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Freedom in electroweak symmetry breaking and mass matrix of fermions in dimensional deconstruction model

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Abstract

There exists a freedom in a class of four-dimensional electroweak theories proposed by Arkani-Hamed et al. relying on deconstruction and Coleman–Weinberg mechanism. The freedom comes from the winding modes of the link variable (Wilson operator) connecting non-nearest neighbours in the discrete fifth dimension. Using this freedom, dynamical breaking of $SU(2)$ gauge symmetry, mass hierarchy patterns of fermions and Cabbibo–Kobayashi–Maskawa matrix may be obtained.

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Dimensional deconstruction [1] (see also subsequent works [2–10]) suggests the natural electroweak (EW) symmetry breaking in four dimensions without using supersymmetry or strong dynamics at the TeV scale physics. The interesting feature of such approach is that perturbative corrections in Higgs sector are finite. In these models, the extra dimensions are the discrete lattice. The simplest version [2] is given by the sites on a circle. By the Coleman–Weinberg mechanism [11], the gauge symmetry can be broken spontaneously. The effective potential of the Higgs field becomes finite. In the naive model for $SU(2)$ gauge theory, the Higgs field is, however, triplet, that is, in the adjoint representation. In the realistic models, of course, the Higgs field should be an $SU(2)$ doublet. In order to introduce the doublet Higgs field, the $N \times N$ torus (moose) of the lattice has been introduced [2] and has been investigated in [6]. Especially the simplest case of $N = 2$ torus case has been constructed in [7] and the most economical case that the Higgs field is pseudo-Goldstone boson in an $SU(5)/SO(5)$ has been presented in [8].

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However, it is quite important (which was not properly realized so far) that there may be more freedom in EW symmetry breaking patterns from dimensional deconstruction due to its dependence from the non-nearest neighbour couplings in theory space. Eventually, it means that bigger number of phenomenologically accepted EW symmetry breakings may be realized by the corresponding choice of boundary condition from latticized dimension. The variety of novel Higgs sectors may emerge. In the context of the $U(1)^N$ model, Wilson-line operators with arbitrary couplings have been discussed in [9], where such operators are generated with finite coefficients by radiative corrections. The non-nearest neighbour couplings in theory space, however, may appear geometrically. We may assume the sites on the lattice corresponding to the branes embedded in the higher-dimensional spacetime. Then the link variables connecting the different sites may correspond to the open string. Hence if the embedding space has non-trivial homotopy (S^1 is most simple but non-trivial case), there will appear several couplings corresponding to the winding mode of the open strings (say, an open string can connect two different branes after winding S^1 several times). When the embedding manifold M is compact but the dimensions (codimension of the brane) is larger than one, the homotopy $\pi_1(M)$ can be more complicated and the non-nearest neighbour couplings appear in general. Another interpretation for such couplings could be that continuum limit of higher-dimensional gauge theory is non-local. Thus, we consider a (non-linear) generalization of the simplest model where the sites of the lattice lies on a circle. Then the Higgs fields are in the adjoint representation. Therefore the theory under consideration is a non-linear, toy model for EW symmetry breaking. More realistic models probably may be constructed by considering the discrete torus as in [3,6,7].

Here we propose a generalization of the model [2], so that it may qualitatively describe the dynamical EW breaking and the mass hierarchy of quarks or leptons between different generations.[2] The model [2] includes N-copies of the gauge field A_μ^n and N link variables $U_{n,n+1}$, following [1]. For the link variables, we impose a periodic boundary condition $U_{n+N,n+N+1} = U_{n,n+1}$ and sometimes we restrict n to be $n = 0, 1, 2, \ldots, N-1$. $U_{n,n+1}$ is assumed to be unitary and $U_{n+1,n}$ is defined by $U_{n+1,n} \equiv U_{n,n+1}^\dagger = U_{n,n+1}^{-1}$.

Before writing our new Lagrangian, it is convenient to define a variable $U_{n,l}$, a link variable connecting "the non-nearest neighbours", by

$$U_{n,l} = \begin{cases} U_{n,n+1}U_{n+1,n+2}\cdots U_{l-2,l-1}U_{l-1,l}, & \text{when } l > n, \\ 1, & \text{when } l = n, \\ U_{n,n-1}U_{n-1,n-2}\cdots U_{l+2,l+1}U_{l+1,l}, & \text{when } l < n. \end{cases} \tag{1}$$

The following Lagrangian is the main starting point of our new model:

$$\mathcal{L} = -\frac{1}{2g^2}\sum_{n=0}^{N-1} \operatorname{tr} F_{\mu\nu}^n F^{n\mu\nu} + \frac{1}{4}\sum_{n,l} a_{nl} \operatorname{tr}\bigl[(D_\mu U_{n,l})^\dagger D^\mu U_{n,l}\bigr]. \tag{2}$$

Here $F_{\mu\nu}^n$ is the field strength given by A_μ^n and a_{nl}'s are constants specifying the couplings including non-nearest neighbours. This kind of couplings was first discussed for gravity in [13], and is useful to obtain the induced positive cosmological constant, which may serve as a quite simple model for the dark energy of our accelerating universe. In the model [2], only nearest neighbour couplings have been introduced. If we assume $U_{n,l}$ connects the branes, in the present model, the branes are connected in a rather complicated way. One may suppose the branes correspond to the site on a circle. Then $U_{n,l}$ connects the branes like a mesh or a net. Such a case might not occur if the codimension of the spacetime is one. We may need to consider more complicated spacetime or the spacetime whose codimension is two or more.

If we denote the gauge group as G, the Lagrangian (2) has G^N gauge symmetry. The non-nearest neighbour couplings in (2) give more degrees of freedom to the model, and are useful to trigger the dynamical breaking of

[2] The model with an infinite number of gauge theories which are linked by scalars has been considered in [12] in order to get an infinite tower of massive gauge fields. The model [12] may be in a same class with that in [2]. We also note that a generalization of the model [2] by using the graph structure has been done in [10].

gauge symmetry and the mass hierarchy. As it will be shown later, the induced Coleman–Weinberg potential and the mass matrix of fermions can include an arbitrary function originating from the non-nearest neighbour couplings. The proper choice of the functions induces the gauge symmetry breaking and the mass hierarchy.

Since $U_{n,l} \neq U_{n,l+N}$ nor $U_{n,l} \neq U_{n+N,l}$ in general, the sums about n and l can be from $-\infty$ to ∞. In (2), the covariant derivative D_μ is defined by

$$D_\mu U_{n,l} \equiv \partial_\mu U_{n,l} - iA^n_\mu U_{n,l} + iU_{n,l}A^l_\mu. \tag{3}$$

In the Lagrangian (2), the terms $\mathcal{L}_{\rm M}$ which do not include derivative can be regarded as mass terms for the gauge fields and $\mathcal{L}_{\rm M}$ is explicitly given by

$$\mathcal{L}_{\rm M} = \frac{1}{4}\sum_{n,l} a_{nl}\,{\rm tr}\big[A^n_\mu A^{n\mu} + A^l_\mu A^{l\mu} - 2A^l_\mu U_{l,n}A^{n\mu}U_{n,l}\big]. \tag{4}$$

By using the gauge transformation, one may impose the unitary gauge condition where $U_{n,n+1}$ does not depend on n as

$$U_{n,n+1} = {\rm e}^{iu}. \tag{5}$$

First we consider the electrodynamics case, where the gauge group is $U(1)$. Since $U^\dagger_{l,n} = U_{n,l}$, and $U_{l,n}$ commutes with $A^{n\mu}$, one obtains

$$\mathcal{L}_{\rm M} = \frac{1}{2}\sum_{n,l} a_{nl}\big[A^n_\mu A^{n\mu} + A^l_\mu A^{l\mu} - 2A^n_\mu A^{l\mu}\big]. \tag{6}$$

There does not appear u-dependence.

As a non-trivial toy model, the case that the gauge group is $SU(2)$ is interesting. Then the action (2) has $SU(2)^N$ gauge symmetry. Writing

$$A^n_\mu = \frac{1}{2}\tau^a A^{na}_\mu, \qquad u = \frac{\upsilon_0}{2}\tau^3 \tag{7}$$

with τ^a's ($a = 1, 2, 3$) being Pauli matrices, we find

$$\begin{aligned}\mathcal{L}_{\rm M} = \frac{1}{2}\sum_{n,l} a_{nl}\big[&A^{na}_\mu A^{na\mu} + A^{la}_\mu A^{la\mu} - 2\cos\big((l-n)\upsilon_0\big)\big(A^{n1}_\mu A^{l1\mu} + A^{n2}_\mu A^{l2\mu}\big) \\ &+ 2\sin\big((l-n)\upsilon_0\big)\big(A^{l1}_\mu A^{n2\mu} - A^{l2}_\mu A^{n1\mu}\big) - 2A^{n3}_\mu A^{l3\mu}\big].\end{aligned} \tag{8}$$

We may assume a_{nl} only depends on the absolute value of the difference between n and l:

$$a_{nl} = a\big(|n-l|\big). \tag{9}$$

We also Fourier transform A^{na}_μ as

$$A^{na}_\mu = \frac{1}{\sqrt{N}}\sum_{L=0}^{N-1}\hat{A}^{La}_\mu {\rm e}^{i\frac{2\pi nL}{N}}. \tag{10}$$

Since A^{na}_μ is real, $\hat{A}^{LA}_\mu = (\hat{A}^{(N-L)A}_\mu)^*$. Then we obtain

$$\mathcal{L}_{\mathrm{M}} = 2\sum_{L=0}^{N-1}\sum_{l=1}^{\infty} a(l)\Bigg[\left(1-\cos(l\upsilon_0)\cos\left(\frac{2\pi l L}{N}\right)\right)\left(\left(\hat{A}_\mu^{L1}\right)^*\hat{A}^{L1\mu}+\left(\hat{A}_\mu^{L2}\right)^*\hat{A}^{L2\mu}\right)$$
$$+\sin(l\upsilon_0)\sin\left(\frac{2\pi l L}{N}\right)\left(\left(\hat{A}_\mu^{L1}\right)^*\hat{A}^{L2\mu}-\left(\hat{A}_\mu^{L2}\right)^*\hat{A}^{L1\mu}\right)$$
$$+\left(1-\cos\left(\frac{2\pi l L}{N}\right)\right)\left(\hat{A}_\mu^{L3}\right)^*\hat{A}^{L3\mu}\Bigg]. \tag{11}$$

As a result the mass matrix for the gauge fields is given by

$$M_n^2(\upsilon_0) = 32g^2\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}\left\{\sum_{l=1}^{\infty}a(l)\begin{pmatrix}2\sin^2\left(l\left(\frac{\upsilon_0}{2}+\frac{\pi L}{N}\right)\right) & 0 & 0\\ 0 & 2\sin^2\left(l\left(\frac{\upsilon_0}{2}-\frac{\pi L}{N}\right)\right) & 0\\ 0 & 0 & 2\sin^2\left(\frac{\pi l L}{N}\right)\end{pmatrix}\right\}$$
$$\times\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0\\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{12}$$

When $\upsilon_0 = 0$, only the mode corresponding to $L = 0$ is massless. Then $SU(2)^N$ gauge symmetry is broken to $SU(2)$. When $\upsilon_0 \neq 0$, the massless gauge field is only $\hat{A}_\mu^{L=0,a=3}$. Thus, the gauge symmetry is broken down to $U(1)$.

In order to obtain non-vanishing υ_0, one may consider the Coleman–Weinberg mechanism [11], where the one-loop induced potential for υ_0 is given by

$$V(\upsilon_0) = \frac{3\Lambda^2}{32\pi^2}\operatorname{tr}\left(M_n^2(\upsilon_0)\right)+\frac{3}{64\pi^2}\operatorname{tr}\left\{\left(M_n^2(\upsilon_0)\right)^2\ln\left(\frac{M_n^2(\upsilon_0)}{\Lambda^2}\right)\right\}. \tag{13}$$

Here Λ^2 is the UV cut-off parameter. As the kinetic term for υ_0 in (2) is given by

$$\mathcal{L}_{\mathrm{K}} = \frac{1}{4}\sum_{l=1}^{\infty}a(l)l^2\partial_\mu\upsilon_0\partial^\mu\upsilon_0, \tag{14}$$

the canonically normalized field ϕ is

$$\phi = \upsilon_0\sqrt{\frac{\sum_{l=1}^{\infty}a(l)l^2}{2}}. \tag{15}$$

It is interesting to consider now some examples. Let $f(x)$ be a function which can be expanded by a Taylor series:

$$f(x) = \sum_{k=0}^{\infty}\alpha_k x^k. \tag{16}$$

$a(l)$ is chosen as

$$a(l) = \begin{cases}\alpha_k, & \text{when } l = Nk+1\ (k=0,1,2,\ldots),\\ 0, & \text{when } l \neq Nk+1.\end{cases} \tag{17}$$

Here k can be regarded as the winding number. As a result

$$2\sum_{l=1}^{\infty}a(l)\sin^2\left(l\left(\frac{\upsilon_0}{2}\pm\frac{\pi L}{N}\right)\right) = f(1)-\frac{1}{2}\left\{\mathrm{e}^{i(\upsilon_0\pm\frac{2\pi L}{N})}f\left(\mathrm{e}^{iN\upsilon_0}\right)+\mathrm{e}^{-i(\upsilon_0\pm\frac{2\pi L}{N})}f\left(\mathrm{e}^{-iN\upsilon_0}\right)\right\}. \tag{18}$$

Therefore we have

$$2\sum_{L=0}^{N-1}\sum_{l=1}^{\infty}a(l)\sin^2\left(l\left(\frac{\upsilon_0}{2}\pm\frac{\pi L}{N}\right)\right)=Nf(1). \tag{19}$$

Then in the potential (13), the term proportional to Λ^2 does not depend on υ_0. We also have

$$\sum_{L=0}^{N-1}\left(2\sum_{l=1}^{\infty}a(l)\sin^2\left(l\left(\frac{\upsilon_0}{2}\pm\frac{\pi L}{N}\right)\right)\right)^2=\begin{cases}N\left\{f(1)^2+\frac{1}{2}f\left(e^{iN\upsilon_0}\right)f\left(e^{-iN\upsilon_0}\right)\right\}, & \text{when } N>2,\\ 2f(1)^2+\frac{1}{2}\left\{e^{i\upsilon_0}f\left(e^{2i\upsilon_0}\right)+e^{-i\upsilon_0}f\left(e^{-2i\upsilon_0}\right)\right\}^2, & \text{when } N=2.\end{cases} \tag{20}$$

When $N>2$, if $f(x)\propto x^I$ by a non-negative integer I, in the potential (13), the term proportional to $\ln\Lambda^2$ does not depend on υ_0. The model with $f(x)\propto x^I$ does not have any essential difference with that in [2], since $f(x)$ is a monomial means not to include different winding modes. On the other hand, in the general case in which $f(x)$ is a polynomial including different winding modes, this term depends on υ_0. In the model [2], there does not appear $\ln\Lambda^2$ terms in the field (corresponding to υ_0 here) dependent part. In the potential we now have included $\ln\Lambda^2$ term in general but in return for it, we have a degrees of freedom of an arbitrary (Taylor expansible) function $f(x)$. As $U_{n,l}$ with $|n-l|>1$ is included, even for the case of $N=2$, the potential can be rather different from that in [2]. The potential (13) for $N=2$ case is explicitly found to be

$$\begin{aligned}V(\upsilon_0)=\frac{96g^4}{\pi^2}\Bigg[&\left\{f(1)^2+\frac{1}{4}\left\{e^{i\upsilon_0}f\left(e^{2i\upsilon_0}\right)+e^{-i\upsilon_0}f\left(e^{-2i\upsilon_0}\right)\right\}^2\right\}\\&\times\ln\left\{\frac{f(1)^2-\frac{1}{4}\{e^{i\upsilon_0}f(e^{2i\upsilon_0})+e^{-i\upsilon_0}f(e^{-2i\upsilon_0})\}^2}{\Lambda^4}\right\}\\&-f(1)\left\{e^{i\upsilon_0}f\left(e^{2i\upsilon_0}\right)+e^{-i\upsilon_0}f\left(e^{-2i\upsilon_0}\right)\right\}\ln\left\{\frac{f(1)-\frac{1}{2}\{e^{i\upsilon_0}f(e^{2i\upsilon_0})+e^{-i\upsilon_0}f(e^{-2i\upsilon_0})\}}{f(1)+\frac{1}{2}\{e^{i\upsilon_0}f(e^{2i\upsilon_0})+e^{-i\upsilon_0}f(e^{-2i\upsilon_0})\}}\right\}\Bigg]\\&+(\upsilon_0\text{ independent terms}).\end{aligned} \tag{21}$$

If we define

$$X^{\pm}\equiv f(1)\pm\frac{1}{2}\left\{e^{i\upsilon_0}f\left(e^{2i\upsilon_0}\right)+e^{-i\upsilon_0}f\left(e^{-2i\upsilon_0}\right)\right\}, \tag{22}$$

$V(\upsilon_0)$ in (21) can be rewritten as

$$\begin{aligned}V(\upsilon_0)&=\frac{96g^4}{\pi^2}\left[\frac{X^{+2}+X^{-2}}{2}\ln\left(\frac{X^+X^-}{\Lambda^4}\right)-\frac{X^{+2}-X^{-2}}{2}\ln\left(\frac{X^+}{X^-}\right)\right]\\&\quad+(\upsilon_0\text{ independent terms})\\&=\frac{96g^4}{\pi^2}\left[\frac{X^{-2}}{2}\ln\left(X^-\right)^2+\frac{(2f(1)-X^-)^2}{2}\ln\left(2f(1)-X^-\right)^2-\frac{(2f(1)-X^-)^2+X^{-2}}{2}\ln\Lambda^4\right]\\&\quad+(\upsilon_0\text{ independent terms}).\end{aligned} \tag{23}$$

We should note that $\upsilon_0=0$ corresponds to $X^-=0$. When υ_0 is small, we find

$$X^-\sim\left(\frac{1}{2}f(1)+4f'(1)-2f''(1)\right)\upsilon_0^2. \tag{24}$$

When υ_0 (X^-) is small, the potential $V(\upsilon_0)$ behaves as

$$V(\upsilon_0)=\frac{96g^4}{\pi^2}\left[(\upsilon_0\text{ independent terms})+\left(-4f(1)\ln\left(2f(1)\right)-2+2f(1)\ln\Lambda^4\right)X^-+\mathcal{O}\left(X^{-2}\ln X^-\right)\right]$$

$$\sim \frac{96g^4}{\pi^2}\bigg[(\upsilon_0 \text{ independent terms}) + \left(-4f(1)\ln\big(2f(1)\big) - 2 + 2f(1)\ln\Lambda^4\right)\left(\frac{1}{2}f(1) + 4f'(1) - 2f''(1)\right)\upsilon_0^2\bigg]. \tag{25}$$

As the term linear in X^- appears, the point $X^- = 0$ ($\upsilon_0 = 0$) is unstable in general. For example, for the choice $f(x) = 1 - \frac{x}{2}$, one gets

$$V(\upsilon_0) \sim \frac{96g^4}{\pi^2}\bigg[(\upsilon_0 \text{ independent terms}) - \frac{3}{2}\left(-2 + \ln\Lambda^4\right)\upsilon_0^2\bigg]. \tag{26}$$

If $\Lambda^4 > \mathrm{e}^2$, the coefficient of υ_0^2 becomes negative and the point of $\upsilon_0 = 0$ is unstable and there could be a non-trivial vacuum expectation value. Note that with the choice $f(x) = 1 - \frac{x}{2}$, $X^\pm$ are given by

$$X^\pm = \mp 2(\cos\upsilon_0 \pm 1)\left(\cos^2\upsilon_0 \mp \cos\upsilon_0 - \frac{1}{4}\right). \tag{27}$$

Since $|\cos\upsilon_0| \leqslant 1$, X^- is bounded as

$$\frac{1}{2}\left\{-\left(\frac{5}{3}\right)^{3/2} + 1\right\} \leqslant X^- \leqslant \frac{1}{2}\left\{\left(\frac{5}{3}\right)^{3/2} + 1\right\}. \tag{28}$$

X^- has maximum at $\cos\upsilon_0 = -\frac{1}{2}\sqrt{\frac{5}{3}}$ and minimum at $\cos\upsilon_0 = \frac{1}{2}\sqrt{\frac{5}{3}}$. In the region given by (28), $V(\upsilon_0)$ (23) is finite, then $V(\upsilon_0)$ is bounded and has finite maximum and minimum. Eq. (26) tells that at the minimum, υ_0 does not vanish.

One sees

$$\frac{1}{2}\sum_{l=1}^{\infty} a(l)l^2 = \frac{1}{2}\sum_{k=0}^{\infty}\alpha_k(Nk+1)^2 = \frac{1}{2}\left\{N^2\big(f''(1) + f'(1)\big) + 2Nf'(1) + f(1)\right\} \tag{29}$$

for general N. Then for the case of $N = 2$, the canonically normalized field ϕ in (15) is given by

$$\phi = \upsilon_0\sqrt{\frac{4f''(1) + 8f'(1) + f(1)}{2}}. \tag{30}$$

In the similar way the coupling with the spinor fields which may be identified with quarks or leptons could be introduced. In order to specify the theory, one may restrict the gauge symmetry to be $SU(2)$ and the spinors of 2-dimensional representation of $SU(2)$ may be considered:

$$\Psi^n = \begin{pmatrix} u^n \\ d^n \end{pmatrix}. \tag{31}$$

If we regard the spinors as quarks, we may identify u^n as up-type quarks and d^n as down-types ones. Then a rather general Lagrangian with general N has the following form:

$$\mathcal{L}_f = i\sum_{n=0}^{N-1}\bar{\Psi}^n D_\mu\left(A_\mu^n\right)\gamma^\mu\Psi^n - \sum_{n=0}^{N-1}\sum_{l=-\infty}^{\infty} b_{nl}\bar{\Psi}^n U_{n,l}\Psi^l. \tag{32}$$

Here $\Psi^{n+N} = \Psi^n$. First we assume as in a_{nl} of (9) that b_{nl} depends on the absolute value of $n - l$:

$$b_{nl} = b\big(|n - l|\big). \tag{33}$$

Choosing $U_{n,n+1}$ as in (5) with (7), one finds

$$\sum_{n=0}^{N-1}\sum_{l=-\infty}^{\infty} b_{nl}\bar{\Psi}^n U_{n,l}\Psi^l = \sum_{L=0}^{N-1}(\bar{\hat{u}}^L, \bar{\hat{d}}^L)\begin{pmatrix} B(L) & 0 \\ 0 & B(L)\end{pmatrix}\begin{pmatrix}\hat{u}^L \\ \hat{d}^L\end{pmatrix}. \tag{34}$$

Here we have written Ψ^n as

$$\Psi^n = \frac{1}{\sqrt{N}}\sum_{L=0}^{N-1}\hat{\Psi}^L e^{i\frac{2\pi nL}{N}} = \frac{1}{\sqrt{N}}\sum_{L=0}^{N-1}\begin{pmatrix}\hat{u}^L \\ \hat{d}^L\end{pmatrix} e^{i\frac{2\pi nL}{N}}, \tag{35}$$

and $B(L)$ is defined by

$$B(L) = b(0) + 2\sum_{l=1}^{\infty}\cos\left(l\left(\frac{\upsilon_0}{2} + \frac{2\pi L}{N}\right)\right)b(l). \tag{36}$$

Let $g(x)$ is an arbitrary even function which can be expanded as a Fourier series

$$g(x) = \sum_{l=-\infty}^{\infty} g_l e^{ilx} = g_0 + 2\sum_{l=1}^{\infty} g_l\cos(lx), \tag{37}$$

and the identification is done:

$$b(l) = g_l. \tag{38}$$

Then

$$B(L) = g\left(\frac{\upsilon_0}{2} + \frac{2\pi L}{N}\right). \tag{39}$$

In order to specify the model we consider $N = 3$ case. Choosing $g(x)$ as an exponential function, for example

$$g = \zeta e^{\eta x} \tag{40}$$

with $\zeta e^{\frac{\eta\upsilon_0}{2}} \sim 10$ MeV and $\eta \sim 2$, we find $B(0) \sim 10$ MeV, $B(1) \sim 1$ GeV, $B(2) \sim 100$ GeV. Hence, we may identify the quark of $L = 0$ as (u, d), $L = 1$ as (c, s), and $L = 2$ as (b, t).

Although we may explain the hierarchy between the generation of the quarks by the mass matrix (34), the mass of the up-type quarks and that of the down-type ones are degenerate in (34) and there does not appear Cabbibo–Kobayashi–Maskawa matrix. In order to solve the problem, the assumption in (33) may be discarded. Now we introduce N arbitrary real functions $g_{nn}(x)$ $n = 0, \ldots, N-1$ and $\frac{N(N-1)}{2}$ arbitrary complex functions $g_{nm}(x)$ $(N-1 \geqslant n > m \geqslant 0)$. The function $g_{nm}(x)$ with $(n < m)$ are defined by a complex conjugate of $g_{nm}(x)$: $g_{nm}(x) \equiv g_{mn}(x)^*$. The natural assumption is that these functions can be expanded as a Fourier series:

$$g_{nm}(x) = \sum_{k=-\infty}^{\infty} g_{nm}^k e^{ikx}. \tag{41}$$

With identification b_{nl} by

$$b_{nm+Nk} = g_{nm}^k \tag{42}$$

one gets

$$\sum_{n=0}^{N-1}\sum_{l=-\infty}^{\infty} b_{nl}\bar{\Psi}^n U_{n,l}\Psi^l = \sum_{n,m=0}^{N-1}\bar{\Psi}^n\tilde{U}_{n,m}\Psi^m = \sum_{n,m=0}^{N-1}(\bar{u}^n, \bar{d}^n)\begin{pmatrix} G_{nm}(\upsilon_0) & 0 \\ 0 & G_{nm}(-\upsilon_0)\end{pmatrix}\begin{pmatrix} u^n \\ d^m\end{pmatrix}. \tag{43}$$

Here

$$\tilde{U}_{nm} \equiv \sum_{k=-\infty}^{\infty} b_{nm+kN} U_{n,m+kN}, \tag{44}$$

and

$$G_{nm}(\upsilon_0) \equiv e^{i\frac{(m-n)\upsilon_0}{2}} g_{nm}\left(\frac{N\upsilon_0}{2}\right). \tag{45}$$

We should note that the loop of the fermion gives an additional contribution to the Coleman–Weinberg potential in (13) by

$$V_f(\upsilon_0) = -\frac{\Lambda^2}{32\pi^2}\operatorname{tr}\big(\tilde{U}(\upsilon_0)\tilde{U}^\dagger(\upsilon_0)\big) - \frac{1}{64\pi^2}\operatorname{tr}\left\{\big(\tilde{U}(\upsilon_0)\tilde{U}^\dagger(\upsilon_0)\big)^2 \ln\left(\frac{\tilde{U}(\upsilon_0)\tilde{U}^\dagger(\upsilon_0)}{\Lambda^2}\right)\right\}. \tag{46}$$

If we regard $G_{nm}(\upsilon_0)$ as an $N \times N$ matrix, $G_{nm}(\upsilon_0)$ is Hermitian: $G_{nm}(\upsilon_0) = G_{mn}(\upsilon_0)^*$. Hence, we can diagonalize $G_{nm}(\upsilon_0)$ by an $N \times N$ unitary matrix $V_{nm}(\upsilon_0)$:

$$\sum_{n',m'=0}^{N-1} V_{nn'}(\upsilon_0) G_{n'm'}(\upsilon_0) V_{m'm}(\upsilon_0)^\dagger = G_n(\upsilon_0)\delta_{nm}. \tag{47}$$

Then the mass eigenstates in (43) are given by

$$\tilde{u}^n = \sum_{n'=0}^{N-1} V_{nn'}(\upsilon_0) u^{n'}, \qquad \tilde{d}^n = \sum_{n'=0}^{N-1} V_{nn'}(-\upsilon_0) d^{n'}. \tag{48}$$

The mass eigenvalues of up-type quarks are given by $G_n(\upsilon_0)$. On the other hand, the mass eigenvalues of down-type quarks are given by $G_n(-\upsilon_0)$. Since $G_n(-\upsilon_0) \neq G_n(\upsilon_0)$ in general, the masses of up-type quarks can be different from those of down-type quarks. The gauge couplings in the Lagrangian (32) are:

$$\bar{u}\big(\gamma^\mu A_\mu\big)d = \sum_{n,m,n'=0}^{N-1} \bar{\tilde{u}}^n\big(\gamma^\mu A_\mu\big) V_{nn'}(\upsilon_0) V_{n'm}(-\upsilon_0)^\dagger \tilde{d}^m. \tag{49}$$

Since $\tilde{u}^n$ and $\tilde{d}^m$ are the eigenstates of the mass, we may identify $M_{nm} \equiv \sum_{n'=0}^{N-1} V_{nn'}(\upsilon_0) V_{n'm}(-\upsilon_0)^\dagger$ as the Cabbibo–Kobayashi–Maskawa (CKM) matrix.

In this Letter we have given a generalization of the 5-dimensional model studied by Arkani-Hamed et al. [2] using N branes and N copies of fields and symmetries [1]. The new point implemented in our Letter is the introduction of the link variables $U_{n,l}$ which connect branes (nth and lth) in the non-nearest neighborhood. Since the link variables in the 5th dimension give the Higgs fields, our model becomes a new model of the Higgs sector. If the 5th dimension is considered as a discrete circle made from N points, possibly, our link variables have winding numbers with respect to the discrete circle. Owing to this non-nearest neighbour link variables, we have obtained the following interesting results:

1. the dynamical breaking of gauge symmetry occurs, and
2. the quark (or lepton) masses and the CKM mixing matrix are dynamically induced, reproducing the mass hierarchy.

More explicitly, in a model with $SU(2)$ gauge symmetry, the set of coefficients $\{\alpha_k\}$ in (17) (k: winding number) of the kinetic terms for $U_{n,n+1+Nk}$ gives a function $f(x)$ which determines the non-vanishing vacuum expectation value of the Higgs scalar, following the Coleman–Weinberg mechanism. Accordingly the $SU(2)$ gauge symmetry is dynamically broken down to $U(1)$. This phenomenon occurs even for $N = 2$, but does not occur without incorporating the different winding sectors.

Regarding the fermion mass matrix, we have studied the $SU(2)$ model. Here the total number N of branes becomes the number of generations of quarks or leptons, and the different waves of fermion fields standing on the discrete N points give the different generations. Therefore, the model with $N = 3$ is a three generation model. The $SU(2)$ symmetry used here is not $SU(2)_L$ nor $SU(2)_R$, but the diagonal group of $SU(2)_V$. The coefficients b^k_{mn} of $\bar{\Psi} U_{m,n+Nk} \Psi$ correspond to the Yukawa couplings. Here the Yukawa couplings also have the winding number k with respect to the discrete circle of N points. Similarly as before, the winding number dependence of the Yukawa couplings gives a function $g_{mn}(x)$ which determines the mass eigenvalues and CKM mixing matrix of the model.

In a simplified case with $SU(2)$ symmetry, the masses $m_1, m_2, m_3, \ldots,$ and m_N of 1st, 2nd, 3rd, ... , and Nth generation fermions, respectively, give the hierarchical structure following

$$m_1 : m_2 : m_3 : \cdots : m_N = g\left(\frac{\upsilon_0}{2} + \frac{2\pi}{N}\right) : g\left(\frac{\upsilon_0}{2} + \frac{4\pi}{N}\right) : g\left(\frac{\upsilon_0}{2} + \frac{6\pi}{N}\right) : \cdots : g\left(\frac{\upsilon_0}{2} + 2\pi\right), \tag{50}$$

where υ_0 is the vacuum expectation value of the Higgs scalar determined dynamically à la Coleman and Weinberg, and N is the number of generations. Since the $SU(2)$ symmetry can be broken dynamically, the mass matrix of up-type quarks, and that of down-type quarks can be determined differently, giving different hierarchical structure.

As was stated above we have used the $SU(2)_V$ symmetry, and the left–right asymmetry is not incorporated, so that the mixing matrices of L-handed current (CC interaction) and the R-handed current (not yet observed) are identical. This result itself is not bad in our study focused on the Higgs sector. We have to eliminate, however, the R-handed current from our model and construct a realistic $SU(2) \times U(1)$ model in order to obtain a realistic gauge sector. For this purpose, we have to introduce the chirality of branes on which the fermions with the same chirality live. Then, the gauge interaction comes from the connection of the branes with the same chiralities and the Higgs interactions connect those with the opposite chiralities (see Ref. [5]).

In summary, we should stress that the toy, non-linear deconstruction model discussed above is not realistic in the same way as first model [2]. Moreover, some properties of such non-linear theory (like UV completion, its continuum interpretation, etc.) are not quite clear and should be further investigated. Nevertheless, in our opinion, the additional freedom which the new deconstruction model may introduce to EM symmetry breaking patterns may be useful in the generalization of more realistic versions [6–8].

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