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The Spectrum of de Bruijn and Kautz Graphs

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We give here a complete description of the spectrum of de Bruijn and Kautz graphs. It is well known that spectral techniques have proved to be very useful tools to study graphs, and we give some examples of application of our result, by deriving tight bounds on the expansion parameters of those graphs.

1. INTRODUCTION

The d-ary de Bruijn directed graph $BD(d, n)$, is a labelled graph, on d^n vertices labelled with all n-tuples over the alphabet $0, 1, \ldots, d-1$, such that there is a directed edge from a vertex $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$, whenever $b_i = a_{i+1}$ for all i in the range $1 \leq i \leq n-1$. By replacing each directed edge by an undirected edge, we obtain the undirected de Bruijn graph $B(d, n)$, which is regular of degree $2d$. Let us note here that this definition allows loops, and double edges. More precisely this last graph has exactly d loops (one for each vertex of the form $(\alpha, \alpha, \ldots, \alpha)$), and there are $\frac{d(d-1)}{2}$ double edges between the vertices of the form $(\alpha, \beta, \alpha, \beta, \ldots)$ and $(\beta, \alpha, \beta, \alpha, \ldots)$ (with $\alpha \neq \beta$).

This definition differs slightly from the more common definition, which deletes these double edges and loops. Our definition has the advantage in tackling regular graphs, and we will see that by using this definition, we will obtain interesting connections between the spectra of these graphs and those of Kautz graphs. Moreover, there is only a 'small difference' between these two definitions, and many results which hold for one type of graph can easily be extended to the other category. We give an illustration of this fact in Section 5.

We define here the Kautz graph $K(d, n)$ as the undirected graph on $d^n + d^{n-1}$ vertices, which is the induced subgraph of $B(d+1, n)$, with vertex set the n-tuples, $(a_1, a_2, \ldots, a_n)$, such that $a_i \neq a_{i+1}$, for all i in the range $1 \leq i \leq n-1$. The directed Kautz graph $KD(d, n)$ has the same set of vertices and is induced by $BD(d+1, n)$. Here again our definition keeps all the double edges of the defining de Bruijn graph (but there are no more loops), and these Kautz graphs are regular of degree $2d$.

These de Bruijn and Kautz graphs have extensively been used in various contexts, including the design of feedback registers [8, 11] or decoders [5], and computer networks and architectures [2, 3, 7, 10, 13]. Nevertheless, many parameters of these graphs are not known precisely and, as explained in [14], it would be desirable to compute the spectrum of these graphs, in order to bring in algebraic tools. The main result that we present here is the explicit spectrum of the de Bruijn graph $B(d, n)$ and of the Kautz graph $K(d, n)$. Moreover, by a covering argument, we show that there is a natural connection between the spectra of de Bruijn and Kautz graphs. Finally we show how one can use this result on the spectrum to obtain other results on the expansion parameters of these graphs.

2. THE TECHNIQUE WE USE TO OBTAIN THE SPECTRUM OF DE BRUIJN AND KAUTZ GRAPHS

The same technique will be used to obtain the spectra of both the de Bruijn and Kautz graphs and it can be described as follows.

(1) We will not study the spectrum of the adjacency matrix of either the de Bruijn graph or the Kautz graph directly, but will proceed by considering the adjacency matrices A_{BD} and A_{KD} of the *directed* de Bruijn $BD(d, n)$ and $KD(d, n)$ Kautz graphs respectively, and we will replace these two matrices by two matrices which are simpler by making several transformations of the form $f_U : A \to U^{-1}AU$ on the adjacency matrix, where U is a unitary matrix. In other words for the de Bruijn graph for example, we will construct a series of transformations $A_{BD} \overset{U_1}{\to} A_1 \overset{U_2}{\to} A_2 \overset{U_3}{\to} \cdots \overset{U_f}{\to} A_f$ such that the final matrix A_f is sparser.

(2) To explain our choice of the unitary matrices U_i which enable us to obtain a matrix A_f, which is very sparse, it is convenient to see those matrices A_i as adjacency matrices of directed weighted graphs (with complex weights in fact). We shall say that a directed weighted graph G is *equivalent* to a graph G' iff we can find a relation of the form $A = U^{-1}A'U$ where A, A' are the adjacency matrices of G and G' respectively, and U is a unitary matrix. This will be denoted $G \equiv G'$. Therefore our series of transformations can be seen as $BD(d, n) \equiv G_1 \equiv G_2 \equiv \cdots \equiv G_f$, where the graphs G_i have A_i as their adjacency matrix.

(3) Once we have obtained A_f it is easy to check that the eigenvalues of the adjacency matrix A_B of the undirected graph $B(d, n)$ are the eigenvalues of the Hermitian matrix $A_f + A_f^*$ (A_f^* is the adjoint of A_f), which will be easy to compute. Another way to express this is to say that with this method we have found a simple equivalent of the undirected de Bruijn $B(d, n)$ graph which will be the *symmetric covering* of G_f whose spectrum can be calculated easily. The symmetric covering of a directed weighted graph G , denotes here the associated undirected graph G' obtained by replacing each directed edge of weight l by an undirected edge of the same weight. In this case if A is the adjacency matrix of G, $A + A^{\mathrm{T}}$ is the adjacency matrix of G', and $B(d, n)$, for example, is the symmetric covering of $BD(d, n)$. By doing so, we achieve our task, because two equivalent graphs clearly have the same spectrum and the same characteristic polynomial.

The reason for considering the adjacency matrix of the directed graph is that it can be expressed in a very simple way as Kronecker products that we are going to simplify by block operations. Let us introduce here the notation which is going to be used in what follows.

NOTATION.

$\mathbf{1}_d$ denotes the row vector of length d filled with 1 s.
$A \otimes B$ is the Kronecker product of A by B, that is the matrix obtained from A by replacing each entry a_{ij} of A by the matrix $a_{ij}B$.
$A^{\otimes n} = A \otimes A \otimes \cdots \otimes A$ (n factors).
I_d is the identity matrix of order d.
ω_d is a primitive dth root of 1.
We will heavily use the Hadamard matrices H_d which are defined by $H_d = (h_{ij})_{\substack{0\le i\le d-1\\ 0\le j\le d-1}}$ with $h_{ij} = \omega_d^{ij}$.
We denote by A^* the adjoint of the square matrix A for the usual hermitian product, that is A^* is the hermitian conjugate of A, i.e. $\bar{A}^{\mathrm{T}}$.

3. The Spectrum of de Bruijn Graphs

We focus in this section on the determination of the spectrum of the de Bruijn graph $B(d, n)$, i.e. the set of eigenvalues of the adjacency matrix A_B of the graph. It is straightforward to check that this matrix is a square matrix of order d^n, whose entry a_{ij} $(0 \leq i, j < d^n)$ is equal to

$$\begin{cases} 0 & \text{if there exists no } \alpha \in \{0, 1, \ldots, d-1\} \text{ such that } i = dj + \alpha \bmod d^n \text{ or } j = di + \alpha \bmod d^n. \\ 2 & \text{if there exists } \alpha \in \{0, 1, \ldots, d-1\} \text{ such that } i = j = \alpha \frac{d^n-1}{d-1}, \text{ or if there exist } \alpha, \beta \in \{0, 1, \ldots, d-1\}\ (\alpha \neq \beta), \text{ such that } \{i, j\} = \{(\alpha + \beta d)\frac{d^n-1}{d^2-1}, (\beta + \alpha d)\frac{d^n-1}{d^2-1}\} \text{ (if } n \text{ is even) or } \{i, j\} = \{\alpha \frac{d^{n+1}-1}{d^2-1} + \beta d \frac{d^n-1}{d^2-1}, \beta \frac{d^{n+1}-1}{d^2-1} + \alpha d \frac{d^n-1}{d^2-1}\} \text{ (if } n \text{ is odd).} \\ 1 & \text{otherwise.} \end{cases}$$

The ith row or column $(0 \leq i < d^n)$ corresponds to the vertex $(a_1, a_2, \ldots, a_n)$ with $i = \sum_{k=1}^{n} a_k d^{n-k}$.

The result we are going to prove here is

THEOREM 1. *The set of eigenvalues of the de Bruijn graph $B(d, n)$ consists of the reals* $\{2d \cos \frac{t\pi}{k}, 0 \leq t < k \leq n+1\}$.

This is a consequence of the more detailed

THEOREM 2. *The characteristic polynomial $P(X)$ of the graph $B(d, n)$ is*

$$\det(XI - A_B) = (X - 2d)P_n(X)^{d-1} \prod_{i=1}^{n-1} P_i(X)^{(d-1)^2 d^{n-i-1}}$$

where the polynomials P_i in one indeterminate X are given by the relation $P_1 = X$, $P_2 = X^2 - d^2$, $P_k = XP_{k-1} - d^2 P_{k-2}$.

As explained in the previous section, the first step consists in expressing the adjacency matrix A_{BD} of the directed de Bruijn graph $B(d, n)$ as a Kronecker product

LEMMA 1. *The adjacency matrix A_{BD} of the directed de Bruijn graph $BD(d, n)$ is*

$$A_{BD} = \mathbf{1}_d{}^* \otimes I_d^{\otimes(n-1)} \otimes \mathbf{1}_d.$$

From this identity, we deduce

LEMMA 2. *Let $H_{d,n} = \frac{1}{d^{n/2}} H_d^{\otimes n}$. $H_{d,n}$ is a unitary matrix*

$$H_{d,n}^{\star} H_{d,n} = I_{d^n}$$
$$H_{d,n}^{-1} A_{BD} H_{d,n} = C_{d,n}$$

where $C_{d,n} = (c_{ij})_{\substack{0 \leq i < d^n \\ 0 \leq j < d^n}}$ is a square matrix of order d^n, with entries

$$\begin{cases} c_{ij} = d & \text{if } j = di, \\ c_{ij} = 0 & \text{otherwise.} \end{cases}$$

PROOF. One can check that $H_d^* H_d = dI_d$. From this, we deduce that $H_{d,n}^* H_{d,n} = I_{d^n}$, owing to the well known formula $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$, which holds whenever the matrix products $A \cdot C$ and $B \cdot D$ are well defined. We have also the adjunction equality $A^* \otimes B^* = (A \otimes B)^*$.

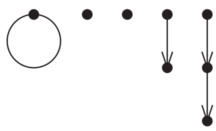
By using this we obtain

$$
\begin{aligned}
H_{d,n}^{-1} A_{BD} H_{d,n} &= H_{d,n}^{*} A_{BD} H_{d,n} \\
&= \frac{1}{d^n}\left(H_d^* \otimes (H_d^{\otimes(n-1)})^*\right)\cdot\left(\mathbf{1}_d{}^* \otimes I_d^{\otimes(n-1)} \otimes \mathbf{1}_d\right)\cdot\left(H_d^{\otimes(n-1)} \otimes H_d\right) \\
&= \frac{1}{d^n}\left(H_d^* \otimes (H_d^{\otimes(n-1)})^*\right)\cdot\left[\left((\mathbf{1}_d{}^* \otimes I_d^{\otimes(n-1)})\cdot H_d^{\otimes(n-1)}\right)\otimes(\mathbf{1}_d \cdot H_d)\right] \\
&= \frac{1}{d^n}\left(H_d{}^* \otimes (H_d^{\otimes(n-1)})^*\right)\cdot\left[(\mathbf{1}_d{}^* \otimes H_d^{\otimes(n-1)})\otimes(d,\underbrace{0,0,\ldots,0}_{d-1})\right] \\
&= \frac{1}{d^n}\left(H_d^* \otimes (H_d^{\otimes(n-1)})^*\right)\cdot\left[\mathbf{1}_d{}^* \otimes \left(H_d^{\otimes(n-1)} \otimes (d,\underbrace{0,0,\ldots,0}_{d-1})\right)\right] \\
&= \frac{1}{d^n}\left(H_d^* \cdot \mathbf{1}_d{}^*\right)\otimes\left[(H_d^{\otimes(n-1)})^* \cdot \left(H_d^{\otimes(n-1)} \otimes (d,\underbrace{0,0,\ldots,0}_{d-1})\right)\right] \\
&= \frac{1}{d^n}(d,\underbrace{0,0,\ldots,0}_{d-1})^* \otimes d^{n-1}\ I_{d^{n-1}} \otimes (d,\underbrace{0,0,\cdots,0}_{d-1}) \\
&= (1,\underbrace{0,0,\ldots,0}_{d-1})^* \otimes I_{d^{n-1}} \otimes (d,\underbrace{0,0,\cdots,0}_{d-1})
\end{aligned}
$$

The result is therefore a square matrix $C_{d,n}$ of order d^n, that we denote as $(c_{ij})_{\substack{0\le i<d^n \\ 0\le j<d^n}}$, for which the only non-zero entries are $c_{i,di} = d$ $(0 \le i < d^{n-1})$.

The crux is that $C_{d,n}$ can be seen as the adjacency matrix of a very simple directed weighted graph $G(d,n)$, which consists of a disjoint union of isolated vertices and directed weighted paths P_i of length $1 \le i \le n-1$ and one graph which is just a vertex with a self-loop. All the directed edges have the same weight d, and there are exactly: $(d-1)^2 d^{n-i-2}$ paths of length i for $1 \le i \le n-2$; $(d-1)$ paths of length $n-1$; $(d-1)^2 d^{n-2}$ isolated vertices; and one self-loop.

For example here is the equivalent $G(2,3)$ of $BD(2,3)$



We finish the proof of Theorem 1 by noting that the equality $A_{BD} = U^* C_{d,n} U$ for a unitary matrix U implies that $A_B = A_{BD} + A_{BD}^* = U^*(C_{d,n} + C_{d,n}^*)U = U^*(C_{d,n} + C_{d,n}^{\mathrm{T}})U$. Therefore the *undirected* de Bruijn graph $B(d,n)$ is equivalent to a disjoint union of isolated vertices and weighted (undirected) paths P_i of length $1 \le i \le n-1$ and one graph which is just a vertex with a self-loop. All the edges have the same weight d, except the self-loop which is of weight $2d$, and there are exactly: $(d-1)^2 d^{n-i-2}$ paths of length i for $1 \le i \le n-2$; $(d-1)$ paths of length $n-1$; $(d-1)^2 d^{n-2}$ isolated vertices; and one self-loop.

Clearly two equivalent graphs have the same spectrum, and the same characteristic polynomial. Therefore the characteristic polynomial of $B(d,n)$ is just the product of the characteristic polynomials of the previous paths, the isolated vertices and the self-loop; the spectrum is the union of the spectrum of these graphs. This implies our two theorems by using the computation of the spectrum and the characteristic polynomial of a path which can be found, for example, in [6, Chapter 2, Section 2.6]. The characteristic polynomial of a path with k vertices and edges of weight d is $P_k(X) = d^k U_k(X/d)$, where U_k is a Tchebychev polynomial of the second kind of degree k. This polynomial satisfies $U_k(2\cos\theta)\sin\theta = \sin((k+1)\theta)$, so that the zeros of P_k are the numbers $2d\cos\frac{t\pi}{k+1}$ (for $1 \le t \le k$). This remark gives us the result on the spectrum, and the result on the characteristic polynomial is a consequence of the relations $U_0(X) = 1$, $U_1(X) = 1$, $U_n(x) = XU_{n-1}(X) - U_{n-2}(X)$.

4. THE SPECTRUM OF KAUTZ GRAPHS

One of the attractive features of keeping the loops and double edges in the definition of de Bruijn and Kautz graphs, besides the fact that we are dealing with regular graphs, lies in the fact that there is a strong connection between them, that is $K(d,n)$ is a *cover* (in the topological sense) of $B(d,n-1)$. In combinatorial terms this means that:

THEOREM 3. *For each $d > 1$, $n > 1$, $B(d,n-1)$ is a divisor of $K(d,n)$. In other words there exists a mapping $\phi : VK \mapsto VB$ from the set VK of vertices of $K(d,n)$ to the set VB of vertices of $B(d,n-1)$, such that*

(i) for every i which belongs to VK, and for every j adjacent to i in $K(d,n)$, the number of edges between i and j is equal to the number of edges between $\phi(i)$ and $\phi(j)$ in $B(d,n-1)$,
(ii) the set of vertices of $B(d,n-1)$ adjacent to $\phi(i)$ is exactly the set of those $\phi(j)$ s, where j is adjacent to i in $K(d,n)$.

PROOF. Let us define the mapping ϕ by

$$\phi : VK \to VB$$
$$(x_1, x_2, \ldots, x_{n-1}, x_n) \mapsto (y_1, y_2, \ldots, y_{n-1})$$

where $y_i = [x_{i+1} - x_i \bmod (d+1)] - 1$. Let us consider a vertex $\boldsymbol{x} = (x_1, x_2, \ldots, x_{n-1}, x_n)$ in VK. The vertices which are adjacent to $\boldsymbol{x}$ are $(x_2, x_3, \ldots, x_n, \alpha)$ (for $\alpha \neq x_n$) and $(\beta, x_1, x_2, \ldots, x_{n-1})$ (for $\beta \neq x_1$). All these edges are simple unless $\boldsymbol{x}$ is of type $(x, y, x, y, \ldots)$ (with $x \neq y$). In this case there is a double edge between $(x, y, x, y, \ldots)$ and $(y, x, y, x, \ldots)$. Let us first consider the case where there are only simple edges. We have $\phi(\boldsymbol{x}) = (y_1, y_2, \ldots, y_{n-1})$, and $\phi(x_2, x_3, \ldots, x_n, \alpha) = (y_2, y_3, \ldots, y_{n-1}, \gamma)$, $\phi(\beta, x_1, x_2, \ldots, x_{n-1}) = (\delta, y_1, y_2, \ldots, y_{n-2})$. Properties (i) and (ii) clearly hold in this case. The case of the double edge is settled in the same way, the point is that the double edge between $(x, y, x, y, \ldots)$ and $(y, x, y, x, \ldots)$ in $K(d,n)$ is mapped by ϕ onto the double edge between $(a, b, a, b, \ldots)$ and $(b, a, b, a, \ldots)$ in $B(d,n-1)$, where $a = [(y-x) \bmod (d+1)] - 1$, and $b = [(x-y) \bmod (d+1)] - 1$. $\square$

This concept of divisors, besides being connected with structural or symmetry properties, has connections with the spectral properties of a graph. For example Theorem 4.5 of [6] asserts that if a graph F is a divisor of a graph G, then the spectrum of F is contained in the spectrum of G. Therefore Theorem 3 has the corollary:

THEOREM 4. *Let us denote by $SK(d,n)$ the spectrum of the Kautz graph $K(d,n)$, and by $SB(d,n)$ the spectrum of the de Bruijn graph $B(d,n)$, then*

$$SB(d, n-1) \subset SK(d,n)$$

and this implies that the reals

$$\left\{2d\cos\frac{t\pi}{k};\, 0 \le t < k \le n\right\}$$

are among the eigenvalues of $K(d,n)$

To describe the remaining part of the spectrum of Kautz graphs we shall use a slightly different description of these graphs. We label the $d^{n-1}(d+1)$ vertices with the following rule.

> Each vertex $a_1, a_2, \ldots, a_n$ is relabelled with $a_1, b_2, \ldots, b_n$ where $b_i = (a_i - a_{i-1} \bmod (d+1)) - 1$ belongs to $\{0 \ldots d-1\}$, and its numerical label is $a_1 d^{k-1} + \sum_{k=2}^{n} b_k d^{n-k}$. Then in the directed Kautz graph, the successors of $a_1, b_2, \ldots, b_n$ are $a_1 + b_2 + 1 \bmod d+1, b_3, \ldots, b_n, \beta$ with arbitrary $0 \le \beta < d$.

Hence the adjacency matrix A_{KD} of the directed Kautz graph $KD(d,n)$ is a matrix with $(d+1) \times (d+1)$ blocks S_{ij} of size d^{n-1}, with $S_{ij} = 0$ if $i = j$ and $S_{ij} = A_{i-j \bmod d+1}$ if $i \neq j$; the matrix A_i is defined by

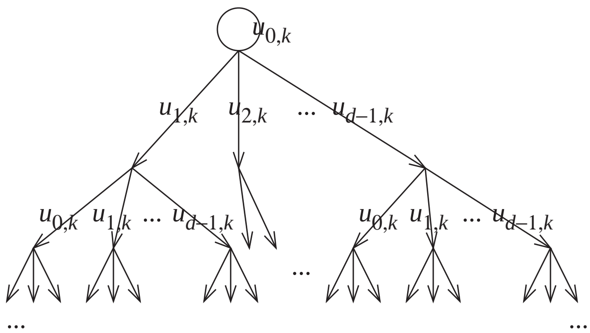
$$A_i = C_i \otimes I_{d^{n-2}} \otimes \mathbf{1}_d \tag{1}$$

C_i is a column vector of length d where all the entries are 0, except the ith entry which is 1. And $\mathbf{1}_d$ is a row vector of length d whose entries are all equal to 1. In other words A_{KD} is the matrix

$$\begin{bmatrix} 0 & A_1 & A_2 & \ldots & A_d \\ A_d & 0 & A_1 & & A_{d-1} \\ \vdots & & & \ddots & \\ A_1 & A_2 & \ldots & & 0 \end{bmatrix}.$$

Straightforward block operations show that the matrix A_{KD} is similar to a block-diagonal matrix, where the blocks are adjacency matrices of directed trees with an additional negative self-loop at the root.

LEMMA 3. *$K(d,n)$ is equivalent to a disjoint union of the de Bruijn directed graph $BD(d, n-1)$ and the opposite of d directed trees T_k ($1 \le k \le d$) with an additional self-loop at the root. What we call here the opposite of a directed graph is the graph with the same set of vertices and each directed edge is replaced by an edge in the opposite direction. The T_k s can be described as follows: they all have d^{n-1} vertices; there is a root with a self-loop of weight $u_{0,k} = -1$, which is the starting point of $d-1$ other directed edges of weight $u_{i,k} = \frac{\omega_d^i - 1}{\omega_{d+1}^k \omega_d^i - 1} - 1$ ($1 \le i \le d-1$); all the other vertices have either d successors of weight $u_{0,k}, u_{1,k}, \cdots, u_{d-1,k}$ or no successors at all (those are called the leaves). The leaves are exactly the vertices which are at distance $n-1$ from the root, and there are $(d-1)d^{n-1}$ of them (as follows)*



PROOF. To obtain this result we will make two transformations on the adjacency matrix of $KD(d,n)$. The first one is suggested by the block-circulant form of the adjacency matrix, i.e. we make the transformation:

$$A_{KD} \to \left(\frac{1}{\sqrt{d+1}} H_{d+1} \otimes I_{d^{n-1}} \right)^* A_{KD} \frac{1}{\sqrt{d+1}} H_{d+1} \otimes I_{d^{n-1}} = D.$$

It is readily seen that D is a block-diagonal matrix with $d+1$ diagonal blocks $B_j = \sum_{i=1}^{d} A_i \omega_{d+1}^{ij}$. Actually B_0 is clearly the adjacency matrix of $BD(d, n-1)$. All the other blocks can be made sparser by the same transformation which worked for the directed de Bruijn $BD(d, n)$, that is

$$B_k \to H_{d,n}^{*} B_k H_{d,n} = C_k$$

We can do exactly the same calculation as in Lemma 2, and use expression (1) of A_i as a Kronecker product to obtain C_k which will be a matrix defined by

$$C_k = (c_{i,j})_{\substack{0 \le i < d^{n-1} \\ 0 \le j < d^{n-1}}}$$

where: $c_{i+\alpha d^{n-2}, di} = \sum_{l=1}^{d} \omega_{d+1}^{lk} \omega_d^{\alpha l} = \frac{\omega_d^{\alpha} - 1}{\omega_{d+1}^{k} \omega_d^{\alpha} - 1} - 1 = u_{\alpha,k}$ for $0 \le \alpha \le d-1$; all the other entries are equal to 0.

Such a matrix is easily seen to correspond to the opposite of T_k which is a tree with an additional self-loop at the root. Therefore the sequence of transformations:

$$A_{KD} \to U_1^{*} A_{KD} U_1 \to U_2^{*} U_1^{*} A_{KD} U_1 U_2$$

where $U_1 = \frac{1}{\sqrt{d+1}} H_{d+1} \otimes I_{d^{n-1}}$ and $U_2 = \begin{bmatrix} I_{d^{n-1}} & 0 \\ 0 & H_{d,n}^{\otimes d} \end{bmatrix}$ show the equivalence of the directed Kautz graph $KD(d, n)$ with the disjoint union of the opposite of the T_k s and the directed de Bruijn graph $B_{d,n}$. □

The symmetries displayed by those weighted graphs T_k enable us to obtain very simple equivalents of them. The key of that is

LEMMA 4. *Assume that we have a weighted directed graph $G(V, E)$ which is the disjoint union of the graph G' and d copies of the same graph H, with d additional edges of weights $u_1, u_2, \ldots, u_d$ which all leave the same vertex v' of G' to attain d copies of a same vertex w of H, in the d copies of H. The adjacency matrix of such a graph is*

$$A = \begin{bmatrix} A(G') & Cu_1 & Cu_2 & \cdots & Cu_d \\ 0 & A(H) & 0 & \vdots & 0 \\ 0 & 0 & A(H) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A(H) \end{bmatrix}$$

where C is a matrix whose entries are all 0, with the exception of one which is 1. $A(G')$ is the adjacency matrix corresponding to G', and $A(H)$ is the adjacency matrix of H.

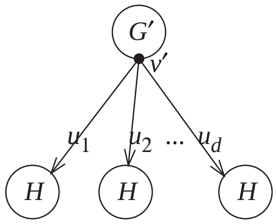
This graph is equivalent to the disjoint union of $d-1$ copies of H and the graph formed by the union of the graph G' and the graph H with an additional edge of weight $s = \sqrt{u_1\bar{u}_1 + u_2\bar{u}_2 + \cdots + u_d\bar{u}_d}$ which goes from v' to w.

PROOF. Let us make the orthonormal basis change given by $P = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}$, where: the block I is identity matrix (with the same size as $A(G)$); the second block U is the product $U' \otimes I'$, where U' is a unitary matrix of size d where the first column is $\frac{1}{s}[\bar{u}_1 \; \bar{u}_2 \; \cdots \; \bar{u}_d]^{\mathrm{T}}$, and I' is the identity matrix with the same size as $A(H)$.

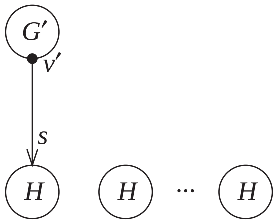
With straightforward block operations we obtain

$$P^{-1}AP = \begin{bmatrix} A(G) & Cs & 0 & \cdots & 0 \\ 0 & A(H) & 0 & \vdots & 0 \\ 0 & 0 & A(H) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & A(H) \end{bmatrix}.$$

This enables us to replace a graph of the form



by the equivalent graph



This lemma gives a recursive method to cut T_k into a disjoint union of small graphs. As a matter of fact, it is easily seen that all the T_k's are equivalent to the same graph which is the disjoint union of: $(d-1)^2 d^{n-i-2}$ directed paths with i vertices each, where all the directed edges have the same weight d, and this for every $1 \leq i \leq n-2$; $d-2$ directed paths with $n-1$ vertices each, all the edges are of the same weight d; and a path with n vertices, with an additional self-loop of weight -1 at one extremity, the other edge leaving this extremity is of weight $\sqrt{d^2-1}$, and all the other edges are of weight d.

The point is that we prune recursively the leaves of T_k with this method, and each pruning involves replacing d directed edges of weights $u_{0,k}, u_{1,k}, \ldots, u_{d-1,k}$ by one directed edge of weight

$$\sqrt{\bar{u}_{0,k}u_{0,k} + \bar{u}_{1,k}u_{1,k} + \cdots + \bar{u}_{d-1,k}u_{d-1,k}} = d.$$

When we attain the root we replace $(d-1)$ directed edges of weights $u_{0,k}, u_{1,k}, \ldots, u_{d-1,k}$ by one directed edge of weight $\sqrt{\bar{u}_{1,k}u_{1,k} + \bar{u}_{2,k}u_{2,k} + \cdots + \bar{u}_{d-1,k}u_{d-1,k}} = \sqrt{d^2-1}$.

Now we can use the same argument as at the end of the previous section, and say that

LEMMA 5. *The undirected Kautz graph $K(d,n)$ is equivalent to the disjoint union of the undirected de Bruijn graph $B(d, n-1)$ and the symmetric covering of the equivalent graphs found for the T_ks which are a disjoint union of: $(d-1)^2 d^{n-i-1}$ (undirected) paths with i vertices each, where all the edges have the same weight d, and this for every $1 \leq i \leq n-2$; $d(d-2)$ paths with $n-1$ vertices each, all the edges are of the same weight d; and a path with n vertices, with an additional self-loop of weight -2 at one extremity, the other edge leaving this extremity is of weight $\sqrt{d^2-1}$, and all the other edges are of weight d. We call this weighted graph $P_{d,n}$.*

To describe the spectrum of the Kautz graph, it remains to compute the characteristic polynomial $\Xi_{d,n}$ of $P_{d,n}$. By using classical methods to compute the characteristic polynomial of a tree (see, for example, Theorem 2.11 in [6]) we obtain that

(1) $\Xi_{d,0}(X) = \frac{d^2-1}{d^2}$
(2) $\Xi_{d,1}(X) = X + 2$
(3) $\Xi_{d,n}(X) = X.\Xi_{d,n-1}(X) - d^2.\Xi_{d,n-2}(X)$.

A straightforward computation shows that

$$\Xi_{d,n}(2d\cos\theta) = \frac{d^{n-2}}{\sin\theta}(d^2\sin(n+1)\theta + 2d\sin n\theta + \sin(n-1)\theta)$$

from which it can be deduced that the largest zero of this polynomial, $\lambda_{d,n}$ satisfies for $n \geq 2$

$$2d\cos\frac{\pi}{n} < \lambda_{d,n} < 2d\cos\frac{\pi}{n+1}.$$

Since equivalent graphs have the same spectrum, we deduce from Lemma 5, Theorems 1 and 2, and from the computation of the characteristic polynomial of $P_{d,n}$:

THEOREM 5. *The spectrum of $K_{d,n}$ is formed by the spectrum of $B_{d,n}$ together with the zeros of $\Xi_{d,n}$ defined above. This implies that for $n \geq 2$ the second largest eigenvalue of the graph lies in the interval $]2d\cos\frac{\pi}{n}, 2d\cos\frac{\pi}{n+1}[$. The characteristic polynomial of the graph is equal to*

$$\det(XI - A_K) = (X - 2d)\Xi_{d,n}^{d}(X)P_{n-1}^{d^2-d-1}\prod_{i=1}^{n-2}P_i(X)^{(d-1)^2(d+1)d^{n-i-2}}$$

where the polynomials P_i are given by the relation $P_1 = X$, $P_2 = X^2 - d^2$, $P_k = XP_{k-1} - d^2P_{k-2}$.

REMARK. One could argue now that $BD(d, n)$ and $KD(d, n)$ can be seen as iterated line graphs of a very simple graph, for example

$$BD(d, n) = \underbrace{L(L(\cdots(L(B_d)\cdots)}_{n} = L^{(n)}(B_d)$$

where $L(G)$ denotes the line graph of G, and B_d is the vertex with d self-loops. This raises the question of obtaining the spectrum of the symmetric covering of iterated line graphs. We do not know a general answer, however, we point out two other cases where the spectrum can be obtained for the symmetric covering of iterated line graphs:

(1) Namely for the symmetric covering of $L^{(n)}(E_d)$, where E_d is the directed graph with two vertices and there are d directed edges going from each vertex to the other. By using the fact that E_d is the double cover of B_d, in other words $E_d = K_2 \times B_d$, and the fact that the cartesian product of graphs commutes with the operation consisting in taking the line graph, it is seen that $L^{(n)}(E_d) = L^{(n)}(B_d) \times K_2$, so that the symmetric covering of $L^{(n)}(E_d)$ is the double cover of $B(d, n)$. The spectrum of this last graph is, therefore, the union of the spectrum of $B(d, n)$ with the opposite of this spectrum.

(2) The spectrum of the symmetric covering of $L^{(n)}(E_{p,q})$, where $E_{p,q}$ is the directed graph with two vertices and there are p directed edges going from the first vertex to the other, and q directed edges going from the second vertex to the first. It can be shown with the same proof technique as that used in the previous sections that this spectrum is also the union of the spectrum of paths. If n is even, then the spectrum consists of $p+q$, $-(p+q)$ and the spectrum of paths of length $< n$ where the edges have the alternating weights p and q. If n is odd, the spectrum consists of $2\sqrt{pq}$ and $-2\sqrt{pq}$ and the spectrum of paths of length $< n$ with edges of weight $\sqrt{pq}$. Some of these values must be removed when p or q is equal to 1.

5. THE EXPANSION PARAMETERS OF DE BRUIJN AND KAUTZ GRAPHS

5.1. Definitions. Spectrum results are particularly relevant to the estimation of expansion parameters of graphs, which are, in general, very hard to obtain by other means (see, for example, [1]). This concerns as well the magnifying coefficient of a graph, as the isoperimetric number. Recall that these expansion parameters are defined as follows.

DEFINITION. The *magnifying coefficient* of a graph $G(V, E)$ (the set of vertices is V, and the set of edges is E) is equal to

$$\min_{X \subset V; |X| \le |V|/2} \frac{|N(X)|}{|X|}$$

where $N(X)$ is the set of vertices out of X adjacent to some vertex in X. The *isoperimetric number* of this graph is

$$\min_{X \subset V; |X| \le |V|/2} \frac{|\partial X|}{|X|}$$

where $\partial X = \{\{x, y\} \in E; x \in X, y \in V - X\}$.

Some bounds on these coefficients will follow from other tools, namely from relations between these expansion coefficients and forwarding parameters: the edge-forwarding index, and the vertex-forwarding index.

DEFINITION. A *routing* R of a graph $G(V, E)$ on n vertices, is a set of $n(n-1)$ paths $R(u, v)$ specified for all ordered pairs u, v of vertices of G. The *load of an edge* e for a routing R is the number $R(e)$ of paths of the routing passing through it. The *load of a vertex* x for a given routing R is the number $R(x)$ of paths of the routing passing through x. The *edge-forwarding index* $\boldsymbol{\pi}$ of G is defined as

$$\boldsymbol{\pi} = \min_R \max_{e \in E} R(e).$$

The *vertex-forwarding index* ξ of G is defined as

$$\xi = \min_R \max_{x \in V} R(x).$$

5.2. Basic connections between the parameters of de Bruijn and Kautz graphs with or without loops and multiple edges. As explained in the introduction, we deal here with de Bruijn and Kautz undirected graphs with loops and multiple edges. A more usual definition of these graphs removes these loops and replaces the double edges by simple edges. We will call these graphs associated to $B(d, n)$ and $K(d, n)$, $B'(d, n)$ and $K'(d, n)$, respectively. It is straightforward to check the following relations between the expansion parameters of $B(d, n)$, $K(d, n)$ and $B'(d, n)$ and $K'(d, n)$

LEMMA 6.

$$\begin{array}{cc} 2i'_B \ge i_B \ge i'_B & 2i'_K \ge i_K \ge i'_K \\ c_B = c'_B & c_K = c'_K \end{array}$$

where: i_B, i_K, i'_B,i'_K are the isoperimetric numbers of $B(d, n)$, $K(d, n)$, $B'(d, n)$, $K'(d, n)$ respectively; and c_B, c_K, c'_B, c'_K are the magnifying coefficients of $B(d, n)$, $K(d, n)$, $B'(d, n)$, $K'(d, n)$ respectively.

5.3. Estimation of the Expansion Parameters

THEOREM 6. *The magnifying coefficient c and the isoperimetric number i of the de Bruijn graph $B(d, n)$ satisfy the following inequalities.*

$$c \le \frac{2\sqrt{d}\pi}{(n+1)\sqrt{1 - \frac{2d\pi^2}{(n+1)^2}}} \qquad i \le \frac{2d\pi}{n+1}.$$

PROOF. It is straightforward to check that Lemma 2.4 given in [1] generalizes to multigraphs (that is graphs with loops and multiple edges), therefore according to [1] we obtain

$$\lambda \geq \frac{c^2}{4+2c^2}$$

where λ is the second smallest eigenvalue of the symmetric matrix $Q = 2dI - A$ (A is the adjacency matrix of $B(d,n)$, and I the identity matrix of order d^n). Using Theorem 1 we obtain that $\lambda = 2d(1-\cos\frac{\pi}{n+1})$, and we deduce our claim. The second inequality follows from Theorem 3.3 given in [4] (this theorem is due to Jerrum and Sinclair [15], and a slightly weaker form can be found in [12]), which implies here that $i \leq \sqrt{(2d)^2 - \lambda_2^2}$. λ_2 denotes here the second largest eigenvalue in absolute value of A. According to Theorem 1, $\lambda_2 = 2d\cos\frac{\pi}{n+1}$, and therefore $i \leq 2d\sqrt{1-(\cos\frac{\pi}{n+1})^2} \leq \frac{2d\pi}{n+1}$. □

Similar inequalities hold for the Kautz graphs

THEOREM 7. *The magnifying coefficient c and the isoperimetric number i of the Kautz graph $K(d,n)$ satisfy*

$$c \leq \frac{2\sqrt{d}\pi}{n\sqrt{1-\frac{2d\pi^2}{n^2}}} \qquad i \leq \frac{2d\pi}{n}.$$

PROOF. We use again Lemma 2.4. Here

$$\lambda \geq \frac{c^2}{4+2c^2}$$

where λ is the second smallest eigenvalue of the symmetric matrix $Q = 2dI - A$ (A is the adjacency matrix of $K(d,n)$, and I the identity matrix of order $(d+1)d^{n-1}$). Using Theorem 4 we obtain that $\lambda \leq 2d(1-\cos\frac{\pi}{n})$, and we deduce our claim. The upper bound on the isoperimetric number comes from Theorem 3.3 of [4]. □

REMARK. It is a beneficial side-effect of the regularity that the matrix Q used in the two previous proofs (usually called the Laplacian matrix of the graph), is that simply obtained from the adjacency matrix A, so that the Laplacian and ordinary spectra are closely related.

When the value of λ is larger than $\frac{1}{2}$, then it gives no upper bound for c; this happens when d is too large with respect to n.

With the notation and results from Lemma 6, we can easily derive the following lower bounds on the de Bruijn $B'(d,n)$ and Kautz $K'(d,n)$:

THEOREM 8. *The following inequalities hold*

$$c'_B \leq \frac{2\sqrt{d}\pi}{(n+1)\sqrt{1-\frac{2d\pi^2}{(n+1)^2}}} \qquad i'_B \leq \frac{2d\pi}{n+1}$$
$$c'_K \leq \frac{2\sqrt{d}\pi}{n\sqrt{1-\frac{2d\pi^2}{n^2}}} \qquad i'_K \leq \frac{2d\pi}{n}.$$

These $O(\frac{1}{n})$ bounds are optimal for fixed d for the magnifying coefficient, and the bounds on the isoperimetric numbers are optimal within a constant factor whether d is fixed or not, according to the following proposition.

THEOREM 9. *The isoperimetric numbers i_B, i'_B, i_K, i'_K of de Bruijn graphs $B(d,n)$, $B'(d,n)$, and Kautz graphs $K(d,n)$, $K'(d,n)$, respectively, satisfy*

$$i_B \geq i'_B \geq \frac{d}{2(n-1)} \qquad i_K \geq i'_K \geq \frac{d}{2(\log_d(d+1)+n-1)}.$$

PROOF. From the proposition in Section 2.5.2 and the relation between isoperimetric number and edge expansion factor given at the beginning of [14, Section 2.4], we obtain $i'_B \geq \frac{d}{2(n-1)}$. From [14, Theorem 1], we have the relation $\beta \geq \frac{2}{\boldsymbol{\pi}}$ between the edge forwarding index $\boldsymbol{\pi}$ of $K'(d,n)$, and the edge expansion factor β of the same graph. This last factor is defined for a graph G with vertex set V by

$$\beta = \min\left\{\frac{|\partial X|}{|X|(|V|-|X|)};\, X \subset V, 1 \leq |X| \leq |V|-1\right\}.$$

Since $\boldsymbol{\pi} \leq 2(d+1)d^{n-2}(\log_d(d+1)+n-1)$ (by Proposition 6.10 of [9]) and $i'_K \geq \beta\frac{(d+1)d^{n-1}}{2}$ we obtain $i'_K \geq \frac{d}{2(\log_d(d+1)+n-1)}$. We conclude by applying Lemma 6.

THEOREM 10. *The (common) magnifying coefficient c_B and c'_B of de Bruijn graphs $B(d,n)$ and $B'(d,n)$ is at least*

$$\frac{1}{n}.$$

The (common) magnifying coefficient c_K and c'_K of Kautz graphs $K(d,n)$ and $K'(d,n)$ is at least

$$\frac{1}{n+\log_d(d+1)}.$$

PROOF. We apply here Theorem 9 of [14] to $B'(d,n)$, thus $c_B \geq \frac{d^n}{d^n+\xi}$ (where ξ is the vertex-forwarding index of $B'(d,n)$). Since by Proposition 6.8 of [9] we have $\xi \leq d^n(n-1)$, we deduce the first part of the theorem. The inequality for Kautz graphs follows by similar arguments (by using Proposition 6.9 of [9] and Theorem 9 of [14]). □

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