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Strong Differential Subordination to Briot–Bouquet Differential Equations

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Let $p(z)$ be analytic in the unit disc D, let $g(z)$ be convex in D, let $f(z)$ be analytic in D such that $z(f'(z)/f(z))$ is analytic and different from zero in D, and let α and β be complex numbers. The authors show that if

$$p(z)+\frac{zp'(z)}{\xi(f'(\xi)/f(\xi))[\alpha p(z)+\beta]}\prec g(z), \qquad z\in D,\ \xi\in D,$$

where $\prec$ denotes subordination and

$$\operatorname{Re}\left\{\frac{f'(\xi)}{f(\xi)}[\alpha g(z)+\beta]\right\}>0 \qquad z\in D,\xi\in D \tag{1}$$

is satisfied, then $p(z)\prec g(z)$.

Further, if the differential equation

$$q(z)+\frac{zq'(z)}{z(f'(z)/f(z))[\alpha q(z)+\beta]}=g(z) \tag{2}$$

verifying (1), has a univalent solution $q(z)$, then sharp subordination $p(z)\prec q(z)$ holds.

INTRODUCTION

Let α and β be complex numbers and $g(z)$ analytic in the unit disc D. In this paper we shall be concerned with determining properties of the solutions of the differential equation (2) with $q(0)=g(0)$. Differential equations of this form are said to be of Briot–Bouquet type [3, p. 403].

Several applications of these equations in the theory of univalent functions have recently appeared in [2, 4, and 5], and in [1] several results involving univalency of the solutions of (2) have appeared. In [2 and 4] results involving dominants of differential subordination have appeared.

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In this paper we obtain, for the equation (2), results involving dominants of differential subordination and we obtain the results of subordination of [2 and 4], when $f(z)=z$.

In [1] the conditions $f(z)$ analytic in D, $f(0)=0$, and $f'(0)\neq 0$, we can replace them by $f(z)$ analytic in D and $z(f'(z)/f(z))$ analytic and different from 0 in D, and these results are valid.

Let $h(z)$ and $h(z)$ be analytic in the unit disc D. The function $h(z)$ is subordinate to $H(z)$, written $h(z)\prec H(z)$, if $H(z)$ is univalent, $h(0)=H(0)$, and $h(D)\subset H(D)$. We shall write $h(z)\prec H(z)$, $z\in D_r$, when $h(0)=H(0)$ and $h(D_r)\subset H(D_r)$, where D_r is the disc of centre 0 and radius r, $0<r<1$.

Dominants of the Briot–Bouquet Differential Subordination

Definition 1. Let $F(\xi, z)$ be analytic in $D\times D$ and let $G(z)$ be analytic in D. The function $F(\xi, z)$ is strongly subordinate to $G(z)$, written $F(\xi, z)\prec G(z)$, $\xi\in D$, $z\in D$, if for each $\xi\in D$ the function of z, $F(\xi, z)$, is subordinate to $G(z)$.

Theorem 1. *Let $p(z)$ be analytic in D, $p(0)=c$, let $g(z)$ be univalent in D and let $f(z)$ be analytic in D such that $z(f'(z)/f(z))$ is analytic and different from zero in D. Let α and β be complex constants that verify* (1).

If $p(z)$ satisfies the differential subordination

$$p(z)+\frac{zp'(z)}{z(f'(z)/f(z))[\alpha p(z)+\beta]}\prec g(z) \tag{2}$$

then $p(z)\prec g(z)$.

Proof. Let us first take a closed disc $\bar{D}_r\subset D$ $(0<r<1)$. Then by the condition (3) and the "Principle of Subordination" (see [6, p. 36]), we can put

$$p(z)+\frac{zp'(z)}{z(f'(z)/f(z))[\alpha p(z)+\beta]}\prec g(z), \qquad z\in D_r.$$

If $p(z)$ is not subordinate to $g(z)$, $z\in D_r$, then there are $z_0\in D_r$, $\xi_0\in\partial D_r$, and $m\geqslant 1$ (see [4, Lemma B]) such that

$$p(z_0)=g(\xi_0)$$

$$z_0 p'(z_0)=m\xi_0 g'(\xi_0)$$

and we can put that

$$p(z_0)+\frac{z_0 p'(z_0)}{z_0(z_0 f'(z_0))/(f(z_0))[\alpha p(z_0)+\beta]}$$
$$=g(\xi_0)+\frac{m\xi_0 g'(\xi_0)}{z_0(z_0 f'(z_0)/f(z_0))[\alpha g(\xi_0)+\beta]}. \tag{4}$$

Now (1) implies

$$\left|\arg\left\{z_0\frac{f'(z_0)}{f(z_0)}[\alpha g(\xi_0)+\beta]\right\}\right|<\pi/2$$

and $\xi_0 g'(\xi_0)$ is in the direction of the outer normal to the convex domain $g(D_r)$, so that the right-hand side of (4) is a complex number outside $g(D_r)$. Because this contradicts (3′), we conclude that $p(z)\prec g(z)$ in D_r. Now r can be as near 1 as we want, and we can conclude the theorem.

THEOREM 2. *Let α and β be complex numbers, and g be convex in D with $g(0)=c$. Let $f(z)$ be analytic in D such that $z(f'(z)/f(z))$ is analytic and different from zero in D and suppose that (1) is verified. If the Cauchy's problem*

$$\begin{cases} u(z)+\dfrac{zu'(z)}{z(f'(z)/f(z))[\alpha u(z)+\beta]}=g(z) & z\in D \\ u(0)=c \end{cases} \tag{5}$$

has a univalent solution $u(z)$, then for all $p(z)$ analytic in D, $p(0)=c$, such that

$$p(z)+\frac{zp'(z)}{\xi(f'(\xi)/f(\xi))[\alpha p(z)+\beta]}$$
$$\prec u(z)+\frac{zu'(z)}{z(f'(z)/f(z))[\alpha u(z)+\beta]}=g(z), \tag{6}$$

$\xi\in D$, $z\in D$, is verified, then $p(z)\prec u(z)\prec g(z)$ and $u(z)$ is the best dominant of (6).

Proof. Let us first take again a closed disc $\bar{D}_r\subset D$ $(0<r<1)$. Then, by the condition (6) and the "Principle of Subordination," we can put

$$p(z)+\frac{zp'(z)}{\xi(f'(\xi)/f(\xi))[\alpha p(z)+\beta]}$$
$$\prec u(z)+\frac{zu'(z)}{\xi(f'(\xi)/f(\xi))[\alpha u(z)+\beta]}=g(z), \tag{7}$$

$\xi \in D$, $z \in D_r$; also, we deduce from (6) and Theorem 1 that $u(z) \prec g(z)$ and $p(z) \prec g(z)$. If $p(z)$ is not subordinate to $u(z)$ in D_r then there are $z_0 \in D_r$, $\xi_0 \in \partial D_r$, and $m \geqslant 1$, such that

$$p(z_0) = u(\xi_0)$$
$$z_0 p'(z_0) = m\xi_0 u'(\xi_0),$$

and we obtain that

$$\begin{aligned} p(z_0) &+ \frac{z_0 p'(z_0)}{\xi(f'(\xi_0)/f(\xi_0))[\alpha p(z_0) + \beta]} \\ &= u(\xi_0) + \frac{m\xi_0 u'(\xi_0)}{\xi_0(f'(\xi_0)/f(\xi_0))[\alpha u(\xi_0) + \beta]} \\ &= u(\xi_0) + m[g(\xi_0) - u(\xi_0)] \in g(D_r) \qquad \text{by (7).} \end{aligned}$$

Since $g(D_r)$ is a convex set, then

$$\frac{1}{m}\{u(\xi_0) + m[g(\xi_0) - u(\xi_0)]\} + \left(1 - \frac{1}{m}\right) u(\xi_0) = g(\xi_0) \in g(D_r),$$

which is a contradiction. Hence we conclude that $p(z) \prec u(z)$ in D_r, too.

Proceeding now as in the above case, we conclude that $p(z) \prec u(z)$ in D.

The existence of a best dominant of the differential subordination, if the condition (6) is verified, is provided by the following theorem, which we can derive from Theorems 1 and 2 and [1, Theorem 2].

THEOREM 3. *Let α and β be complex numbers with $\alpha \neq 0$ and let $g(z)$ be convex in D. Let $f(z)$ be analytic in D such that $z(f'(z)/f(z))$ is analytic and different from zero in D. If we let*

$$S = \alpha g(z) + \beta, \qquad P = \xi \frac{f'(\xi)}{f(\xi)}[\alpha g(z) + \beta], \qquad R = z\frac{f'(z)}{f(z)}[\alpha g(z) + \beta],$$

(a) $\operatorname{Re} P > 0$ *and* $\operatorname{Re} S > 0$, $\xi \in D$, $z \in D$;

(b) $T = 1/R$ *and* Q *as defined by* $Q' = RS'/S^2$, *are convex in D.*

Then, the solution $u(z)$ of the differential equation

$$u(z) + \frac{zu'(z)}{z(f'(z)/f(z))[\alpha u(z) + \beta]} = g(z) \qquad (u(0) = g(0)),$$

is the best dominant of (6).

EXAMPLE. Let the differential equation be

$$u(z)+zu'(z)\Big/\left(\frac{r}{1-r}\frac{1}{rz+R+1}\left[u(z)+R\right]\right)=g(z) \tag{8}$$

where $g(z)=z+1$, $R>1$, $r<1$, and

$$z\frac{f'(z)}{f(z)}=\frac{r}{1-r}\frac{1}{rz+R+1}.$$

Furthermore, it satisfies

$$\operatorname{Re}\frac{z+R+1}{r\xi+R+1}>0, \qquad z\in D,\ \xi\in D.$$

On the other hand, let $p(z)=\rho z+R$ be with $\rho=Rr/(R+1-r^2)$, then

$$p(z)+zp'(z)\Big/\left(\frac{r}{1-r}\frac{1}{r\xi+R+1}\left[p(z)+R\right]\right)\prec z+1, \qquad z\in D,\ \xi\in D.$$

Calculating the solution of (8) with $u(0)=1$ (see [1]), we obtain the univalent function $u(r)=z+1$ and, according to Theorems 1 and 2, we have that $p(z)\prec u(z)\prec g(z)$.

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