On the metrizability of spaces with a sharp base

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Abstract

A base $B$ for a space $X$ is said to be \textit{sharp} if, whenever $x \in X$ and $(B_n)_{n \in \omega}$ is a sequence of pairwise distinct element of $B$ each containing $x$, the collection $\{\bigcap_{j \leq n} B_j : n \in \omega\}$ is a base at the point $x$. We answer questions raised by Alleche et al. and Arhangel’skii et al. by showing that a pseudocompact Tychonoff space with a sharp base need not be metrizable and that the product of a space with a sharp base and $[0, 1]$ need not have a sharp base. We prove various metrization theorems and provide a characterization along the lines of Ponomarev’s for point countable bases.

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The notion of a uniform base was introduced by Alexandroff who proved that a space (by which we mean $T_1$ topological space) is metrizable if and only if it has a uniform base and is collectionwise normal [1]. This result follows from Bing’s metrization theorem since a space has a uniform base if and only if it is metacompact and developable. Recently Alleche et al. [2] introduced the notions of sharp base and weak development. These fit very naturally into the hierarchy of strong base conditions, which includes weakly uniform bases, introduced by Heath and Lindgren [10], and point countable bases (see Fig. 1 below). In this paper we look at the question of when a space, with a sharp base is

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metrizable. In particular, we show that a pseudocompact space with a sharp base need not be metrizable, but generalize various situations where a space with a sharp base is seen to be metrizable.

**Definition 1.** Let $B$ be a base for a space $X$.

1. $B$ is said to be **sharp** if, whenever $x \in X$ and $(B_n)_{n \in \omega}$ is a sequence of pairwise distinct elements of $B$ each containing $x$, the collection $\{\bigcap_{j \leq n} B_j : n \in \omega\}$ is a base at the point $x$.
2. $B$ is said to be **uniform** if, whenever $x \in X$ and $(B_n)_{n \in \omega}$ is a sequence of pairwise distinct elements of $B$ each containing $x$, then $(B_n)_{n \in \omega}$ is a base at the point $x$.
3. $B$ is said to be **weakly uniform** if, whenever $B'$ is an infinite subset of $B$, then $\bigcap B'$ contains at most one point.
4. $B$ is said to be a **weak development** if $B = \bigcup_{n \in \omega} B_n$, each $B_n$ a cover of $X$ and, whenever $x \in B_n \in B_n$ for each $n \in \omega$, then $\{\bigcap_{j \leq n} B_j : n \in \omega\}$ is a base at the point $x$.

Arhangel’skiǐ et al. prove that a space with a sharp base has a point countable sharp base [2,4] and is meta-Lindelöf. Moreover a weakly developable space has a $G_δ$-diagonal and a submetacompact space with a base of countable order is developable [2].

We note in passing that the obvious definition of 'uniform weak developability' (having a base $G = \bigcup \{G_n : n \in \omega\}$ such that each $G_n$ is a cover and whenever $x \in G_n \in G_n$, $\{G_n\}_n$ is a base at $x$) is simply a restatement of developability. We also note that a space with a $σ$-disjoint base need not have a sharp base: Bennett and Lutzer [7] construct a first countable (and a Lindelöf) example of a non-metrizable LOTS with $σ$-disjoint bases (and continuous separating families), which cannot have a sharp base by Theorem 2.

When is a space with a sharp base metrizable? We summarize relevant the results of [2, 4,6] in the following theorem.
Theorem 2. Let $X$ be a regular space with a sharp base, then $X$ is metrizable if any of the following hold:

1. $X$ is separable;
2. $X$ is locally compact (so a manifold with sharp base is metrizable);
3. $X$ is countably compact;
4. $X$ is pseudocompact and CCC;
5. $X$ is a GO space.

A space is pseudocompact if every continuous real valued function is bounded. Every (Tychonoff) pseudocompact space with a uniform base is metrizable (see [18,15] or [17]), whilst a pseudocompact space with a point-countable base need not be metrizable [16]. Moreover pseudocompact Tychonoff spaces with regular $G_δ$-diagonals are metrizable [13], whilst Mrowka’s $Ψ$ space is an example of a pseudocompact, non-metrizable Moore space. So it is natural to ask (see [2,4]) whether every pseudocompact space with a sharp base is metrizable. The space $P$ of Example 3 shows that the answer to this question is ‘no’. In addition, $P$ answers a number of other questions in the negative: Alleche et al. ask whether the product $X \times [0,1]$ has a sharp base if $X$ does; Heath and Lindgren [10] ask whether a space with a weakly uniform base has a $G_δ^*$-diagonal; and $P$ is another example (see [16, 19]) of a pseudocompact space with a point countable base that is not compact, and is a non-compact pseudocompact space with a weakly uniform base, answering questions of Peregudov [14].

Example 3. There exists a Tychonoff, non-metrizable pseudocompact space with a sharp base but without a $G_δ^*$-diagonal whose product with the closed unit interval does not have a sharp base.

Proof. Our example $P$ is a modification of the example of a non-developable space with a sharp base [2]. We add extra points to a (non-separable) metric space $B$ in such a way that the resulting space is pseudocompact, has a sharp base but is not compact, hence not metrizable.

Let $B = ω_1$ be the Tychonoff product of countably many copies of the discrete space of size continuum with the usual Baire metric. For each finite partial function $f \in <ω_1$, let $[f]$ denote the basic open subset of $B$,

$[f] = \{g \in ω_1 : f \subseteq g\}$

(so $[f]$ is the collection of all elements of $B$ which agree with $f$ on dom $f$). Note that, if dom $f \subseteq$ dom $g$, then the two basic open sets $[f]$ and $[g]$ have non-empty intersection if and only if $f \subseteq g$ if and only if $[g] \subseteq [f]$. If $[f] \cap [g] = \emptyset$ then the functions $f$ and $g$ are incompatible (we write $f \perp g$) and neither $f \subseteq g$ nor $g \subseteq f$.

Let

$S = \{S \in ω_1 (<ω_1 c) : S(m) \perp S(n), \text{ for each } m \text{ and } n\},$

so that each $S$ in $S$ codes for a sequence of disjoint basic open sets in $B$. Enumerate $S$ as $\{S_\alpha : \alpha \in c\}$ in such a way that each $S$ in $S$ occurs $c$ times. To ensure that our space is
pseudocompact, we recursively add limit points (to some of) these sequences of open sets. These limit points \( s_\alpha \) will have basic open neighbourhoods of the form

\[
N(\alpha, n) = \{ s_\alpha \} \cup \bigcup_{m \geq n} [T_\alpha(m)],
\]

where \( T_\alpha \in \omega(\omega^\omega) \) is defined depending on \( S_\alpha \).

Suppose that for each \( \alpha < \gamma \) we have either defined if possible a sequence \( T_\alpha \in \omega(\omega^\omega) \) such that

(1\( \gamma \)) for \( i \neq j \), \( T_\alpha(i) \perp T_\alpha(j) \),

(2\( \gamma \)) for \( \beta < \gamma \), \( \beta \neq \alpha \), \( T_\beta \) defined, \( \text{ran } T_\beta \cap \text{ran } T_\beta = \emptyset \), and

(3\( \gamma \)) for \( \beta < \gamma \), \( \beta \neq \alpha \), \( T_\beta \) defined, if \( T_\alpha(i) \supseteq T_\beta(j) \), then \( T_\alpha(i') \perp T_\beta(j') \) for all \( i', j' \neq (i, j) \)

or we have not defined \( T_\alpha \). We now define \( T_\gamma \).

First note that if \( S_\gamma(i) \) extends \( S_\gamma(i) \), then the open set \( \{ S_\gamma(i) \} \) is a subset of \( \{ S_\gamma(i) \} \), so any limit of the sequence of open sets \( \{ [S_\gamma(i)]: i \in \omega \} \) will be a limit of the sequence \( \{ [S_\gamma(i)]: i \in \omega \} \).

Since each \( T_\alpha(j) \) is finite, there is some \( \delta < \omega \) which is not in \( \bigcup \{ T_\alpha(j): \alpha < \gamma, j \in \omega \} \). For each \( i \in \omega \), let \( S_\gamma(i) = S_\gamma(i) \cap \{ \delta \} \) extend \( S_\gamma(i) \). Then for all \( i, j \in \omega \) and \( \alpha < \gamma \), \( S_\gamma(i) \supseteq T_\gamma(j) \) and \( T_\alpha(j) \subseteq S_\gamma(i) \) if \( T_\alpha(j) \subseteq S(i) \). Notice that this implies that \( T_\gamma(j) \subseteq \{ S_\gamma(i) \} \) and that \( \{ S_\gamma(i) \} \subseteq T_\gamma(j) \) only if \( \{ S_\gamma(i) \} \subseteq T_\gamma(j) \).

Case 1. Suppose that there exists some \( \alpha < \gamma \) for which \( T_\alpha \) was defined, such that for infinitely many \( i \in \omega \) there exists some \( j \in \omega \) such that \( S_\gamma(i) \supseteq S_\gamma(i) \supseteq T_\gamma(j) \). In this case we do not define \( T_\gamma \) (since infinitely many of the basic open sets \( [T_\gamma(j)] \) contain an open set \( S_\gamma(i) \) and the limit point \( s_\alpha \) will deal with the sequence \( S_\gamma \)).

Case 2. Now suppose that case 1 does not hold and that hence

(\( * \)) for each \( \alpha < \gamma \) there are at most finitely many \( i \) for which \( S_\gamma(i) \supseteq T_\alpha(j) \) for some \( j \).

Suppose further that for each \( i \leq k \), we have chosen natural numbers \( 0 = r_0 < r_1 < \cdots < r_k \) and defined \( T_\gamma(i) \) to be \( S_\gamma(r_i) \).

Since each \( T_\gamma(i) \) is a finite partial function, there are at most finitely many possible partial functions such that \( f \subseteq T_\gamma(i) \) for some \( i \leq k \). By condition (2\( \gamma \)) there are at most finitely many \( \alpha < \gamma \) with such an \( f \) in \( \text{ran } T_\alpha \). List these \( \alpha \) as \( \alpha(1), \ldots, \alpha(m) \). By (\( * \)), for each \( \alpha(m) \), there is a \( j_m \) such that for all \( i \geq j \), \( S_\gamma(i) \) does not extend any \( T_\alpha(m)(j) \). Now let \( r_{k+1} = \max j_m \) and \( T_\gamma(k + 1) = S_\gamma(r_{k+1}) \).

We now claim that conditions (1\( \alpha \)), (2\( \alpha \)) and (3\( \alpha \)) hold. Suppose that \( T_\beta \) and \( T_\alpha \) were defined for some \( \beta < \alpha < \gamma \). Condition (1\( \alpha \)) is obvious since each \( T_\alpha \) is a subsequence of \( S_\gamma \) each term of which extends the corresponding term of \( S_\gamma \), and \( S_\gamma \) is a sequence of pairwise incompatible partial functions. (2\( \alpha \)) holds since, if \( \beta < \alpha \), then the extension \( S_\gamma(i) \) was chosen to ensure that \( T_\beta(j) \supseteq S_\gamma(i) \) for any \( j \), so in particular \( T_\beta(j) \neq T_\alpha(i) \) and \( \text{ran } T_\alpha \cap \text{ran } T_\beta \). To see that (3\( \alpha \)) holds, note first that \( S_\gamma(i) \) was chosen so that \( S_\gamma(i) \subseteq T_\beta(j) \) for any \( j \), which implies that \( T_\alpha(i) \subseteq T_\beta(j) \) for any \( \{ i, j \} \). On the other hand, suppose that \( i \) is least such that for some \( j \), \( T_\beta(j) \subseteq T_\alpha(i) \). If \( k > i \), then \( T_\gamma(k) = S_\gamma(r_k) \) and \( r_k \) was
chosen precisely so that $S'_n(r_k) \nsubseteq T_\beta(l)$ for any $l \in \omega$. Moreover, there can be at most one $j$ such that $T_\alpha(l) \supseteq T_\beta(j)$, since by (1c), $T_\beta(j) \perp T_\beta(l)$, $j \neq l$. This completes the recursion.

Let $L = \{s_\omega: \mathcal{T}_\omega \text{ has been defined}\}$ be a set of pairwise distinct points disjoint from $B$ and let $P = B \cup L$. We topologize $P$ by letting $B$ be an open subspace with the usual Baire metric topology and declaring the $n$th basic open set about the point $s_\omega$ to be the set $N(\alpha, n) = \{s_\omega\} \cup \bigcup_{m \geq n} \{T_\alpha(m)\}$.

If $\mathcal{T}_\omega = \{\mathcal{T}_\alpha(n): n \in \omega\}$, then condition (1c) ensures that each $\mathcal{T}_\alpha$ is a pairwise disjoint collection. (2c) ensures that each basic open set $\{f\}$ occurs in at most one $\mathcal{T}_\alpha$, and (3c) ensures that if $N(\alpha, n)$ meets $N(\beta, m)$, then $N(\alpha, n) \cap N(\beta, m) = \mathcal{T}_\alpha(j) \cap \mathcal{T}_\beta(k)$ for some $j \geq n$ and $k \geq m$.

That $P$ has a sharp base follows exactly as for the example due to Alleche et al. Let $\mathcal{B}_B$ be a sharp base for $B$ and let $\mathcal{B} = \mathcal{B}_B \cup \{N(\alpha, n): s_\omega \in L \text{ and } n \in \omega\}$. Suppose $x \in \bigcap_{k \in \omega} B_k$ for some (injective) sequence $\{B_k \in \mathcal{B}: k \in \omega\}$. Since $\mathcal{B}_B$ is a sharp base and $s_\omega \in N \in \mathcal{B}$ if and only if $N = (\alpha, n)$ for some $n$, the only case that is not obvious is when $x \in B$ and $B_k = N(\alpha_k, m_k)$ for all but finitely many $k$. But in this case condition (3c) implies that, for $n \geq 1$, $\bigcap_{k \leq n} B_k = \bigcap_{k \leq n} \{T_\alpha(j_k)\}$, and $\{\bigcap_{k \leq n} B_k : n \in \omega\}$ contains a strictly decreasing subsequence and is therefore a base at $x$.

Since the set $\{s_\alpha: \alpha \in c\}$ is infinite, closed discrete, $P$ is not compact. On the other hand, $P$ is pseudocompact (so $P$ is not metrizable). To see this, suppose that $\varphi$ is a continuous real-valued function on $P$ taking values in $[n, \infty)$ for each $n \in \omega$. Since $B$ is dense in $P$, for each $n \in \omega$, there is some $s_\alpha$ in $B$ such that $\varphi(s_\alpha) > n$. By continuity, $\{s_\alpha: n \in \omega\}$ does not have a limit point in $B$. Since $\varphi$ is continuous and $B$ is metrizable, there are basic open sets $\{f_n\}$ for each $n \in \omega$ such that $s_\alpha \in \{f_n\} \subseteq \varphi^{-1}(n, \infty)$ and $\{\{f_n\} : n \in \omega\}$ is a disjoint collection. But in this case $f_n \perp f_m$ when $n \neq m$ so that $\{f_n: n \in \omega\} = S_\alpha$ for some $\alpha \in c$. In which case, either $s_\alpha$ and $T_\alpha$ were defined or $s_\alpha$ was not defined and, for some $\beta < \alpha$, $T_\beta(j) \subseteq S_\alpha(n) = f_n$ for infinitely many $n$. In the second case, each basic open neighbourhood $N(\beta, n)$ of $s_\beta$ contains infinitely many of the sets $\{f_n\}$. In the first case, $T_\alpha$ was chosen so that $T_\alpha(l) \supseteq f_l$ for each $l \in \omega$, so that $\{T_\alpha(l)\} \subseteq \{f_l\}$. In either case, each neighbourhood of $s_\beta$ or $s_\alpha$ contains points which take arbitrarily large values under $\varphi$, contradicting continuity.

Now suppose for a contradiction that $P \times [0, 1]$ has a sharp base. We shall show that this would imply that $P$ has a $\sigma$-point finite base, which is impossible since Uspenski [17] shows that a pseudocompact space with a $\sigma$-point finite base is metrizable.

To this end, let $\mathcal{W}$ be a sharp base for $P \times [0, 1]$ and let $\mathcal{C}$ be a countable base for $[0, 1]$. For each $x$ in $L$ choose $W_x^\alpha$ in $\mathcal{W}$, $B_x^\alpha$ in $\mathcal{B}$ (the sharp base for $P$), and $C_x^\alpha$ in $\mathcal{C}$ such that $B_x^\alpha \times C_x^\alpha \subseteq W_x^\alpha$, $\{W_x^\alpha: n \in \omega\}$ (and hence $\{B_x^\alpha \times C_x^\alpha: n \in \omega\}$) is a base at the point $(x, 1/2)$ and $W_x^\alpha \cap (L \times [0, 1]) \subseteq \{x\} \times [0, 1]$, which is possible since $L$ is a closed discrete subset of $P$.

Let $\mathcal{B}_C = \{B \in \mathcal{B}: \text{ for some } n \in \omega \text{ and some } x \in L, B = B_x^\alpha \text{ and } C = C_x^\alpha\}$. If $\mathcal{B}_C$ is not point finite then for some $y$ in $P$, $y \in \bigcap_{j \in \omega} B_j$ for some pairwise distinct $B_j \in \mathcal{B}_C$. By definition, for each $j$ there is some $x_j \in L$ and $n_j \in \omega$ such that $B_j = B_{x_j}^{n_j}$ and $C = C_{x_j}^{n_j}$. But then
\[
\{y\} \times C \subseteq \bigcap_{j \in \omega} (B_{x_j}^{n_j} \times C_{x_j}^{n_j}) \subseteq \bigcap_{j \in \omega} W_{x_j}^{n_j}.
\]
Since $B_j \neq B_k$, either there is an infinite set $J \subseteq \omega$ such that $x_j \neq x_k$, for distinct $j, k \in J$, or there is an infinite set $K \subseteq \omega$ such that $x_j = x_k = x$ but $n_j \neq n_k$ for some $x \in L$ and distinct $j, k \in K$. In the first case, $\{W_{n_j}^{x_j} : j \in J\}$ is a pairwise distinct subset of the sharp base $\mathcal{W}$ and $\bigcap_{j \in J} W_{n_j}^{x_j}$ contains at most one point. In the second case

$$\bigcap_{k \in K} (B_{x_k}^{n_k} \times C_{x_k}^{n_k}) = (x, 1/2),$$

since $\{B_x^{n} \times C_x^{n} : n \in \omega\}$ is a base at $(x, 1/2)$. In either case, $\{y\} \times C$ contains at most one point, which is not the case, and $B_C$ is point finite.

Since $\{B_x^{n} \times C_x^{n} : n \in \omega\}$ is a base at $(x, 1/2)$ and $C$ is countable, $B = \bigcup_{C \in C} B_C$ is a $\sigma$-point finite base for points of $L$. But $P = B \cup L$ and $B$ is a metric space, so $P$ has a $\sigma$-point finite base: a contradiction.

By Theorem 4, $P$ does not have a $G^*_\delta$ diagonal, nor indeed is it submetacompact. We also note that $P$ is dense-in-itself. $\blacksquare$

So when is a pseudocompact space with a sharp base metrizable? As mentioned above, a pseudocompact, CCC regular space with a sharp base is metrizable [4, Theorem 21]. Pseudocompact, Moore spaces are CCC. Moreover, in proving that a pseudocompact Tychonoff space with a regular $G_\delta$-diagonal is metrizable, McArthur [13] proves that a pseudocompact space with a $G^*_\delta$-diagonal is developable. Hence we have

**Theorem 4.** A pseudocompact regular space $X$ with a sharp base is metrizable if either of the following hold:

1. $X$ is developable, or;
2. $X$ has a $G^*_\delta$-diagonal.

A pseudocompact space with a $G_\delta$-diagonal is Čech complete [4, Lemma 20], hence Baire, so the following theorem is a strengthening of Theorem 21 of [4]. A space is strongly quasi-complete if there is a map $g$ assigning to each $x \in X$ and $n \in \omega$ an open set $g(n, x)$ containing $x$ such that $\{x_n\}$ clusters at $x$ whenever $\{x, x_n\} \subseteq \bigcap_{l \in \omega} g(l, y_l)$. Weakly developable spaces are clearly strongly quasi-complete.

**Theorem 5.** A regular, locally CCC, locally Baire space with a sharp base is metrizable.

**Proof.** Let $X$ be a regular, locally CCC, locally Baire space with a sharp base. Since $X$ has a weak development, it is strongly quasi-complete. Hodel [11] shows that every regular, quasi-complete CCC Baire space with either a $G_\delta$-diagonal or a point countable separating open cover is separable. Since $X$ has a sharp base, $X$ has a point countable base, a $G_\delta$-diagonal and is quasi-complete. Hence $X$ is locally separable. But every locally separable regular space with a point countable base is a disjoint union of clopen subspaces each of which has a countable base (see Theorem 7.2 of [9]). Hence $X$ is metrizable. $\blacksquare$

A space is $\omega_1$-compact if every subset of cardinality $\omega_1$ has a limit point. Generalizing the fact that a countably compact space with a sharp base is metrizable we have:
Theorem 6. A regular, $\omega_1$-compact space with a sharp base is metrizable.

Proof. Since $X$ is $\omega_1$-compact, every point-countable open cover of $X$ has a countable subcover [9, Lemma 7.5]. Since $X$ has a sharp base, it has a point countable base and therefore is Lindelöf. A metacompact space with a sharp base is developable [2] and so a Lindelöf space with a sharp base is metrizable. \qed

Not surprisingly a monotonically normal space with a sharp base is metrizable (cf. [6] where it is shown that a GO-space with a sharp base is metrizable).

Theorem 7. For a monotonically normal $X$ space the following are equivalent:

1. $X$ is metrizable;
2. $X$ has a sharp base;
3. $X$ has a weak development;
4. $X$ is strongly quasi-complete;
5. $X$ has a base of countable order and a $G_\delta$-diagonal.

Proof. Since (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) (that (4) implies (5) follows from Theorems 2.2 and 2.3 of [8]), it remains to show that a monotonically normal space with a base of countable order and a $G_\delta$-diagonal is metrizable. By the Balogh–Rudin theorem [5], since a stationary set of a regular cardinal does not have a $G_\delta$-diagonal, a monotonically normal space with a $G_\delta$-diagonal is paracompact. The result then follows since a paracompact space with a base of countable order is metrizable [3]. \qed

The proof that $P \times [0, 1]$ does not have a sharp base does not quite extend to a proof that if the product of a space $X$ with $[0, 1]$ has a sharp base then $X$ has a $\sigma$-point finite base. The converse however is easily seen to be true.

Proposition 8. If a space $X$ has a $\sigma$-point finite sharp base then $X \times [0, 1]$ has a sharp base.

Proof. Suppose that $B = \bigcup B_n$ is a $\sigma$-point finite sharp base for $X$ and $C = \bigcup C_n$ is a development for $[0, 1]$ such that each $C_{n+1}$ is finite and refines $C_n$ (so that $C$ is also a sharp base for $[0, 1]$). For each $n \in \omega$ let $W_n = \{B \times C: B \in B_n, C \in C_n\}$ and let $W = \bigcup W_n$.

Firstly note that $W$ is a base for $X \times [0, 1]$. If $(x, r)$ is in some open set $U$, choose $n$ and $B \in B_m$ such that $(x, r) \in B \times st(r, C_n) \subseteq U$. Now for some $k \geq \max\{m, n\}$, there is $B' \in B_k$, $x \in B' \subseteq B$. But then, since $C_k$ refines $C_n$, if $r \in C \in C_k$, $B' \times C \in W_k$ and $(x, r) \in B' \times C \subseteq B' \times st(r, C_k) \subseteq B \times st(r, C_n) \subseteq U$.

Now suppose that $(x, r) \in B_j \times C_j = W_j \in W$ for distinct $W_j$, $j \in \omega$. Each $W_n$ is a point finite family since both $B_n$ and $C_n$ are point finite and so both $\{B_j\}_{j \in \omega}$ and $\{C_j\}_{j \in \omega}$ are infinite. Since $B$ and $C$ are sharp bases, this implies that $\{\bigcap_{j \leq n} B_j \times C_j: n \in \omega\}$ is a base at the point $(x, r)$ and $W$ is a sharp base as required. \qed
Ponomarev, see [9], characterized those spaces with a point countable base as precisely
the open s-images of metric spaces (a map is an s-map if it has separable fibres). There is
a similar characterization for sharp bases.

**Theorem 9.** A space $X$ has a sharp base if and only if there is a metric space $M$ with
a base $\mathcal{B}$ and a continuous open mapping $f : M \to X$ such that, whenever $x \in X$ and
$\{B_n \in \mathcal{B} : n \in \omega\}$ is a pairwise distinct collection, if $f^{-1}(x) \cap B_n \neq \emptyset$ for each $n \in \omega$,
then there exists $n_0$ such that for each $y \in X$, if $f^{-1}(y) \cap B_j \neq \emptyset$, for each $j \leq n_0$, then
$f^{-1}(y) \cap B_0 \neq \emptyset$.

**Proof.** Suppose that $\mathcal{G}$ is a sharp base for the space $X$. Let

$$M = \{(G_n) \in \mathcal{G}^\omega : x \in \bigcap_{n \in \omega} G_n \text{ for some } x \in X\}$$

be the subspace of the Baire metric space $\mathcal{G}^\omega$, with metric $d((G_n), (H_n)) = 1/2^k$ where $k$ is
least such that $G_n \neq H_n$. Let $f : M \to X$ be defined letting $f((G_n))$ be the unique element
of $\bigcap_{n \in \omega} G_n$ and let $\mathcal{B}$ be the base for $M$ consisting of all $1/2^n$-balls about points of $M$.
Then $f$ is easily seen to be a continuous, open mapping onto $X$ and the condition on $\mathcal{B}$
in the statement of the theorem is merely a translation of the fact that $\mathcal{G}$ is a sharp base. $\Box$

It is clear from the proof that, in the statement of the theorem, we can take $\mathcal{B}$ to be the
collection of $1/2^n$ balls for any $n$ rather than a base for $M$. Since a space with a sharp base
has a point countable sharp base, we can also assume that the map in the statement of the
theorem is an s-map. However, it is not immediately clear that we can prove that a space
with a sharp base has a point countable base directly from the theorem.

We conclude with some open problems. Since every collectionwise normal Moore space
is metrizable, the following is a natural and intriguing question.

**Question 1.** Is every collectionwise normal space with a sharp base metrizable?

Example 4 of [2] shows that weakly developable, collectionwise normal spaces do not
have to be metrizable and the Heath V-space over a Q-set is an example of a normal space
with a uniform base that is not metrizable. On the other hand, the answer is ‘yes’ if the
space is also submetacompact (since it is then a Moore space) or a strict p-space. We might
also ask whether a perfect, collectionwise normal space with a sharp base is metrizable.
It is interesting to note that it is not known whether a collectionwise normal space with a
point countable base need be paracompact.

Since the Heath V-space over a $\Delta_1$-set is countably paracompact but not normal [12], at
least consistently a countably paracompact, (Moore) space with a sharp base need not be
normal. What about the converse?

**Question 2.** Is there a Dowker space with a sharp base?

**Question 3.** Is every perfect, regular space with a sharp base developable? Is every normal
space with a sharp base developable? Is every perfectly regular, pseudocompact space with
a sharp base metrizable?
Not every Moore space with a weakly uniform base has a uniform base (see [2]) so we ask:

**Question 4.** Does every Moore space with a sharp base have a uniform base?

Every pseudocompact space with a $G_δ$-diagonal is Čech complete [4], and every pseudocompact Moore space with a sharp base is metrizable.

**Question 5.** Is every Čech complete Moore space with a sharp base metrizable? What about Baire instead of Čech complete?

**Question 6.** If $X \times [0, 1]$ has a sharp base, does $X$ have a $σ$-point finite sharp base?

As the referee points out, the open, perfect pre-image of a space with a sharp base need not have a sharp base (the projection map from $P \times [0, 1]$ to $P$ is open and perfect), so we ask:

**Question 7.** Does the image of a space with a sharp base under a perfect map (closed and open map, open map with compact, countable or finite fibres) have a sharp base?

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**References**