ASYMPTOTIC BEHAVIOR OF RANDOM DISCRETE EVENT SYSTEMS

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In this paper we discuss some aspects of the asymptotic behavior of Discrete Event Dynamic Systems (DEDS), in which the activity times are random variables. The main result is that a central limit theorem holds for DEDS and consequently that the cycle time of the system is asymptotically normally distributed.

1. Introduction

A large class of dynamic systems—such as the material flow in production or assembly lines, the message flow in communication networks, and jobs in computer systems—can be modeled by Discrete Event Dynamic Systems (DEDS). These are systems where the occurrence of events is determined by the system itself and not described by time. Examples of such events are the beginning or completion of a task in an assembly line or the arrival of a message in a communication network.

Current research on DEDS uses a number of methods. Among these are the logical approach to automata (see, e.g., Wonham and Ramadge, 1988), the perturbation analysis of trajectories (see, e.g., Ho, 1987), simulation, and the temporal approach, which we shall follow in this article (see, e.g., Cohen et al., 1985).

In these models, activity times at a node of, for instance, a production network are successively determined by combining the activity times at other nodes during previous activity cycles with delay and/or transport times. The aim is then to describe the dynamic temporal behavior of the network, given the knowledge of the nature of the delay and transport times and the initial state of the system. An important aspect of the temporal approach is that it permits a conceptual simplification by use of the so-called max-algebra to describe the models, yielding an analogy to conventional system theory. The elements of this max-algebra are the real numbers (together with \(-\infty\)), and the only admissible operations are maximization and addition.

In Cuninghame-Green (1979), a systematic theory parallel to linear algebra has been developed for the max-algebra, and in Cohen et al. (1984, 1985) the use of the max-algebra in the temporal approach to DEDS has been discussed and illustrated.
In Section 2, the models for DEDS and the max-algebra will be introduced and some results on DEDS will be recalled.

At present, only deterministic DEDS have been studied using the temporal approach via the max-algebra. However, in practice, delay and transport times are often of a stochastic nature, either inherently or because of lack of information concerning the precise nature of the system. Here we shall concentrate on models which take this random behavior into account. In particular, we derive for a class of random discrete event systems (see Section 2 for a definition) the expected cycle time and we show that this cycle time is asymptotically normal (see Section 3). Calculations for the expectation and variance of the cycle time are given in several examples in Section 4. Finally, in Section 5 we consider reducible random DEDS and we compare the asymptotic behavior of random DEDS with deterministic DEDS.

2. Discrete event dynamic systems

In this section we shall first introduce the models we use for Discrete Event Dynamic Systems (DEDS) and show how these models can arise. Then we introduce the concept of random DEDS. Finally, some known results on the asymptotic behavior of deterministic DEDS are recalled.

Consider a production network with the following functional description. There is a fixed number, $n$, of nodes in the network. We shall be interested in the time point at which node $i$ ($1 \leq i \leq n$) becomes active (i.e., starts production) for the $k$th time. This time point will be denoted by $x_i(k)$. In order to start the $(k+1)$st activity at node $i$, it is necessary to wait until each node $j$ has finished its $k$th activity and 'supplied' node $i$. As soon as all necessary supplies from the $k$th production cycle are available at node $i$, it becomes active for the $(k+1)$st time. Let $a_{ij}(k)$ denote the sum of the production time at node $j$ in the $k$th cycle and the transport time from node $j$ to node $i$.

Then the above description gives rise to the formula

$$x_i(k+1) = \max_{1 \leq j \leq n} (x_j(k) + a_{ij}(k)).$$

At this stage it is both intuitive and convenient to introduce the max-algebra notation. Following Cuninghame-Green (1979), we define for real numbers $r$ and $s$ the operations $\oplus$ and $\otimes$ respectively by

$$r \oplus s = \max(r, s), \quad r \otimes s = r + s.$$

The reason for using the symbols $\oplus$ and $\otimes$ is that a number of results from conventional linear algebra and system theory can be 'transferred' to the max-algebra and DEDS by replacing the $+$ and $\times$ signs by $\oplus$ and $\otimes$, respectively. Formula (2.1) for the $n$-vector $x(k+1)$ of $(k+1)$st activity times then becomes

$$x(k+1) = A(k) \otimes x(k),$$

with the natural definition of matrix multiplication.
For many purposes the above autonomous formulation of a dynamic system is too restrictive: one must add a vector \( u(k) \) of outside resource times and one should consider a general output vector \( y(k) \), obtained from \( x(k) \) by adding production and transport times. In this manner one arrives at the general form:

\[
\begin{align*}
x(k+1) &= (A(k) \otimes x(k)) \oplus (B(k) \otimes u(k)), \\
y(k) &= C(k) \otimes x(k)
\end{align*}
\]

(2.3)
of a linear DEDS. The reader is referred to Cohen et al. (1984, 1985) for a detailed description and discussion.

In this paper we shall be interested in the asymptotic behavior of the time vector \( x(k) \). Therefore we assume that \( C(k) \) is the identity matrix in the max-algebra and that all outside resources are available at the start of the process, so that the term \( B(k) \otimes u(k) \) disappears. This allows us to concentrate on the behavior of the system as a function of the matrices \( A(k) \), which we shall assume to be of a stochastic nature.

**Definition 2.1.** Let \((A(k))_{k \geq 0}\) in (2.2) be a sequence of independent, identically distributed (i.i.d.) real \( n \times n \) matrices and let an initial random vector \( x(0) \) be given independent of \((A(k))_{k \geq 0}\). Then the system, which is described by (2.2), will be called a *random discrete event dynamical system*. (A similar definition can be given when the general form (2.3) is used.)

We shall assume that the matrices \( A(k) \) are real-valued and finite with probability one. Our goal is to show under suitable conditions that the sequence \( x(k) \) is asymptotically normal and to give examples of explicit calculations (for \( n = 2 \)) of the asymptotic mean and variance, thereby determining the average cycle time and the nature of the deviation from this average during long-term operation of the system.

Next, we briefly describe some results of Cohen et al. (1984, 1985) concerning the behavior of deterministic DEDS, which can be seen as a special case of the above definition in which the i.i.d. sequence \( A(k) \) is simple a constant matrix \( A = (a_{ij}) \) with real entries.

An eigenvalue of the matrix \( A \) (in the max-algebra sense) is a real number \( \lambda \) such that the equation

\[
A \otimes x = \lambda \otimes x
\]

possesses a solution \( x \in \mathbb{R}^n \). Then the following results can be formulated.

1. Every (real-valued) matrix \( A \) possesses a unique eigenvalue \( \lambda = \lambda(A) \).
2. For each sequence \( \gamma = (i_1, \ldots, i_j, i_{j+1} = i_1) \) of nodes, the average weight

\[
\omega(\gamma) = \frac{a_{i_1i_2} \otimes \cdots \otimes a_{i_ji_1}}{j}
\]

(here, division by \( j \) is the conventional algebraic operation, not a max-algebra operation) satisfies

\[
\omega(\gamma) \leq \lambda.
\]
(3) There is a $\gamma = (i_1, \ldots, i_d, i_1)$, with $i_j \neq i_k$ if $j \neq k$, such that $w(\gamma) = \lambda$. Such a sequence $\gamma$ is then called a critical circuit.

(4) There exist $d$ and $k_0$ such that, for all $k \geq k_0$,
\[ x(k + d) = \lambda^d \otimes x(k). \]
If the critical circuit $\gamma$ is unique, then $d$ equals the number of distinct nodes of $\gamma$.

Remarks. An interpretation of (1)–(4) is that the asymptotic behavior of the system is completely determined by the 'slowest' circuit (i.e., the circuit with maximal average weight), other circuits playing no role, after finite time.

If the arc from node $j$ to node $i$ is absent from the system, this can be modeled mathematically by setting $a_{ij} = -\infty$. This is in a certain sense convenient, since $-\infty$ is the zero element of the max-algebra. An interpretation in the production network is that node $j$ does not have to supply node $i$ to start the next activity at node $i$. If $-\infty$ entries are allowed in $A$, it is necessary to place an irreducibility assumption on the underlying graph to ensure the validity of the given results.

3. Asymptotic normality

For ease of exposition we assume that $n = 2$, although the results remain valid in principle for general $n$. Let a random DEDS be given; our notation is as in the preceding section. We define
\[ z(k) := x_2(k) - x_1(k), \quad k \geq 0. \tag{3.1} \]

Proposition 3.1. The process $z(0), z(1), \ldots$ is a Markov chain. For fixed $z \in \mathbb{R}$, the jump probability measure $P(z, \cdot)$ of this Markov chain is the distribution of the random variable
\[ (a_{21} \oplus (a_{22} \otimes z)) - (a_{11} \oplus (a_{12} \otimes z)) \]
where
\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]
has the distribution of $A(k)$.

Proof. We have
\begin{align*}
x_1(k+1) &= a_{11}(k) \otimes x_1(k) \oplus a_{12}(k) \otimes x_2(k), \\
x_2(k+1) &= a_{21}(k) \otimes x_1(k) \oplus a_{22}(k) \otimes x_2(k),
\end{align*}
so that
\begin{align*}
z(k+1) &= a_{21}(k) \otimes x_1(k) \oplus a_{22}(k) \otimes x_2(k) - a_{11}(k) \otimes x_1(k) \oplus a_{12}(k) \otimes x_2(k) \\
&= a_{21}(k) \otimes a_{22}(k) \otimes z(k) - a_{11}(k) \oplus a_{12}(k) \otimes z(k)
\end{align*}
and the claim of the proposition follows. \qed

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Now define
\[ d(k) := x_i(k) - x_i(k-1), \quad k \geq 1. \quad (3.2) \]
Then we have
\[ x_i(k) = x_i(0) + \sum_{j=1}^{k} d(j), \quad k \geq 1, \]
\[ x_2(k) = x_2(0) + (z(k) - z(0)) + \sum_{j=1}^{k} d(j), \quad k \geq 1. \]

**Proposition 3.2.** For each \( k \geq 1 \), the distribution of \((d(k), z(k))\) given \( z(0), d(1), z(1), \ldots, d(k-1), z(k-1)\) depends only on \( z(k-1) \). If \( z(k-1) = z \), this distribution is equal to the distribution of the random vector \((a_{11} \oplus (a_{12} \otimes z), a_{21} \oplus (a_{22} \otimes z) - a_{11} \oplus (a_{12} \otimes z))\), where
\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]
has the distribution of \( A(k) \).

**Proof.** We have
\[
d(k) = a_{11}(k-1) \otimes x_i(k-1) \oplus a_{12}(k-1) \otimes x_2(k-1) - x_i(k-1)
\]
\[
= a_{11}(k-1) \oplus a_{12}(k-1) \otimes z(k-1),
\]
which, together with Proposition 3.1, yields the desired result. \( \square \)

**Remarks.** (1) Proposition 3.2 says that the joint distribution of \((d(k), z(k))\) depends only on the value of \( z(k-1) \) and not on other values of \( d \) or \( z \) in the past.

(2) For similar results when \( n \geq 3 \), see Olsder et al. (1988).

It should be clear now that \( x_i(k) - x_i(0) \) is equal in distribution to a sum of random variables with distributions depending only on an underlying Markov chain, and that we are in the situation studied in O'Brien (1974) for discrete state space chains, and in Grigorescu and Oprisan (1976) for general state space chains. We shall formulate (a special case of) a theorem of Grigorescu and Oprisan (1976) and use it in the sequel. For a definition of uniform \( \Phi \)-recurrence and a sketch of the proof of this theorem, we refer to Appendix A.

**Theorem 3.3.** Suppose that the Markov chain \( z(k), k \geq 0 \), is aperiodic and uniformly \( \Phi \)-recurrent, and denote by \( \pi \) its unique invariant probability measure. If the entries of \( A \) have finite first moments, then
\[
\lim_{k \to \infty} \left( \frac{x_i(k)}{k}, \frac{x_2(k)}{k} \right) = (\mu, \mu)
\]
(3.4)
exists almost surely for any initial activity time \((x_1(0), x_2(0))\), and we have
\[ \mu := E_\pi(d(1)), \]
the expectation of \(d(1)\), given that the distribution of \(z(0)\) equals \(\pi\). Moreover, if the entries of \(A\) and the initial activity time \((x_1(0), x_2(0))\) have finite second moments, then
\[ 0 \leq \sigma^2 := E_\pi((d(1) - \mu)^2) + 2 \sum_{l=2}^{\infty} E_\pi((d(1) - \mu)(d(l) - \mu)) < \infty \]
and if \(\sigma^2 > 0\), then the sequence
\[ \frac{(x_1(k), x_2(k)) - k(\mu, \mu)}{\sigma \sqrt{k}}, \quad k \geq 1, \]
converges in distribution to the random vector \((N, N)\), where \(N\) is a standard normal random variable. \(\square\)

4. Examples

In Sections 4.1–4.3 we present three examples illustrating the theory of the preceding section. In all examples, the dimension of the system equals two \((n = 2)\), and the entries of the matrix \(A(k)\) are i.i.d.

4.1. Bernoulli delays

Assume that for each \(k \geq 0\), \(A(k)\) has i.i.d. entries \(a_{ij}(k), \ i, j \in \{1, 2\}\), which take the values 0 or 1 only, each with probability \(\frac{1}{2}\). For any initial vector \(x(0)\) there exists an index \(k_0\) such that for \(k \geq k_0\) the Markov chain \(z(k)\) takes values in the finite set \((-1, 0, 1)\). The transition probabilities on \((-1, 0, 1)\) are easily seen from Proposition 3.1 to be given by the matrix
\[ P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{16} & \frac{5}{16} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}. \]
The Markov chain \(z(k)\) is aperiodic (all entries of \(P\) are positive) and uniformly \(\Phi\)-recurrent (the state space is finite). It follows from \(\pi'P = \pi'\) that the discrete measure \(\pi\) on \(\mathbb{R}\) defined by
\[ \pi\{-1\} = \frac{2}{14}, \quad \pi\{0\} = \frac{8}{14}, \quad \pi\{1\} = \frac{3}{14} \]
will be the unique invariant probability measure. From Proposition 3.2 we find
\[ \mu = E_\pi(d(1)) = \frac{3}{14} \times \frac{1}{2} + \frac{8}{14} \times \frac{3}{4} + \frac{3}{14} \times \frac{3}{2} = \frac{5}{6}. \]
It is difficult to calculate \(\sigma^2\) directly through (3.5), because \(E_\pi(d(1) - \mu)(d(l) - \mu)\) involve the evolution of the Markov chain \(z(k), \ k \geq 0, \) from time 0 up to time \(l\). However, it is possible to calculate \(\sigma^2\) via a detour. To this end we first note that
in cases where for each \( k, a_y(k), i, j \in \{1, 2\} \), are i.i.d. it is convenient to work with \( \tilde{z}(k) \) and \( \tilde{d}(k) \) where

\[
\tilde{z}(k) := |x_2(k) - x_1(k)|, \quad (4.1)
\]
\[
\tilde{d}(k) := x_1(k) \oplus x_2(k) - x_1(k-1) \oplus x_2(k-1). \quad (4.2)
\]

The transition probabilities \( P\{z(k) = i, \tilde{d}(k) = j \mid z(k-1) = z\} \) for \( z \in \{-1, 0, 1\} \) are given by

\[
(i, j)
\]

\[
\begin{array}{cccccc}
  & (-1, 0) & (0, 0) & (1, 0) & (-1, 1) & (0, 1) & (1, 1) \\
-1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & \frac{1}{16} & 0 & \frac{3}{16} & \frac{9}{16} & \frac{3}{16} \\
1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{array}
\quad (4.3)
\]

Of course when we add the entries with \( z(k) \) fixed and \( \tilde{d}(k) = 0 \) or \( \tilde{d}(k) = 1 \) we obtain the matrix \( P \). We also note from (4.3) that the row with \( z = -1 \) is identical to the row with \( z = 1 \). This implies that \( \tilde{z}(k) \) itself is a Markov chain. The transition probabilities \( P\{\tilde{z}(k) = i, \tilde{d}(k) = j \mid \tilde{z}(k-1) = z\} \) for \( z \in \{0, 1\} \) are given by

\[
(i, j)
\]

\[
\begin{array}{cccc}
  & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\
0 & \frac{1}{16} & \frac{9}{16} & 0 & \frac{6}{16} \\
1 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\
\end{array}
\quad (4.4)
\]

Now we shall show how we can use the more simple variables \( \tilde{z}(k) \) and \( \tilde{d}(k) \) to calculate \( \sigma^2 \). It follows from Theorem 3.3, and the continuous mapping theorem (Billingsley, 1968, Theorem 5.1) that

\[
\frac{\max(x_1(k), x_2(k)) - \mu k}{\sigma \sqrt{k}}
\]

converges in distribution to a standard normal random variable \( N \). Consequently,

\[
\lim_{k \to \infty} \text{var} \left( \frac{\max(x_1(k), x_2(k)) - \mu k}{\sigma \sqrt{k}} \right) = 1.
\]

Also from (4.2),

\[
\max(x_1(k), x_2(k)) = \max(x_1(0), x_2(0)) + \sum_{j=1}^{k} \tilde{d}(j).
\]

These two equations together imply that

\[
\sigma^2 = \lim_{k \to \infty} \frac{1}{k} \text{var} \left( \sum_{j=1}^{k} \tilde{d}(j) \right),
\quad (4.5)
\]
because the result of Theorem 3.3 is independent of \( x(0) \), which value we therefore take to be equal to 0. The right-hand side of (4.5) can be evaluated in this simple example. To this end define \( \delta(k) := (\bar{z}(k-1), \bar{z}(k)) \), \( k \geq 1 \). The Markov chain \( \delta(k) \), \( k \geq 1 \), has state space \{(0,0),(0,1),(1,0),(1,1)\} and, according to (4.4), its transition matrix equals

\[
P_\delta = \begin{pmatrix}
\frac{5}{8} & \frac{3}{8} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{3}{8} & \frac{3}{8} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

The invariant probability vector \( b' \) of the matrix \( P_\delta \) is \( b' = \frac{1}{10} (5,3,3,3) \). Furthermore, it is not difficult to verify that

\[
\begin{align*}
f_1 &= P\{\bar{d}(k) = 1 | \delta(k) = (0,0)\} = \frac{9}{10}, \\
f_2 &= P\{\bar{d}(k) = 1 | \delta(k) = (0,1)\} = 1, \\
f_3 &= P\{\bar{d}(k) = 1 | \delta(k) = (1,0)\} = \frac{1}{2}, \\
f_4 &= P\{\bar{d}(k) = 1 | \delta(k) = (1,1)\} = 1.
\end{align*}
\]

Now introduce the limit matrix \( B \) of \( P_\delta \), defined as the \( 4 \times 4 \) matrix with all 4 rows equal to the equilibrium vector \( b' \), and define the fundamental matrix \( Z \) by

\[
Z := (I - (P_\delta - B))^{-1},
\]

which exists according to Kemeny and Snell (1970, Theorem 4.3.1). It follows from their Theorem 4.6.3 (reformulated in our notation) that

\[
\lim_{k \to \infty} \frac{1}{k} \text{var} \left( \sum_{i=1}^{k} \bar{d}(l) \right) = 2 \left\{ \sum_{i,j=1}^{4} f_i f_j b_i (z_i - \delta_{ij}) \right\} - \mu (\mu - 1),
\]

where \( \delta_{ij} \) denotes the Kronecker delta. Easy but tedious calculations show that

\[
Z = \frac{1}{49} \begin{pmatrix}
64 & 9 & -12 & -12 \\
-20 & 37 & 16 & 16 \\
15 & 9 & 37 & -12 \\
-20 & -12 & 16 & 65
\end{pmatrix},
\]

and hence \( \sigma^2 = 33/343 \).

### 4.2. Exponential delays

Let, for each \( k \geq 0 \), \( A(k) \) have i.i.d. entries \( a_{ij}(k) \), \( i,j \in \{1,2\} \), with distribution

\[
P\{a_{ij}(k) < x\} = (1 - e^{-\lambda x})1_{(0,\infty)}(x), \quad \lambda > 0.
\]

As in Section 4.1, we find that, for all \( z \geq 0 \) and \( s \in \mathbb{R} \),

\[
P\{z(k) < s | z(k-1) = -z\} = P\{z(k) < s | z(k-1) = z\}.
\]
This follows from Proposition 3.1, since the i.i.d. assumptions on \( a_j \) imply

\[
a_{21} \oplus (a_{22} \otimes z) - a_{11} \oplus (a_{12} \otimes z) \overset{d}{=} a_{21} \oplus (a_{22} \otimes -z) - a_{11} \oplus (a_{12} \otimes -z),
\]

where \( \overset{d}{=} \) denotes equality in distribution. Hence in this example \( \tilde{z}(k) \) is a Markov chain with state space \([0, \infty)\). We first calculate the transition kernel

\[
p(z, [0, s)) := P\{\tilde{z}(k) < s | \tilde{z}(k-1) = z\}.
\]

Let us define random variables \( u_z \) and \( v_z \) such that

\[
u_z = \max(a_{11}(k), a_{12}(k) - z), \quad v_z = \max(a_{21}(k), a_{22}(k) - z).
\]

Then

\[
p(z, [0, s)) = P\{|z(k)| < s | z(k-1) = z\} = P\{|u_z - v_z| < s\}
\]

\[
= \int_0^s P\{v_z < y + s\} \, dP\{u_z < y\} + \int_s^\infty P\{y - s < v_z < y + s\} \, dP\{u_z < y\},
\]

and \( u_z, v_z \) have common distribution

\[
P\{u_z < y\} = (1 - e^{-\lambda y})(1 - e^{-\lambda(y+s)}), \quad y \geq 0.
\]

Hence we find

\[
p(z, [0, s)) = 1 - e^{-\lambda z} - \frac{2}{3} e^{-\lambda(s+z)} + \frac{2}{3} e^{-\lambda(2s+z)}
\]

\[
+ \frac{1}{3} e^{-\lambda(s+2z)} - \frac{1}{3} e^{-\lambda(2s+2z)}, \quad s, z \geq 0.
\]

(4.4)

**Lemma 4.1.** The Markov chain \( \tilde{z}(k), k \geq 0 \), is aperiodic and uniformly \( \Phi \)-recurrent.

**Proof.** Aperiodicity is clear. To prove uniform \( \Phi \)-recurrence we use part (i) of Theorem A.1. Let \( \Phi(A) := \int_A \lambda e^{-\lambda s} \, ds \). Then it is easily seen, using the inequalities

\[
0 \leq \frac{2}{3} e^{-\lambda z} - \frac{1}{3} e^{-2\lambda z} \leq \frac{1}{3} \quad \forall z \in [0, \infty),
\]

\[
|1 - 2 e^{-\lambda s}| \leq 1 \quad \forall s \in [0, \infty),
\]

that

\[
p(z, A) = \int_A \frac{d}{ds} p(z, [0, s)) \, ds
\]

\[
= \int_A \lambda e^{-\lambda s} (1 + (\frac{2}{3} e^{-\lambda z} - \frac{1}{3} e^{-2\lambda z})(1 - 2 e^{-\lambda s})) \, ds
\]

\[
\geq \frac{2}{3} \int_A \lambda e^{-\lambda s} \, ds = \frac{2}{3} \Phi(A).
\]

Thus, for \( 0 < \varepsilon < \frac{2}{3} \Phi(A) \), we have \( \lambda P^{(1)}(z, A) = p(z, A) > \varepsilon \) uniformly in \( z \). \( \Box \)
It is evident from the definition $\tilde{z}(k) = |z(k)|$, $k \geq 0$, that the conclusion of Lemma 4.1 also holds for $z(k)$. Hence, the assumptions of Theorem 3.3 are satisfied. To calculate $\mu = E_\pi(d(1))$ observe from the derivation in the previous example that

$$\mu = E_\pi(\tilde{d}(1)),$$

where $\tilde{d}$ is the invariant probability measure of $\tilde{z}(k)$ and with $\tilde{d}(k)$ defined by (4.2). The invariant probability measure of $\tilde{z}(k)$ is given by the unique distribution $\tilde{\pi}(\cdot)$ for which, for all $s \in [0, \infty)$,

$$\tilde{\pi}(s) = \int_0^\infty p(z, [0, s)) \, d\tilde{\pi}(z).$$

So, defining the Laplace-Stieltjes transform of $\tilde{\pi}(\cdot)$ by

$$\tilde{\pi}(\rho) := \int_0^\infty e^{-\rho s} \, d\tilde{\pi}(s),$$

we get

$$\tilde{\pi}(\rho) = \int_0^\infty \left( \int_0^\infty e^{-\rho s} \, \frac{d}{ds} p(z, [0, s)) \, ds \right) \, d\tilde{\pi}(z).$$

Using (4.6) we find

$$\int_0^\infty e^{-\rho s} \, \frac{d}{ds} p(z, [0, s)) \, ds$$

$$= \frac{\lambda}{\lambda + \rho} + \frac{2}{3} \frac{\lambda}{\lambda + \rho} e^{-\lambda s} - \frac{4}{3} \frac{\lambda}{2\lambda + \rho} e^{-\lambda s}$$

$$- \frac{1}{3} \frac{\lambda}{\lambda + \rho} e^{-2\lambda s} + \frac{2}{3} \frac{\lambda}{2\lambda + \rho} e^{-2\lambda s},$$

and hence

$$\tilde{\pi}(\rho) = \frac{\lambda}{\lambda + \rho} + \frac{2}{3} \frac{\lambda}{\lambda + \rho} \tilde{\pi}(\lambda) - \frac{4}{3} \frac{\lambda}{2\lambda + \rho} \tilde{\pi}(\lambda) - \frac{1}{3} \frac{\lambda}{\lambda + \rho} \tilde{\pi}(2\lambda)$$

$$+ \frac{2}{3} \frac{\lambda}{2\lambda + \rho} \tilde{\pi}(2\lambda).$$

Substituting $\rho = \lambda$ and $\rho = 2\lambda$ gives

$$\tilde{\pi}(\lambda) = 53/114, \quad \tilde{\pi}(2\lambda) = 17/57,$$

and so

$$\tilde{\pi}(\rho) = \frac{23}{19} \frac{\lambda}{\lambda + \rho} - \frac{8}{19} \frac{\lambda}{2\lambda + \rho}$$

or, equivalently,

$$\tilde{\pi}(s) = 1 - \frac{23}{19} e^{-\lambda s} + \frac{4}{19} e^{-2\lambda s}.$$
Using that, under the condition $\tilde{z}(0) = z$,

$$\tilde{d}(1) \overset{d}{=} \max(a_{11}(0), a_{21}(0), a_{12}(0) - z, a_{22}(0) - z),$$

we find after some calculations

$$\mu = E_{\tilde{d}}(\tilde{d}(1)) = \int_0^\infty E(\tilde{d}(1) | \tilde{z}(0) = z) \, d\tilde{\pi}(z)$$

$$= \int_0^\infty \frac{1}{\lambda} \left( \frac{3}{5} + \frac{2}{3} e^{-\lambda z} - \frac{1}{12} e^{-2\lambda z} \right) \, d\tilde{\pi}(z) = \frac{1}{\lambda} \frac{407}{228}.$$

### 4.3. Uniform delays

Let $a_{ij}(k)$ be mutually independent random variables uniformly distributed on the interval $[0, 1]$. Once again we consider the Markov chain $\tilde{z}(k) := |x_2(k) - x_1(k)|$ and calculate the transition kernel

$$p(z, [0, s)) = P\{\tilde{z}(k) < s | \tilde{z}(k-1) = z\}, \quad 0 \leq z, s \leq 1.$$

Let us define $u, v$ as in Section 4.2. Then one can easily check that the joint probability density $p_z$ of $(u, v)$ is given by

$$p_z(u, v) = \begin{cases} (2u + z)(2v + z), & 0 < u < 1 - z, 0 < v < 1 - z, \\ 2u + z, & 0 < u < 1 - z, 1 - z < v < 1, \\ 2v + z, & 1 - z < u < 1, 0 < u < 1 - z, \\ 1, & 1 - z < u < 1, 1 - z < v < 1. \end{cases}$$

Furthermore we have the following relation between the transition density $(d/ds)p(z, [0, s))$ and $p_z$:

$$\frac{d}{ds} p(z, [0, s)) = 2 \int_0^{1-s} p_z(u, u + s) \, du.$$

For the calculation of this density we have to consider the following cases:

**Case A**: $0 \leq s \leq \min(z, 1 - z)$.

$$\frac{d}{ds} p(z, [0, s)) = 2 \left( \int_0^{1-z-s} (2u + z)(2u + 2s + z) \, du + \int_{1-z-s}^{1-z} (2u + z) \, du + \int_{1-z}^{1} \, du \right)$$

$$= \frac{8}{3} - 2z + 2z^2 - \frac{2}{3} z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3} s^3.$$

**Case B**: $z < s < 1$.

$$\frac{d}{ds} p(z, [0, s)) = 2 \left( \int_0^{1-z-s} (2u + z)(2u + 2s + z) \, du + \int_{1-z-s}^{1-z} (2u + z) \, du \right)$$

$$= \frac{8}{3} + 2z^2 - \frac{2}{3} z^3 - 4s - 2sz^2 + \frac{4}{3} s^3.$$
Case C: $1-z \leq s \leq z$.

$$\frac{d}{ds} p(z, [0, s)) = 2 \left( \int_0^{1-z} (2u+z) \, du + \int_{1-z}^{1-s} \, du \right) = 2 - 2s.$$ 

Case D: $\max(z, 1-z) \leq s \leq 1$.

$$\frac{d}{ds} p(z, [0, s)) = 2 \int_0^{1-z} (2u+z) \, du = 2 - 4s + 2s^2 + 2z - 2zs.$$ 

Lemma 4.2. The Markov chain $\tilde{z}(k)$, $k \geq 0$, is aperiodic and uniformly $\Phi$-recurrent.

Proof. Aperiodicity is clear. For the proof of uniform $\Phi$-recurrence we shall use the following claim.

Claim 4.2.1. Define

$$f(s) :=\begin{cases} 
2(1-s), & 0 \leq s \leq -\frac{1}{2} + \frac{1}{2}\sqrt{3} \\
\frac{8}{3} - 4s + \frac{4}{3}s^3, & -\frac{1}{2} + \frac{1}{2}\sqrt{3} \leq s \leq 1.
\end{cases}$$

Then for all $z \in [0, 1]$ we have $(d/ds)p(z, [0, s)) \geq f(s)$.

The claim follows from easy, but tedious calculations in Cases A, B, C and D. At this point we have found a function $f(s)$, continuous on $[0, 1]$, and positive on $(0, 1)$, such that $f(s) \leq (d/ds)p(z, [0, s))$ for all $z \in [0, 1]$, and all $s \in (0, 1)$.

Now define on the Borel subsets of $(0, 1)$,

$$\Phi(A) := \int_A f(s) \, ds.$$ 

Using this measure $\Phi$ it is easily checked that the Markov chain $\tilde{z}(k)$, $k \geq 1$, is uniformly $\Phi$-recurrent. This concludes the proof of Lemma 4.2. 

We now proceed as in Section 4.2 and calculate the invariant probability measure $\tilde{\pi}$ of $\tilde{z}(k)$ and from this the expectation $E_{\tilde{\pi}}(\tilde{d}(1))$. Let us denote the density of the stationary distribution of $\tilde{z}(k)$ by $g$. To obtain a numerical approximation for $g$ we shall subdivide Cases A, B, C and D once more. We want to distinguish between $s \leq \frac{1}{2}$ and $s \geq \frac{1}{2}$, and $z \leq \frac{1}{2}$ and $z \geq \frac{1}{2}$. Hence we get eight subcases as shown in Fig. 1. Furthermore, for convenience of notation we replace $z$ by $1-z$ if $\frac{1}{2} \leq z \leq 1$ and $s$
by $1 - s$ if $\frac{1}{2} \leq s \leq 1$. If we use these replacements, calculations yield the following eight polynomials for the densities of the transition probabilities:

<table>
<thead>
<tr>
<th>Subcase</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1:</td>
<td>$\frac{8}{3} - 2z + 2z^2 - \frac{2}{3}z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3}s^3$</td>
</tr>
<tr>
<td>1.2:</td>
<td>$\frac{8}{3} + 2z^2 - \frac{2}{3}z^3 - 4s - 2sz^2 + \frac{4}{3}s^3$</td>
</tr>
<tr>
<td>1.3:</td>
<td>$2sz + 2s^2$</td>
</tr>
<tr>
<td>1.4:</td>
<td>$-\frac{2}{3}z^3 + 2sz^2 + 4s^2 - \frac{4}{3}s^3$</td>
</tr>
<tr>
<td>2.1:</td>
<td>$2 + \frac{2}{3}z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3}s^3$</td>
</tr>
<tr>
<td>2.2:</td>
<td>$2 - 2s$</td>
</tr>
<tr>
<td>2.3:</td>
<td>$2s - 2sz + 2s^2$</td>
</tr>
<tr>
<td>2.4:</td>
<td>$2s$</td>
</tr>
</tbody>
</table>

In order to find the stationary density $g$ of $\hat{z}(k)$, i.e., the normalized solution of

$$g(s) = \int_0^1 \frac{d}{ds} p(z, [0, s]) g(z) \, dz,$$

we put

$$g_0(s) := g(s), \quad 0 \leq s \leq \frac{1}{2},$$

$$g_1(s) := g(1 - s), \quad 0 \leq s \leq \frac{1}{2}.$$  

Then, if we denote by $P_{ij}(z, s)$ ($i = 1, 2, j = 1, 2, 3, 4$) the polynomial of subcase $ij$, $g_0$ and $g_1$ must satisfy the equations

\begin{align*}
g_0(s) &= \int_0^s P_{12}(z, s) g_0(z) \, dz + \int_s^{1/2} P_{11}(z, s) g_0(z) \, dz \\
&\quad + \int_0^s P_{22}(z, s) g_1(z) \, dz + \int_s^{1/2} P_{21}(z, s) g_1(z) \, dz, \quad 0 \leq s \leq \frac{1}{2}, \quad (4.7) \\
g_1(s) &= \int_0^s P_{14}(z, s) g_0(z) \, dz + \int_s^{1/2} P_{13}(z, s) g_0(z) \, dz \\
&\quad + \int_0^s P_{24}(z, s) g_1(z) \, dz + \int_s^{1/2} P_{23}(z, s) g_1(z) \, dz, \quad 0 \leq s \leq \frac{1}{2}. \quad (4.8)
\end{align*}

Now assume

$$g_0(s) = \sum_{n=0}^\infty c_n s^n, \quad g_1(s) = \sum_{n=0}^\infty d_n s^n. \quad (4.9)$$

From (4.7), (4.8) and (4.9) and the formulas for $P_{ij}(z, s)$ we can deduce the following recurrence relations for $c_n$ and $d_n$:

\begin{align*}
c_n &= c_{n-2} \frac{-2}{n(n-1)} + c_{n-3} \frac{2}{(n-1)(n-2)} \\
&\quad + d_{n-3} \frac{2}{(n-1)(n-2)} + d_{n-4} \frac{-2(n+1)}{n(n-1)(n-3)}, \quad n \geq 4,
\end{align*}

\begin{align*}
d_n &= c_{n-3} \frac{2}{(n-1)(n-2)} + c_{n-4} \frac{-2(n+1)}{n(n-1)(n-3)} \\
&\quad + d_{n-3} \frac{-2}{(n-1)(n-2)}, \quad n \geq 4.
\end{align*}
Using $\tilde{c}_n := c_n n!$ and $\tilde{d}_n := d_n n!$ instead of $c_n$ and $d_n$ gives
\begin{align*}
\tilde{c}_n &= -2 \tilde{c}_{n-2} + 2n \tilde{c}_{n-3} + 2n \tilde{d}_{n-3} - 2(n+1)(n-2) \tilde{d}_{n-4}, \quad n \geq 4, \\
\tilde{d}_n &= 2n \tilde{c}_{n-3} - 2n \tilde{d}_{n-3} - 2(n+1)(n-2) \tilde{c}_{n-4}, \quad n \geq 4.
\end{align*}
(4.10)

For the terms $\tilde{c}_n$ and $\tilde{d}_n$, $n = 0, 1, 2, 3$, we find
\begin{align*}
\tilde{c}_0 &= \frac{3}{2}(4 - 3\beta^{(1)} + 3\beta^{(2)} - \beta^{(3)}), \quad \tilde{d}_0 = 0, \\
\tilde{c}_1 &= -2 + 2\beta^{(1)} - 2\beta^{(2)}, \quad \tilde{d}_1 = 2\beta^{(1)}, \\
\tilde{c}_2 &= -4 + 2\tilde{c}_0, \quad \tilde{d}_2 = 4, \\
\tilde{c}_3 &= 8 - 2\tilde{c}_1 + 6\tilde{c}_0, \quad \tilde{d}_3 = 6\tilde{c}_0,
\end{align*}
(4.11)

where $\beta^{(n)}$ is the $n$th moment of the stationary distribution of $\tilde{z}(k)$, i.e.,
\begin{equation}
\beta^{(n)} := \int_0^1 z^n g(z) \, dz.
\end{equation}
(4.12)

To calculate the mean cycle time $\mu$ we use that, under the condition $\tilde{z}(0) = z$,
\begin{equation*}
\tilde{d}(1) = \max(a_{11}(0), a_{21}(0), a_{12}(0) - z, a_{22}(0) - z),
\end{equation*}
with $\tilde{d}(1)$ as defined in (4.2). Hence,
\begin{equation*}
P(\tilde{d}(1) < y | \tilde{z}(0) = z) = \begin{cases}
    y^2(y + z)^2, & 0 \leq y \leq 1 - z, \\
y^2, & 1 - z \leq y \leq 1,
\end{cases}
\end{equation*}
and so
\begin{align*}
E(\tilde{d}(1) | \tilde{z}(0) = z) &= \int_0^1 y \, dP(\tilde{d}(1) < y | \tilde{z}(0) = z) \\
&= \int_0^{1-z} (4y^4 + 6y^3z + 2y^2z^2) \, dy + \int_{1-z}^1 2y^2 \, dy \\
&= \frac{4}{3} - \frac{1}{2}z + \frac{2}{3}z^2 - \frac{1}{3}z^3 + \frac{1}{30}z^5.
\end{align*}

It follows that
\begin{equation*}
\mu = \frac{4}{3} - \frac{1}{2}\beta^{(1)} + \frac{2}{3}\beta^{(2)} - \frac{1}{3}\beta^{(3)} + \frac{1}{30}\beta^{(5)}.
\end{equation*}

From (4.9) and (4.12) we have
\begin{equation*}
\beta^{(i)} = \int_0^{1/2} s^i \sum_{n=0}^{\infty} c_n s^n \, ds + \int_0^{1/2} (1 - s)^i \sum_{n=0}^{\infty} d_n s^n \, ds.
\end{equation*}

Using this relation together with the recurrence relations (4.10) and the initial conditions (4.11) we can calculate $\beta^{(i)}$ numerically for $i = 1, 2, 3, 5$. This yields
\begin{align*}
\beta^{(1)} &= 0.284, \quad \beta^{(2)} = 0.124, \quad \beta^{(3)} = 0.067, \quad \beta^{(5)} = 0.027.
\end{align*}
So for the mean cycle time, we obtain

\[ \mu = 0.719. \]

An alternative way of calculating \( \mu \) can be obtained from discrete approximations of the uniform distribution. Let us assume that, for each \( k \), \( a_{ij}(k) \) are mutually independent with distribution

\[ P(a_{ij}(k) = l/(m-1)) = 1/m \quad \text{for } l = 0, 1, \ldots, m-1, \ m \in \mathbb{N}, \ m \geq 2. \]

Then \( \tilde{z}(k) \) \( (k \geq 0) \) is a Markov chain with state space \( S = \{ l/(m-1), l = 0, 1, \ldots, m-1 \} \). The transition probabilities \( p_{ij} \) of this Markov chain are given by

\[
p_{ij} := P(\tilde{z}(k) = l/(m-1) | \tilde{z}(k-1) = j/(m-1))
\]

where

\[
b_{jh} := P(x_i(k) = x + (m-h-1)/(m-1) \mid \max(x_i(k-1), x_j(k-1)) = x, \]
\[
\tilde{z}(k-1) = j/(m-1)).
\]

Easy calculations show that

\[
b_{jh} = \left\{ \begin{array}{ll}
1/m, & j > h, \\
(j + 1 + 2(m - h - 1))/m^2, & j \leq h.
\end{array} \right.
\]

Further,

\[
f_{ij} := E(\tilde{a}(1) \mid \tilde{z}(0) = j/(m-1), \tilde{z}(1) = l/(m-1))
\]

\[
= \frac{1}{m-1} \left( \sum_{h=1}^{m-1} h b_{jh} b_{j, m-h-1} b_{j, m-h-1} \right) / \left( \sum_{h=0}^{m-1} b_{jh} b_{j, m+h} \right).
\]

Now fix \( m \); from the transition probabilities \( p_{ij} \) it is possible to calculate numerically the stationary distribution \( \{ \tilde{\pi}_j : j = 0, \ldots, m - 1 \} \) of \( \tilde{z}(k) \) and from this

\[
\mu = \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \tilde{\pi}_j p_{ij} f_{ij}.
\]

In (4.13) we show for increasing \( m \) the approximate values of \( \mu \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>50</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>0.8571</td>
<td>0.7661</td>
<td>0.7414</td>
<td>0.7337</td>
<td>0.7299</td>
<td>0.7276</td>
<td>0.7232</td>
<td>0.7217</td>
</tr>
</tbody>
</table>

(4.13)
5. Expectation of cycle times

In this section the emphasis will be on mean cycle times. As in the previous sections we shall consider two-dimensional systems.

In Section 5.1 we assume that $a_{21}(k) = -\infty$. We show an example where the state space of the Markov chain $z(k) = x_2(k) - x_1(k)$ becomes countably infinite and where no invariant probability measure exists for $z(k)$.

In Section 5.2 we show that a main result from the theory of deterministic DEDS, i.e., the fact that the slowest circuit in the network determines the asymptotic behavior of the system, does not necessarily remain true for random DEDS.

5.1. Reducible systems

In this section we assume that $P(a_{21}(k) = -\infty) = 1$ while the other entries are real-valued and finite with probability one. The system description then becomes

\[ x_1(k+1) = a_{11}(k) \otimes x_1(k) + a_{12}(k) \otimes x_2(k), \]
\[ x_2(k+1) = a_{22}(k) \otimes x_2(k). \quad (5.1) \]

Instead of $d(k)$ given by (3.2), we use once again (see (4.2))

\[ \tilde{d}(k) = x_1(k) \oplus x_2(k) - x_1(k-1) \oplus x_2(k-1). \quad (5.2) \]

The reason for this is that $\lim_{k \to \infty} x_1(k)/k$ is not necessarily equal to $\lim_{k \to \infty} x_2(k)/k$ for systems of the form (5.1), because Theorem 3.3 is no longer valid.

In Example 5.1 below, the state space of $z(k) = x_2(k) - x_1(k)$ remains finite, so that $\mu = E_x(\tilde{d}(1))$ can be computed as before. In Example 5.2, the state space of $z(k)$ becomes countably infinite. Depending on the parameter $p$, the Markov chain $z(k)$ will be positive recurrent, null-recurrent or transient. Only in the first case it is possible to obtain the mean cycle time from the invariant probability measure.

Example 5.1. Consider the following distributions of the transition times:

\[ P(a_{11}(k) = 0) = P(a_{11}(k) = 1) = \frac{1}{2}, \]
\[ P(a_{12}(k) = 0) = P(a_{12}(k) = 1) = \frac{1}{2}, \]
\[ P(a_{21}(k) = -\infty) = 1, \]
\[ P(a_{22}(k) = 1) = P(a_{22}(k) = 2) = \frac{1}{2}. \]

In the stationary situation, the state space of $z(k)$ equals $\{0, 1, 2\}$ and the Markov transition matrix is given by

\[ P = \begin{pmatrix}
\frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\
\frac{1}{4} & \frac{3}{8} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{pmatrix}. \]
which has as invariant probability vector \( \pi = \left( \frac{4}{14}, \frac{7}{14}, \frac{3}{14} \right) \). We find in this case
\[ E_\pi(\bar{d}(1)) = \frac{3}{2} \]. Note that this value is equal to \( E(a_{22}) \). This is not surprising since, in equilibrium, \( P(x_3(k) = x_1(k)) = 1 \) and hence, from (5.2), \( \bar{d}(k) = x_3(k) - x_2(k - 1) = a_{22}(k - 1) \).

**Example 5.2.** Consider the following distributions of the transition times:

\[ P(a_{11}(k) = 0) = 1 - P(a_{11}(k) = 1) = 1 - p, \quad p \in (0, 1), \]
\[ P(a_{12}(k) = 0) = P(a_{12}(k) = 1) = \frac{1}{2}, \]
\[ P(a_{21}(k) = -\infty) = 1, \]
\[ P(a_{22}(k) = 0) = P(a_{22}(k) = 1) = \frac{1}{2}. \]

In the stationary situation, the Markov chain \( z(k), k \geq 0 \), has state space \( \{1, 0, -1, -2, \ldots \} \), and transition matrix given by

\[
P = \begin{pmatrix}
\frac{4}{14} & \frac{3}{14} & \frac{3}{14} & 0 & 0 & 0 & \cdots \\
\frac{1}{14}(1 - p) & \frac{1}{2} & \frac{1}{14}(1 + p) & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{2}(1 - p) & \frac{1}{2} & \frac{1}{2}p & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{2}(1 - p) & \frac{1}{2} & \frac{1}{2}p & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}.
\]

The Markov chain is positive recurrent for \( p < \frac{1}{2} \), null recurrent for \( p = \frac{1}{2} \) and transient (drifts away to \( -\infty \)) for \( p > \frac{1}{2} \). Only in the first case a unique invariant probability distribution exists. Some calculations show that this distribution is given by

\[
\pi_1 = \frac{1 - 2p}{2(3 - p)}, \quad \pi_0 = \frac{3}{1 - p} \pi_1, \]

\[
\pi_j = \frac{2 + p}{(1 - p)^2} \left( \frac{p}{1 - p} \right)^{j-1} \pi_1, \quad j = 1, 2, \ldots
\]

Using this distribution we find

\[ E_\pi(\bar{d}(1)) = \frac{1}{2}, \quad (5.3) \]

which value is independent of \( p \).

For \( p > \frac{1}{2} \) the chain is transient. The Markov chain \( z(k) \) drifts away to \( -\infty \) and hence, for \( k \) large enough, \( \bar{d}(k) = x_1(k) - x_1(k - 1) \). Consequently,

\[ E(\bar{d}(k)) = E(x_1(k) - x_1(k - 1)) = E(a_{11}(k)) = p. \quad (5.4) \]

This also holds for \( p = \frac{1}{2} \), because for a null-recurrent Markov chain we have for all states \( \lim_{k \to \infty} P(z(k) = j) = 0 \) and hence \( \lim_{k \to \infty} P(z(k) \in \{1, 0, -1\}) = 0 \).

The following intuitive explanation of the answer in (5.3) can be given. Since \( \mu \) can be expected to be increasing in \( p \), we conclude from (5.4) that \( \mu \leq \frac{1}{2} \) for \( p < \frac{1}{2} \). However, \( \mu \) can not be smaller than \( E(x_3(k) - x_3(k - 1)) \) and hence also \( \mu \geq \frac{1}{2} \).
5.2. The slowest circuit

In Section 2 it was pointed out that for deterministic DEDS the asymptotic behavior of
the system is completely determined by the slowest circuit in the network. For
random DEDS, in general this is not the case. We shall show that the mean cycle
time of the process is at least equal to the maximum of the average weights of
the circuits. We shall show with some examples that equality holds only for very few
cases.

Let again

\[ A(k) = \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{pmatrix} \]

be a sequence of i.i.d. real-valued random matrices and let \( d(k) = x_1(k) - x_1(k-1) \).

\[ \text{Proposition 5.3. Suppose that the Markov chain } z(k), k \geq 0, \text{ is aperiodic and uniformly } \Phi\text{-recurrent, and that the entries of } A(k) \text{ have finite first moment. The mean cycle time } \mu \text{ satisfies} \]

\[ \mu \geq \max\{E(a_{11}), E(a_{21}), E(\frac{1}{2}(a_{12} \otimes a_{21}))\}, \quad (5.5) \]

where \( a_q \) denotes a random variable with the same distribution as \( a_q(k) \).

\[ \text{Proof. We have } d(k+1) = a_{11}(k) \oplus (a_{12}(k) \otimes z(k)) \text{ (see (3.3)) and hence } \mu \geq E(a_{11}). \]

According to (3.4), we also have \( \mu = E_{x}(x_2(k+1) - x_2(k)) \) and so, by symmetry,
\( \mu \geq E(a_{22}). \) From the system equations it can be derived that

\[ x_1(k+1) - x_1(k-1) = (a_{11}(k) \otimes a_{11}(k-1)) \]

\[ \quad \oplus (a_{11}(k) \otimes a_{12}(k-1) \otimes z(k-1)) \]

\[ \quad \oplus (a_{12}(k) \otimes a_{21}(k-1)) \]

\[ \quad \oplus (a_{12}(k) \otimes a_{22}(k-1) \otimes z(k-1)). \]

This implies that \( \mu \geq E(\frac{1}{2}(a_{12} \otimes a_{21})). \) \( \square \)

We shall present some examples. Only in the first example equality in (5.5) holds. In all examples we assume the entries of \( A(k) \) to be mutually independent.

\[ \text{Example 5.4. Let} \]

\[ P(a_{11} = 2) = P(a_{11} = 3) = \frac{1}{2}, \]

\[ P(a_{12} = 2) = P(a_{12} = 3) = \frac{1}{2}, \]

\[ P(a_{21} = 0) = P(a_{21} = 1) = \frac{1}{2}, \]

\[ P(a_{22} = 0) = P(a_{22} = 1) = \frac{1}{2}. \]

It can be computed that \( \mu = E(a_{11}) = \frac{5}{2}. \) This can be explained as follows. The
Markov chain \( z(k) \) has \( \{-1, -2, -3\} \) as its state space in the stationary situation
and thus \( a_{11}(k) \oplus a_{12}(k) \otimes z(k) \) is always equal to \( a_{11}(k) \) in the stationary situation.
Then, according to (3.3), \( d(k+1) = a_{11}(k) \) for all \( k \) and thus \( \mu = E(a_{11}). \) The above
property holds in general if \( P(a_{11} = a_{22}; a_{11} \geq \frac{1}{2}(a_{12} \otimes a_{21})) = 1. \)
Example 5.5. Let
\[ P(a_{11} = 1) = P(a_{11} = 2) = \frac{1}{2}, \]
\[ P(a_{12} = 1) = P(a_{12} = 2) = \frac{1}{2}, \]
\[ P(a_{21} = 0) = P(a_{21} = 1) = \frac{1}{2}, \]
\[ P(a_{22} = 0) = P(a_{22} = 1) = \frac{1}{2}. \]

Just as in Example 5.4 we have that \( E(a_{11}) > E(a_{22}) \) and \( E(a_{11}) > E(\frac{1}{2}(a_{12} \otimes a_{21})) \). Some computations, however, show that \( \mu = \frac{53}{32} \), which is larger than \( E(a_{11}) = \frac{5}{2} \). This can be explained by the fact that in this case \( P(a_{11} \geq \frac{1}{2}(a_{12} \otimes a_{21})) < 1 \). The state space of \( z(k) \) is equal to \( \{-2, -1, 0\} \) in the stationary situation and thus \( P(a_{12}(k) \otimes z(k) > a_{11}(k)) > 0 \). This implies that \( \mu > E(a_{11}) \).

Example 5.6. Let
\[ P(a_{11} = 0) = P(a_{11} = 1) = \frac{1}{2}, \]
\[ P(a_{12} = 1) = P(a_{12} = 2) = \frac{1}{2}, \]
\[ P(a_{21} = 1) = P(a_{21} = 2) = \frac{1}{2}, \]
\[ P(a_{22} = 0) = P(a_{22} = 1) = \frac{1}{2}. \]

In this case, the circuit with maximal average weight is \( 1 \rightarrow 2 \rightarrow 1 \) and this average weight equals
\[ \frac{1}{2}E(a_{12} \otimes a_{21}) = \frac{5}{2}. \]

Furthermore, \( P(\frac{1}{2}(a_{12} \otimes a_{21}) \geq a_{11}; \frac{1}{2}(a_{12} \otimes a_{21}) \geq a_{22}) = 1 \) (cf. Example 5.4). The Markov chain \( z(k) \) has in the stationary situation as state space \( \{-2, -1, 0, 1, 2\} \), while its invariant probability distribution \( \pi \) is given by
\[ \pi(-2) = \pi(2) = \frac{1}{24}, \quad \pi(-1) = \pi(1) = \frac{1}{4}, \quad \pi(0) = \frac{5}{12}. \quad (5.6) \]

From (5.6) we conclude that with positive probability the term \( a_{11}(k) \otimes a_{12}(k-1) \otimes z(k-1) \) is larger than the term \( a_{12}(k) \otimes a_{21}(k-1) \). Hence \( \mu \) will be larger than \( E(\frac{1}{2}(a_{12} \otimes a_{21})) \). Some calculations show that \( \mu = \frac{27}{48} \) which is indeed larger than \( \frac{3}{2} \). The crucial point in this example is that the Markov chain \( z(k) \) can, with positive probability, get into states, in which the ‘faster’ circuits do influence the behavior of the system.

In the previous examples it was shown that in contrast to the theory on deterministic DEDS the asymptotic behavior of random DEDS is not necessarily determined by the slowest circuit only.
Appendix A

A.1. Uniform $\Phi$-recurrence

A Markov chain $(X_k)_{k \geq 0}$ with state space $\mathbb{R}$ is called uniformly $\Phi$-recurrent if there exists a $\sigma$-finite measure $\Phi$ on the Borel sets $\mathcal{B}$ of $\mathbb{R}$ such that, for each $A \in \mathcal{B}$ with $\Phi(A) > 0$,

$$
\sum_{m=1}^{k} \lambda^m(x, A) \to 1 \quad \text{for } k \to \infty
$$

uniformly in $x$, where $\lambda^m(x, A)$ is defined as the taboo probability

$$
\lambda^m(x, A) := P\{X_m \in A, X_i \notin B, 1 \leq i \leq m-1 | X_0 = x\}, \quad A, B \in \mathcal{B}.
$$

**Theorem A.1.** (i) Suppose a Markov chain with state space $\mathbb{R}$ satisfies the following condition: for each Borel set $A$ with $\Phi(A) > 0$ there exist $k > 0$, $\epsilon > 0$ such that

$$
\sum_{m=1}^{k} \lambda^m(x, A) > \epsilon
$$

for all $x \in \mathbb{R}$. Then the chain is uniformly $\Phi$-recurrent.

(ii) A uniformly $\Phi$-recurrent chain has an invariant probability measure $\pi$. Moreover, there exists a finite constant $a$ and a number $p < 1$ such that, for each initial probability measure $\mu$,

$$
\| (\mu - \pi) P^k \| \leq a p^k
$$

if the chain is aperiodic. Here, $\| \cdot \|$ denotes the total variation norm.

For the proof, see Orey (1971).

A.2. Central limit theorem for stationary mixing processes

For a stationary sequence $\xi_1, \xi_2, \ldots$ of random variables on some basic space $(\Omega, \mathcal{F}, P)$ we define $\mathcal{F}_n$ as the $\sigma$-field generated by $\xi_1, \ldots, \xi_n$ and $\mathcal{F}_n^{\infty}$ as the $\sigma$-field generated by $\xi_n, \xi_{n+1}, \ldots$. Let $\phi : \mathbb{N} \to [0, \infty)$ be a given function. We call the sequence $\xi_1, \xi_2, \ldots$ $\phi$-mixing if $n \geq 1$, $k \geq 1$, $E_1 \in \mathcal{F}_n$, and $E_2 \in \mathcal{F}_{n+k}$ together imply

$$
|P(E_1 \cap E_2) - P(E_1) P(E_2)| \leq \phi(k) P(E_1).
$$

**Theorem A.2.** Suppose that $\{\xi_k\}$ is $\phi$-mixing with $\sum_{k=1}^{\infty} \sqrt{\phi(k)} < \infty$ and that $E\xi_1 = 0$, $E\xi_1^2 < \infty$. Then the series

$$
\sigma^2 = E(\xi_1^2) + 2 \sum_{i=2}^{\infty} E(\xi_1 \xi_i)
$$

converges absolutely; if $\sigma^2 > 0$, then $X_k := S_k / \sigma \sqrt{k}$, where $S_k = \xi_1 + \xi_2 + \cdots + \xi_k$, converges in distribution to a standard normal random variable $N$. 

For the proof, see Billingsley (1968, Theorem 20.1).

Billingsley’s Theorem 20.1 actually quotes that $S_{[tk]}/\sigma\sqrt{k}, 0 \leq t \leq 1$, converges in distribution to standard Brownian motion on $[0, 1]$. Theorem A.2 follows after applying the continuous mapping theorem (Billingsley, 1968, Theorem 5.1) to the projection at time $t = 1$. Observe that via the continuous mapping theorem many other similar results can be obtained.

A.3. **Proof of Theorem 3.1**

The pair $(x_1(k), x_2(k))$ can be written as

$$(x_1(k), x_2(k)) = \left(x_1(0) + \sum_{j=1}^{k} d(j), x_2(0) + z(k) - z(0) + \sum_{j=1}^{k} d(j)\right).$$

Since the Markov chain $z(k)$ is uniformly $\Phi$-recurrent we have $k^{-1}(z(k) - z(0)) \to 0$ almost surely, and hence (3.4) is a direct consequence of Grigorescu and Oprisan (1976, Theorem 1). In order to prove the remainder of our theorem we would like to apply Theorem A.2 to the sequence $\{d(k) - \mu\}, k = 1, 2, \ldots$. This involves two difficulties:

(i) The sequence $\{d(k)\}_{k \geq 1}$ is not stationary because the initial distribution of $z(0)$ is in general not equal to the invariant measure $\pi$.

(ii) What conditions on $z(k)$ should be imposed to ensure the $\phi$-mixing condition with a function $\phi$ that decreases so rapidly that $\sum \sqrt{\phi(k)} < \infty$?

The answer to both questions was given in the paper by Grigorescu and Oprisan (1976). Theorem A.1(ii) shows that, if $z(k)$ is aperiodic and uniformly $\Phi$-recurrent, it has a unique stationary probability measure $\pi$, such that for each bounded measurable function $f$ on $\mathbb{R}$ and for all $y \in \mathbb{R}$ there exist a constant $C > 0$ and a real number $\rho \in (0, 1)$ with

$$\left| \int_{\mathbb{R}} f(x) \rho^k(y, dx) - \int_{\mathbb{R}} f(x) \pi(dx) \right| \leq (\sup f) C \rho^k.$$

According to the proof of Grigorescu and Oprisan (1976, p. 68) this shows that, if the initial distribution of $z(k)$ equals $\pi$, the stationary sequence $\{d(k)\}_{k \geq 1}$ is $\phi$-mixing with $\phi(k) = C \rho^k$. This answers question (ii), since obviously $\sum \sqrt{\phi(k)} < \infty$.

The answer to question (i) is rather technical. Although $d(k)$ is not stationary, we have seen above that the distribution of $z(k)$ converges geometrically fast to its stationary distribution $\pi$. Hence the trick is to introduce a sequence $\{p_k\}$ of integers going to infinity slowly enough to allow $p_k/\sqrt{k} \to 0$. The section

$$(\sigma \sqrt{k})^{-1} \sum_{l \leq p_k} d(l)$$

will not influence the asymptotic behavior of $(\sigma \sqrt{k})^{-1} \sum_{l=1}^{k} d(l)$, whereas for $l > p_k$ the distribution of $z(l)$ is sufficiently close to the stationary distribution $\pi$ to ensure Theorem A.2 to hold through. Precise mathematical details can be found in Grigorescu and Oprisan (1976, p. 70).
References