if \( (B_1, \ldots, B_k) \) is a finite family of disjoint sets from \( \Sigma \) contained in \( A \), then

\[
\sum_{i=1}^k |\sigma(B_i)| = \sum_{i=1}^k \sup_{\|x^*\| \leq 1} \left| \lim_{n \to \infty} \int_{B_i} \langle h_n, x^* \rangle \, d\mu \right| \\
\leq \sum_{i=1}^k \lim \inf_{n \to \infty} \int_{B_i} |h_n| \, d\mu \\
\leq \lim \inf_{n \to \infty} \sum_{i=1}^k \int_{B_i} |h_n| \, d\mu \\
\leq \sup_{n \to \infty} \int_A |h_n| \, d\mu \leq \sup_{n \to \infty} \|h_n\|_p \|\varphi_A\|_q.
\]

For \( 1 < p < \infty \), \( |\sigma(A)| \leq M\mu(A)^{1/p} < \infty \), and it follows that \( \sigma \leq \mu \) on \( \Sigma_f \).

For \( p = 1 \), the finiteness of \( |\sigma(A)| \) also follows and the uniform \( \sigma \) additivity of \( K \) yields the absolute continuity of \( \sigma \) with respect to \( \mu \) on \( \Sigma_f \). If \( E \) has property \( R-N \) (or if \( L_{E^*}^p(\mu) = I_{E^*} \)), we can apply the Radon–Nikodym theorem for every set \( A \in \Sigma_f \) of finite measure, and find a Bochner integrable function \( f_A : X \to E \) vanishing outside \( A \) and satisfying \( \sigma(B) = \int_B f_A \, d\mu \), for \( B \in A \cap \Sigma \). Since \( X \) is a countable union of sets of finite measure, we can find a function \( f : X \to E \), such that \( h = f_A \) a.e. \( \mu \) for every \( A \in \Sigma_f \). We shall prove now that \( h \in L_{E^*}^p(\mu) \).

For every set \( A \in \Sigma \) of finite measure and every \( x^* \in E^* \), we have

\[
\lim_{n \to \infty} \int_A \langle h_n, x^* \rangle \, d\mu = \sigma(A) x^* = \int_A \langle h, x^* \rangle \, d\mu;
\]
hence, for every step function \( g = \sum A_i x_i^* \) with \( \mu(A_i) < \infty \) we have
\[
\lim_n \int \langle h_n, g \rangle \, d\mu = \int \langle h, g \rangle \, d\mu,
\]
therefore
\[
\left| \int \langle h, g \rangle \, d\mu \right| = \lim_n \left| \int \langle h_n, g \rangle \, d\mu \right| \leq M \| g \|_q .
\]

If \( g \in L^q_E(\mu) \), we can find a sequence \( (g_n) \) of step functions of the above form with
\[
|g_n| \leq |g| \quad \text{and} \quad g_n \to g \text{ a.e. } \mu.
\]
Then \( \langle h, g_n \rangle \to \langle h, g \rangle \), a.e. \( \mu \) and
\[
\liminf_n \int |\langle h, g_n \rangle| \, d\mu \leq M \| g_n \|_q \leq M \| g \|_q < \infty .
\]

By Fatou's lemma, \( |\langle h, g_n \rangle| \) is \( \mu \)-integrable. Since \( h \in L^p_E(\mu) \) if and only if
\( \langle h, g \rangle \in L^q(\mu) \) for each \( g \in L^q_E(\mu) \) [10, p. 233], we deduce that \( h \in L^p_E(\mu) \). From
\[
\lim_n \int_A \langle h_n, x^* \rangle \, d\mu = \int_A \langle h, x^* \rangle \, d\mu, \quad \text{for } A \in \Sigma_f, \quad x^* \in E^*,
\]
and from step (B'), we conclude that
\[
\lim_n \int \langle h_n, g \rangle \, d\mu = \int \langle h, g \rangle \, d\mu, \quad \text{for } g \in L^q_E(\mu).
\]

Since the hypothesis that \( E^* \) has property \( R-N \) (or that \( L^p_E(\mu) = L^p(\mu) \)) implies that \( L^p_E(\mu)^* = L^q_E(\mu) \), we deduce that \( h_n \to h \) weakly in \( L^p_E(\mu) \), hence \( K \) is relatively weakly compact. This proves statements I(b) and II(b). Later in this proof we shall need statements I(b) and II(b) in the particular case of the spaces
\( L^p_E \), which we record in Theorem 2, infra.

I(a), II(a). We shall now prove statements I(a) and II(a). Assume \( K \subset L^p_E(\mu), 1 < p < \infty \) is bounded; if \( p > 1 \), \( K \) satisfies conditions (2) and when \( p = 1 \), also assume that \( K \) satisfies condition (1).

(E) If \( \pi = (A_i)_{i \in I} \) is a \( \mu \)-partition, then the mapping \( \theta : U_\pi(L^p_E(\mu)) \to L^p_E(I) \) defined by
\[
\theta(f_\pi) = (c_i \int_{A_i} f \, d\mu)_{i \in I}, \quad c_i = (\mu(A_i))^{-1/p},
\]
where \( c_i = (\mu(A_i))^{-1/p} \) is a surjective isometry. In fact
\[
\| \theta f_\pi \|_p \leq \sum_i |c_i|^{-p} \left| \int_{A_i} f \, d\mu \right|^p = \left| \int \left[ \sum_i (\mu(A_i))^{-1/p} \int_{A_i} f \, d\mu \right] \varphi_{A_i} \right|^p \, d\mu = \| f_\pi \|_p .
\]

Hence \( \theta \) is an isometry. \( \theta \) is also surjective since for \( x = (x_i)_{i \in I} \) from \( L^p_E(I) \) choose \( f = \sum_{i \in I} \varphi_{A_i} x_i(\mu(A_i))^{-1/p} \). It follows that \( f_\pi = f \in L^p_E(\mu) \) and \( \theta(f_\pi) = x \). Thus \( U_\pi(L^p_E(\mu)) \) is a Banach space.

(F) For any \( \mu \)-partition \( \pi = (A_i)_{i \in I} \), the set \( K_\pi = \{ f_\pi : f \in K \} \) is conditionally (respectively relatively) weakly compact in \( L^p_E(\mu) \). In fact the set \( \theta(K_\pi) \) in \( L^p_E \) satisfies the conditions of Theorem 2 infra as follows. Since \( \theta(K_\pi)(i) = 607/24/2-6 \)
\{c_i \int_{A_i} f \, d\mu : f \in K \} = c_i K(A_i)$, the set $\theta(K_n)$ satisfies condition (2) of Theorem 2; in the case $p = 1$, $\theta(K_n)$ satisfies condition (1) of Theorem 2:

$$\sum_{i \geq k} \left| \int_{A_i} f \, d\mu \right| \leq \sum_{i \geq k} \int_{A_i} |f| \, d\mu = \int_{U_{i \geq k}} |f| \, d\mu \to 0$$

uniformly for $f \in K$. Finally, if $1 < p < \infty$, $\theta(K_n)$ is conditionally (respectively relatively) weakly compact in $l^p$.

Since $\theta$ is an isometry, $K_n$ has the same property in $U_p((L^p)(\mu))$. Recall that $U_p((L^p)(\mu))$ is complete. It then follows that $K_n$ is conditionally (respectively relatively) weakly compact in $l^p$. Note that if $\pi$ is finite and $p = 1$, the set $\theta(K_n)$ is conditionally (respectively relatively) weakly compact without requiring condition (1) of Theorem 2.

(G1) Assume now that $K$ also satisfies condition (3). We shall now show that $K$ is conditionally (respectively relatively) weakly compact.

To see this, consider the following situation. Suppose $S$ is a set and $f, f_n : S \to Y$ are functions, $n = 1, 2, ..., $ such that each $f_n$ has conditionally weakly compact range. We assert that if $f_n \to f$ uniformly in the weak topology of $Y$, where $Y$ is a Banach space, then $f(S)$ is conditionally weakly compact. To prove this, let $(s_i)$ be a sequence of points from $S$. We shall show that $(f(s_i))$ contains a weak Cauchy subsequence. Since $f_k(S)$ is conditionally compact, there is a subsequence such that $(f_k(s_i'))$ is weak Cauchy; using the diagonal process, we can extract from $(s_i)$ a subsequence $(t_i)$ such that for every $n$, $(f_n(t_i))$ is weak Cauchy. Hence $\lim_i \langle f_n(t_i), x^* \rangle$ exists for each $x^* \in Y^*$ and each $n$. By hypothesis, for each $x^* \in Y^*$, we have $\lim_i \langle f_n(t_i), x^* \rangle = \langle f(t_i), x^* \rangle$ uniformly for $i$. By the standard interchange of limit theorem, the following limit exists:

$$\lim_i \langle f(t_i), x^* \rangle ;$$

hence $(f(t_i))$ is a weak Cauchy sequence, which in turn shows that $f(S)$ is conditionally weakly compact.

(G2) Suppose in the above setting we further impose the conditions that each $f_n$ has relatively weakly compact range and $f_n(s) \to f(s)$ uniformly for $s \in S$ in the norm topology of $Y$. We assert that $f(S)$ is relatively weakly compact. To see this, by the above argument we can assume that given $(s_i)$ from $S$ there exists a subsequence $(t_i)$ and $(x_n)$ from $Y$ such that $\lim_i \langle f_n(t_i), x^* \rangle = \langle x_n, x^* \rangle$ for each $n$ and $x^* \in Y^*$; hence $\lim_i \langle f_n(t_i), x^* \rangle = \lim_n \langle x_n, x^* \rangle$ for each $x^*$. We now show that $(x_n)$ is Cauchy in the norm topology. Let $\epsilon > 0$ and let $n, m \geq n$ be such that $| f_n(s) - f_m(s) | < \epsilon$ for $n, m \geq n$ and $s \in S$. Then for $| x^* | \leq 1$, we have $| \langle f_n(t_i) - f_m(t_i), x^* \rangle | \leq \epsilon$ for all $t_i$ and $n, m \geq n$. Letting $i \to \infty$, we obtain $| \langle x_n - x_m, x^* \rangle | \leq \epsilon$ for all $n, m \geq n$, and $| x^* | \leq 1$, therefore $| x_n - x_m | \leq \epsilon$ for $n, m \geq n$. Let $\lim_n x_n = x$. Then $\lim_i \langle f(t_i), x^* \rangle = \lim_n \langle x_n, x^* \rangle$, hence $(f(t_i))$ converges weakly to $x$.

(G3) Let $f_n = U_n / K_0$, where $K_0$ is a countable subset of $K$ and $f$ is
the identity map on $K_0$. From $(G_1)$ and $(G_2)$ we conclude that $f(K_0)$ is conditionally (respectively relatively) weakly compact; hence $K$ has the same property. Thus I(a) and II(a) are proved.

I(c), II(c). To prove statements I(c) and II(c), assume that $K$ is conditionally (respectively relatively) weakly compact. Then $K$ is bounded. From the continuity of the mapping $f \mapsto \int_A f \, d\mu$ of $L^p_\Sigma(\mu)$ into $E$, where $A \in \Sigma_f$, we see that $K$ satisfies condition (2). This proves II(c).

(H) In the case $p = 1$, we have to prove that $K$ satisfies condition (1). If we deny it, there exists an $\epsilon > 0$, a decreasing sequence $(A_n)$ with $A_n \searrow \emptyset$, and a sequence $(f_n)$ from $K$ satisfying

$$
\int_{A_n} |f_n| \, d\mu > \epsilon
$$

for all $n$. (*)

Since $K$ is conditionally weakly compact $(f_n)$ contains a weak Cauchy subsequence; so we may consider $(f_n)$ itself a weak Cauchy sequence. If the $f_n$ are scalar valued, for every $A$ we have $\varphi_A \in L^\infty(\mu)$ and the limit $\sigma(A) = \lim_n \int_A f_n \, d\mu$ existing; by the Nikodym theorem the indefinite integrals $\int_A f_n \, d\mu$ are uniformly $\sigma$-additive; hence the indefinite integrals $\int_A |f_n| \, d\mu$ are also uniformly $\sigma$ additive, which contradicts (*). Consider now the general case of Banach-valued functions $f_n$. The sets $B_n = A_n - A_{n+1}$ are disjoint. Let $\epsilon > 0$. Since $\varphi_{B_n} f_n \in L^1(\mu)$, there exists a step function $g_n \in L^\infty(\mu)$ with $\|g_n\| \leq 1$ and

$$
\int_{B_n} \langle f_n, g_n \rangle \, d\mu > \int_{B_n} |f_n| \, d\mu - \epsilon/2^{n+1},
$$

by [10, p. 228, Theorem 5]. The function $g = \sum_{n=1}^{\infty} h_n \varphi_{B_n}$ belongs to $L^\infty(\mu)$ and for each $n$ we have

$$
\int_{B_n} \langle f_n, g \rangle \, d\mu = \int_{B_n} \langle f_n, g_n \rangle \, d\mu > \int_{B_n} |f_n| \, d\mu - \epsilon/2^{n+1};
$$

consequently, since $A_n = \bigcup_{i \geq n} B_i$,

$$
\int_{A_n} \langle f_n, g \rangle \, d\mu = \sum_{i \geq n} \int_{B_n} \langle f_n, g \rangle \, d\mu > \int_{A_n} |f_n| \, d\mu - \epsilon/2 > \epsilon.
$$

Note that $\langle f_n, g \rangle$ is a weak Cauchy sequence in $L^1(\mu)$; therefore, by the first part of the proof of this step, the indefinite integrals $\int_A \langle f_n, g \rangle \, d\mu$ are uniformly $\sigma$ additive, which contradicts (**). Thus I(c) is proved.

I(d), II(d). We finally prove statements I(d) and II(d).

(J) We shall first prove that condition (4) implies condition (3) with weak convergence. Let $K_0 \subset K$ be a countable set. We may assume, as we mentioned at the beginning, that $\Sigma$ is generated by a countable ring $R$ of sets of
finite measure. The net \((\pi)\) of all finite partitions with sets from \(R\) is countable. Consider it arranged in a sequence \((\pi_n')\). We choose \(\pi_1 = \pi_1'\), then \(\pi_2 \geq \pi_1'\), \(i = 1, 2\), and in general \(\pi_n \geq \pi_i'\), \(i = 1, \ldots, n\). Then \((\pi_n)\) is a sequence cofinal with respect to the net \((\pi)\). Consequently if condition (4) holds, then \(f_n \to f\) weakly in \(L_{E^\infty}(\mu)\), uniformly for \(f \in K_0\); hence \(f_n \to f\) weakly uniformly for \(f \in K_0\), so that condition (3) is satisfied with weak convergence.

\((K)\) Assume now that \(K \subseteq L_{E^\infty}(\mu)\) is conditionally weakly compact and that \(E^\infty\) has property \(R \to N\); we shall show that condition (4) holds. By I(c), the indefinite integrals \(|K| = \{\int f \, d\mu : f \in K\}\) are uniformly \(\sigma\) additive, therefore there exists a bounded control measure \(\lambda \geq 0\), such that \(K \subseteq \lambda\) uniformly, and 
\(\lambda(A) = 0\) if and only if \(\int_A f \, d\mu = 0\) for all \(f \in K\). In particular \(\lambda \leq \mu\), so that \(\mu\) measurability implies \(\lambda\) measurability. Let \(g \in L_{E^\infty}(\mu)\) and let \((g_n)\) be a sequence of step functions from \(S_{\infty}(R)\) with \(\|g_n\| \leq \|g\|\) and \(g_n \to g\) a.e. Let \(\epsilon > 0\) and let \(\delta > 0\) be such that \(A \in \Sigma\) and 
\(\lambda(A) < \delta\) imply \(\int_A |f| \, d\mu < \epsilon\) for all \(f \in K\). Applying Egorov’s theorem, we can find a set \(X_0 \subseteq X\) with 
\(\lambda(X - X_0) < \delta\) such that \(g_n \to g\) uniformly on \(X_0\). Then \(\int_{X \setminus X_0} |f| \, d\mu < \epsilon\) for all \(f \in K\).

Let \(g_\varepsilon\) be a function from the sequence \((g_n)\) with \(\|g - g_\varepsilon\| < \epsilon\) on \(X_0\). Let \(\pi_\varepsilon = (A_1, \ldots, A_n)\) be the partition determined by \(g_\varepsilon\) (note that \(A_i \in R\)). For any partition \(\pi \geq \pi_\varepsilon\) we have

\[
\left| \int \langle f_\pi, g - g_\varepsilon \rangle \, d\mu \right| \leq \int_{X \setminus X_0} |f_\pi| \, |g - g_\varepsilon| \, d\mu + \int_{X_0} |f_\pi| \, |g - g_\varepsilon| \, d\mu
\leq \|g - g_\varepsilon\|_\infty \int_{X \setminus X_0} |f| \, d\mu + \epsilon \int |f| \, d\mu
\leq 2 \|g\|_\infty \epsilon + M\epsilon.
\]

Similarly,

\[
\left| \int \langle f, g - g_\varepsilon \rangle \, d\mu \right| \leq 2 \|g\|_\infty \epsilon + M\epsilon.
\]

Since

\[
\int \langle f_\pi, g_\varepsilon \rangle \, d\mu = \int \langle f, g_\varepsilon \rangle \, d\mu,
\]

we deduce that

\[
\left| \int \langle f_\pi, g \rangle \, d\mu - \int \langle f, g \rangle \, d\mu \right|
\leq \left| \int \langle f_\pi, g - g_\varepsilon \rangle \, d\mu \right| + \left| \int \langle f_\pi, g_\varepsilon \rangle \, d\mu - \int \langle f, g_\varepsilon \rangle \, d\mu \right|
+ \left| \int \langle f, g - g_\varepsilon \rangle \, d\mu \right| \leq 4 \|g\|_\infty \epsilon + M\epsilon.
\]
Consequently, for any \( g \in L_{E^*}(\mu) \) we have

\[
\lim_{\pi} \int \langle f, g \rangle \, d\mu = \int \langle f, g \rangle \, d\mu,
\]

uniformly for \( f \in K \), where \( (\pi) \) is any net containing the finite partitions over \( R \). Since \( E^* \) has property \( R-N \), this means that \( f_{\pi} \to f \) weakly uniformly for \( f \in K \). Thus condition (4) is proved. From step (J) it follows that statement I(c) is proved.

(I.) Assume \( K \subset L_{E^p}(\mu) \), \( 1 < p < \infty \) is bounded and \( E^* \) has property \( R-N \); we shall prove that condition (4) holds. Let \( g \in L_{E^p}(\mu) \) and let \( \epsilon > 0 \). Let \( g_\epsilon \in S_{E^p}(R) \) be a step function with \( \| g_\epsilon - g \|_q < \epsilon \). If \( \pi_\epsilon \) is the partition determined by \( g_\epsilon \) and if \( \pi \supset \pi_\epsilon \), then

\[
\left| \int \langle f, g \rangle \, d\mu - \int \langle f, g_\epsilon \rangle \, d\mu \right| \leq \left| \int \langle f, g - g_\epsilon \rangle \, d\mu \right| + \left| \int \langle f, g - g_\epsilon \rangle \, d\mu \right| 
\leq \| f \|_p \epsilon + \| f \|_p \epsilon \leq 2M\epsilon,
\]

since \( \int \langle f, g_\epsilon \rangle \, d\mu = \int \langle f, g \rangle \, d\mu \). It follows that \( \lim_{\pi} \int \langle f, g \rangle \, d\mu = \int \langle f, g \rangle \, d\mu \), uniformly for \( f \in K \), that is, condition (4) is satisfied. From step (J) we see that II(c) is proved. This concludes the proof of the theorem.

4. Consequences of Theorem 1

Theorem 2. (I) A set \( K \subset l_{E^1} \) is conditionally (respectively relatively) weakly compact if and only if the following two conditions are satisfied:

(i) \( \sum_{i=1}^{\infty} |x_i| \to 0 \) as \( k \to \infty \), uniformly for \( x \in K \);
(ii) \( K(i) = \{ x_i : x = (x_i) \in K \} \) is conditionally (respectively relatively) weakly compact in \( E \) for each \( i \).

(II) A set \( K \subset l_{E^p} \), \( 1 < p < \infty \), is conditionally (respectively relatively) weakly compact if and only if \( K \) is bounded and satisfies condition (ii).

Proof. We remark first that conditions (i) and (ii) imply that \( K \) is bounded in \( l_{E^1} \). Note also that \( (l_{E^p})^* = l_{E^1} \), \( 1 \leq p < \infty \), without any assumption on \( E^* \), and that \( l_{E^p} = L_{E^p}(\mu) \), where \( \Sigma \) is the power set of the integers and \( \mu \) is the counting measure. The Radon–Nikodym theorem holds for any measure \( \psi : \Sigma \to E \) of finite variation. Finally, (i) and (ii) are equivalent to (1) and (2), respectively, of Theorem 1. With these remarks in mind, I follows from I(b) and II(c) of Theorem 1; II follows from II(b) and II(c) of Theorem 1.

Corollary 3. The space \( l_{E^p} \), \( 1 \leq p < \infty \), is weakly sequentially complete if and only if \( E \) is weakly sequentially complete.
Proof. We shall just sketch the proof for a particular case. Suppose that \((\tau_n)\) is a weak Cauchy sequence in \(l^1\). Then it follows that the set \(K = (\tau_n)\) is conditionally compact in \(l^1\), hence \(K\) satisfies (i), and since if we assume that \(E\) is weakly sequentially complete, condition (ii) with relative weak compactness is satisfied. Hence \(K\) is relatively weakly compact by Theorem 2. There exists an element \(\tau\) in \(l^1\) and a subsequence \((n_i)\) such that \(\tau_{n_i} \rightarrow \tau\) weakly in \(l^1\). It then follows that \(\tau_{n_i} \rightarrow \tau\) weakly. We omit the rest of the proof.

Remarks. In general, if \(L^p(\mu)\) is weakly sequentially complete for some \(p \geq 1\), then \(E\) is weakly sequentially complete. We do not know if the converse is true. If \(E\) is reflexive, then \(L^1(\mu)\) is weakly sequentially complete (see [17]). If \(E\) is reflexive and \(1 < p < \infty\), then \(L^p(\mu)\) is reflexive, hence weakly sequentially complete. Also, if \(E = L^1(\nu)\), then \(L^1(\mu)\) is weakly sequentially complete since \(L^1(\mu) = L^1(\nu \times \mu)\).

From the proof of Theorem 1, we can extract the following convergence criteria for sequences in \(L^p(\mu)\), endowed with the pairing topology \(\sigma(L^p(\mu), L^q(\mu))\), \((1/p) + (1/q) = 1\).

Theorem 4. Let \(A\) be an algebra generating \(\Sigma\).

(i) Let \(f_n, f \in L^1(\mu)\). The sequence \((\int f_n \cdot g \, d\mu)\) is convergent (respectively converges to \(\int f \cdot g \, d\mu\)) for every \(g \in L^1(\mu)\), if and only if:

1. \(\int |f_n| \, d\mu\) are uniformly \(\sigma\) additive on \(\Sigma\);
2. \((\int_A f_n \, d\mu)\) is a weak Cauchy sequence in \(E\) (respectively converges weakly to \(\int_A f \, d\mu\)) for every \(A \in A\) with \(\mu(A) < \infty\).

(ii) Let \(f_n, f \in L^p(\mu)\). The sequence \((\int f_n \cdot g \, d\mu)\) is convergent (respectively converges to \(\int f \cdot g \, d\mu\)) for every \(g \in L^q(\mu)\) if and only if \((f_n)\) is bounded and satisfies condition (2).

Corollary 5. (i) A sequence \((x^n)\) from \(l^1\) is weak Cauchy (respectively converges weakly to \((x_i) \in l^1\)) if and only if:

1. \(\sum_{k \geq 1} |x_{i_k}^n| \rightarrow 0\) as \(k \rightarrow \infty\), uniformly for \(n = 1, 2, \ldots\);
2. \((x_{i_k}^n)_{1 \leq k \leq n}\) is weak Cauchy in \(E\) (respectively converges weakly to \(x_i\)) for every \(i\).

(ii) A sequence \((x^n)\) from \(l^p\), \(1 < p < \infty\) is weak Cauchy (respectively converges weakly to \((x_i) \in l^p\)) if and only if \((x^n)\) is bounded and satisfies condition (2).

The proof of the following lemma is contained in steps \((G_1)\) and \((G_2)\) of the proof of Theorem 1.

Lemma 6. Let \(S\) be a set, \(E\) a Banach space, and \(f_n, f : S \rightarrow E\) functions, \(n =
Assume that each $f_n$ has conditionally weakly compact range $f_n(S)$. If $f_n \to f$ uniformly on $S$ in the weak topology of $E$, then $f(S)$ is conditionally weakly compact. If, in addition, each $f_n(S)$ is relatively weakly compact and $f_n \to f$ on $S$ in the norm topology, then $f(S)$ is relatively weakly compact.

Remarks. (1°) If $f_n \to f$ weakly uniformly on $S$ and each $f_n(S)$ is relatively weakly compact but $E$ is not weakly sequentially complete, then $f(S)$ is not necessarily relatively weakly compact, as the following example shows. Let $E = c_0$ and let $S$ be the closed unit ball of $c_0$, which is not weakly compact. Let $f_n((a_1, a_2, \ldots)) = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ and let $f$ be the identity map. Then $f_n \to f$ weakly uniformly on $S$ and each $f_n(S)$ is weakly compact, but $f(S)$ is not relatively weakly compact.

5. The Spaces of Measures and Applications

Let $R$ be a ring of subsets of $X$ and $\text{fabv}(R, E)$ will denote the space of finitely additive measures $m: R \to E$ with bounded variation. Endowed with the total variation norm, $\|m\| = |m|(X)$, $\text{fabv}(R, E)$ is a Banach space. Let $\mu \geq 0$ be a finitely additive finite measure on $R$ (not necessarily bounded). We shall consider $\mu$ partitions $\pi = (A_i)_{i \in I}$ of disjoint sets from $R$ with $\mu(A_i) > 0$. Let $\pi = (A_i)_{i \in I}$ be a $\mu$ partition and let $m \in \text{fabv}(R, E)$. We set $U_\pi(m) = m_\pi = \sum_{i \in I} (\mu(A_i))^{-1} m(A_i) \mu_{A_i}$, where $\mu_{A_i}$ is the measure defined on $R$ by $\mu_{A_i}(A) = \mu(A \cap A_i)$, for $A \in R$. The preceding sum is well defined and one can show that the family $\{(\mu(A_i))^{-1} m(A_i) \mu(A \cap A_i): i \in I\}$ is absolutely summable for every $A \in R$. The measure $m_\pi$ belongs to $\text{fabv}(R, E)$, $|m_\pi| \leq |m|$, and $\|m_\pi\| \leq \|m\|$.

Theorem 7. Let $K \subset \text{fabv}(R, E)$. Consider the following conditions:

1. $K = \{m \mid m \in K\}$ is uniformly strongly additive; that is, $m(A_i) \to 0$ uniformly for $m \in K$, whenever $(A_i)$ is a disjoint sequence.

2. $K(A) = \{m(A) \mid m \in K\}$ is conditionally (respectively relatively) weakly compact in $E$, for every $A \in R$;

3. For every countable set $K_0 \subset K$, there exists a sequence $(\pi_n)$ of partitions such that $m_{\pi_n} \to m$ weakly (respectively strongly) in $\text{fabv}(R, E)$, uniformly for $m \in K_0$.

4. $m_n \to m$ weakly in $\text{fabv}(R, E)$, uniformly for $m \in K$, where $(\pi)$ is the net of all finite partitions over $R$.

Then:

(a) If $K$ satisfies conditions (1), (2), and (3), then $K$ is conditionally (respectively relatively) weakly compact.
(b) If \( K \) is bounded and satisfies conditions (1) and (2) and if \( E \) and \( E^* \) have property R–N, then \( K \) is conditionally (respectively relatively weakly compact).

Conversely:

(c) If \( K \) is conditionally (respectively relatively) weakly compact, then \( K \) is bounded and satisfies conditions (1) and (2).

(d) If \( K \) is conditionally weakly compact and \( E \) and \( E^* \) have property R–N, then condition (3) (with weak convergence) and conditions (4) are satisfied.

The proof of this theorem uses the device of passing to the Stone algebra and Theorem 1. We remark that in passing from the Stone algebra \( \Sigma_1 \) to the \( \sigma \) algebra \( \Sigma_2 \) generated by \( \Sigma_1 \), the following technique is employed: Suppose that \( K(A) \) satisfies condition (2), where \( A \in \Sigma_1 \); we must show condition (2) also holds for \( K_2 \), the extensions of \( K \) to \( \Sigma_2 \). To do this let \( \lambda \) be a control measure for \( K_2 \), and let \( A \in \Sigma_2 \). Choose \( A_0 \in \Sigma_1 \) such that \( \lambda(A \Delta A_0) \to 0 \). By using Lemma 6, we can conclude that \( K_2(A) \) is conditionally or relatively weakly compact. We omit the proof of the above theorem and refer the reader to [4] for similar constructions.

As an application, we give criteria for operators on the space of continuous functions to be weakly compact. Let \( X \) be a compact space, let \( E \) and \( F \) be Banach spaces, and let \( B \) be the family of Borel sets of \( X \). For each finitely additive measure \( m: B \to L(E, F) \), where \( L(E, F) \) is the space of operators from \( E \) into \( F \), we consider the semivariation \( m(A) = \sup |\sum_i m(A_i) x_i| \), where the supremum is extended over all finite partitions of \( A \) into measurable sets \( A_i \) and all \( x_i \) in the unit ball of \( E \). We set \( \| m \|_v = \tilde{m}(X) \). Note that \( \tilde{m}(X) = \sup m_z |(X) \), where the supremum is extended over all \( z \) in the unit ball of \( F^* \), and \( m_z : B \to F^* \) is the measure defined by \( \langle x, m_z(A) \rangle = \langle m(A) X, z \rangle \), for \( A \in B \), \( x \in E \), and \( z \in F^* \). We have \( m_z(A) = m(A)^*z \). Let \( C_E(X) \) be the space of continuous \( E \)-valued functions defined on \( X \), endowed with the uniform norm. Let \( T: C_E(X) \to F \) be a continuous operator and \( m \) is its representing measure (in our case \( m: B \to L(E, F) \) since \( m \) will be strongly bounded) such that \( T(f) = \int f \, d m \), for \( f \in C_E(X) \). See [6] for the theory of representations in this setting.

**Theorem 8.** Let \( T: C_E(X) \to F \) be a continuous operator with representing measure \( m \). Suppose the following conditions hold:

1. \( \tilde{m}(A_n) \to 0 \), whenever \( A_n \searrow \phi \);
2. \( m(A) \) is a weakly compact operator in \( L(E, F) \) for \( A \in B \);
3. there exists a net of partitions \( (\pi) \) such that \( \lim_{\pi} \| m_\pi - m \|_v = 0 \) (where \( m \) is defined with respect to a control measure of \( m \) [4]).

Then \( T \) is a weakly compact operator. If \( (\pi) \) consists of finite partitions, then condition (1) is superfluous.
Conversely, if $T$ is weakly compact, then conditions (1) and (2) are satisfied. If, in addition, $E^*$ and $E^{**}$ have property R-N, then for some control measure $\lambda$ for $\mu$ we have $\lim_{\pi} (m_\pi) = m_\pi$ weakly in $cabv(B, E^*)$, uniformly for $z$ in the unit ball of $F^*$, where $(\pi)$ is the net of partitions.

**Proof.** The adjoint operator $T^* : F^* \to cabv(B, E^*)$ maps the unit ball of $F^*, F^*_1$, onto the set $K = (m_\pi)_{\pi \in F^*_1}$. From (1) it follows that $K$ is uniformly $\sigma$ additive; from (2) we see that $m(A)^* : F^* \to E^*$ is weakly compact, hence the set

$$K(A) = \{m_\pi(A) : z \in F^*_1\} = \{m(A)^*_z : z \in F^*_1\}$$

is relatively weakly compact in $E$ for every $A \in B$. Finally, from (3), we deduce that

$$\lim_{\pi} (m_\pi) = m_\pi,$$

uniformly for $z \in F^*_1$.

From Theorem 7 it follows that $K$ is relatively weakly compact; hence $T^*$ is weakly compact; consequently $T$ is weakly compact.

Conversely, if $T$ is weakly compact, then $K$ is relatively weakly compact, and we again use Theorem 7.

The last application deals with operators from a Banach space $F$ into $L^p_{E^*}(\mu)$. We shall say that an operator $L : F \to L^p_{E^*}(\mu)$ is a finite rank operator if $L$ is of the form $L(x) = \sum_{i=1}^n a_i(x) \phi_{E_i}$ for each $x \in F$, where the $a_i : F \to E$. $F_1$ denotes the unit ball of $F$.

**THEOREM 9.** Let $T : F \to L^p_{E^*}(\mu)$, $1 \leq p < \infty$, be a continuous operator.

(I) Suppose that $\Sigma$ is generated by a countable algebra of sets and that $E^*$ has property R-N. In the case $p = 1$, also assume that $T$ is weakly compact. Then there exists a sequence of finite rank operators $T_n : F \to L^p_{E^*}(\mu)$ such that $T_n(x) \to T(x)$ weakly in $L^p_{E^*}(\mu)$ uniformly for $x \in F_1$.

(II) Suppose that $\{\int_A T(x) \, d\mu : x \in F_1\}$ is relatively weakly compact for each $A \in \Sigma$. Suppose that there exists a sequence of operators of the form $T_n(x) = \sum_{A \in \pi_n} [\mu(A_i)]^{-1} \int_{A_i} T(x) \, d\mu \phi_{A_i}$, where $(\pi_n)$ is a sequence of finite $\mu$ partitions, and $T_n(x) \to T(x)$ weakly uniformly for $x \in F_1$. Then $T$ is weakly compact.

**Proof.** Part I follows from Theorem 1 and its proof. In fact, the set $K = T(F_1)$ is relatively weakly compact in $L^p_{E^*}(\mu)$ for $p = 1$ and bounded for $p > 1$. The family of all finite $\mu$ partitions with sets coming from the countable algebra of sets generating $\Sigma$ is also countable. By steps (J), (K), and (L) in the proof of Theorem 1, we can construct a sequence of partitions $\pi_n$ which is cofinal in the net of all finite partitions from the algebra generating $\Sigma$ such that $(T(x))_{\pi_n} \to T(x)$ weakly in $L^p_{E^*}(\mu)$ uniformly with respect to $x \in F_1$. For each $n$, set $T_n(x) = (T(x))_{\pi_n}$. Then the $T_n$ have finite rank and satisfy the required condition. The proof of II follows from Lemma 6 since for each $n$ the restriction of $T_n$ to $F_1$ has relatively weakly compact range.