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Soliton dynamics for a general class of Schrödinger equations ☆

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ABSTRACT

The soliton dynamics for a general class of nonlinear focusing Schrödinger problems in presence of non-constant external (local and nonlocal) potentials is studied by taking as initial datum the ground state solution of an associated autonomous elliptic equation.

1. Introduction and main result

1.1. Introduction

The aim of this paper is to study a general class of scalar and vectorial Schrödinger equations in presence of local and nonlocal potentials, modelling an electric and magnetic field and a Newtonian type interaction, respectively. This class of problems includes various physically meaningful particular cases, that will be individually described in details later in this section. In fact, we would also like to discuss the latest developments available in literature for this kind of issue, particularly when approached via the technique initiated by the 2000 work of R. Jerrard and J. Bronski [4]. More precisely, let $m \geqslant 1$, $N \geqslant 1$, $0 < p < 2/N$, $\varepsilon > 0$ and let

$$V : \mathbb{R}^N \to \mathbb{R}, \qquad A : \mathbb{R}^N \to \mathbb{R}^N, \qquad \Phi : \mathbb{R}^N \to \mathbb{R}, \tag{1.1}$$

be $C^3(\mathbb{R}^N)$ functions satisfying suitable assumptions that will be stated in the following. Then, if i denotes the complex imaginary unit, consider the Schrödinger equation

$$\begin{cases} -\mathrm{i}\varepsilon\partial_t\zeta_\varepsilon^j + L_A\zeta_\varepsilon^j + V(x)\zeta_\varepsilon^j = |\zeta_\varepsilon|_j^{2p}\zeta_\varepsilon^j + \dfrac{1}{\varepsilon^N}\Phi * |\zeta_\varepsilon|_j^2\zeta_\varepsilon^j & \text{in } \mathbb{R}^N \times (0,\infty), \\ \zeta_\varepsilon^j(x,0) = \zeta_0^j(x) & \text{in } \mathbb{R}^N, \\ j = 1,\dots,m, \end{cases} \tag{S}$$

where $\zeta_\varepsilon = (\zeta_\varepsilon^1,\dots,\zeta_\varepsilon^m) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C}^m$ is the unknown, the magnetic operator L_A is defined as

$$L_A\zeta := -\frac{\varepsilon^2}{2}\Delta\zeta - \frac{\varepsilon}{\mathrm{i}}A(x)\cdot\nabla\zeta + \frac{1}{2}\left|A(x)\right|^2\zeta - \frac{\varepsilon}{2\mathrm{i}}\,\mathrm{div}_x A(x)\zeta,$$

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the convolution is denoted by $(\Phi * v)(x) := \int \Phi(x-y)v(y)\,dy$, and

$$|\zeta|_j^{2p} := \alpha_j \left|\zeta^j\right|^{2p} + \sum_{i\neq j}^{m} \gamma_{ij}\left|\zeta^i\right|^{p+1}\left|\zeta^j\right|^{p-1}, \qquad |\zeta|_j^2 := \beta_j\left|\zeta^j\right|^2 + \sum_{i\neq j}^{m}\omega_{ij}\left|\zeta^i\right|^2,$$

for some nonnegative constants α_i, β_i, γ_{ij}, ω_{ij} such that $\gamma_{ij} = \gamma_{ji}$ and $\omega_{ij} = \omega_{ji}$, for all $i, j = 1, \dots, m$. By rescaling problem (S) with $\phi_\varepsilon(x,t) = \zeta_\varepsilon(\varepsilon x, \varepsilon t)$, we reach the following system, where ε appears now only in the arguments of the potentials V, A and Φ

$$\begin{cases} -\mathrm{i}\partial_t \phi_\varepsilon^j + L_A\phi_\varepsilon^j + V(\varepsilon x)\phi_\varepsilon^j = |\phi_\varepsilon|_j^{2p}\phi_\varepsilon^j + \Phi(\varepsilon x) * |\phi_\varepsilon|_j^2\phi_\varepsilon^j & \text{in } \mathbb{R}^N \times (0,\infty), \\ \phi_\varepsilon^j(x,0) = \phi_0^j(x) & \text{in } \mathbb{R}^N, \\ j = 1,\dots,m, \end{cases} \tag{P}$$

with $\phi_\varepsilon = (\phi_\varepsilon^1, \dots, \phi_\varepsilon^m) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C}^m$ and

$$L_A\phi := -\frac{1}{2}\Delta\phi - \frac{1}{\mathrm{i}}A(\varepsilon x)\cdot\nabla\phi + \frac{1}{2}\left|A(\varepsilon x)\right|^2\phi - \frac{1}{2\mathrm{i}}\,\mathrm{div}_x\, A(\varepsilon x)\phi. \tag{1.2}$$

As we have already recalled, here $V : \mathbb{R}^N \to \mathbb{R}$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ are an *electric* and *magnetic* potentials, respectively. The magnetic field B is $B = \nabla \times A$ in $\mathbb{R}^3$ and can be thought (and identified) in general dimension as a 2-form $\mathbb{H}^B$ of coefficients $(\partial_i A_j - \partial_j A_i)$. We will keep using the notation $B = \nabla \times A$ in any dimension N.

We point out that the general Schrödinger problem (S) we aim to investigate contains, as particular cases, the following physically meaningful situations.

Class I. If $m = 1$, $A = 0$, $\beta_j = \omega_{ij} = \gamma_{ij} = 0$ and $\alpha_j = 1$, one finds:

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\zeta_\varepsilon - V(x)\zeta_\varepsilon + |\zeta_\varepsilon|^{2p}\zeta_\varepsilon = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ \zeta_\varepsilon(x,0) = \zeta_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

This is the classical *Schrödinger equation* with a spatial potential. For general results about local and global existence of solutions, regularity, orbital stability and instability, we refer the reader to [6] and to the references therein. From the point of view of the semi-classical analysis of standing wave solutions $\zeta_\varepsilon(x,t) = u_\varepsilon(x)e^{-\mathrm{i}Et}$ for $E \in \mathbb{R}$, the Schrödinger equation reduces to a semi-linear elliptic equation. In the last few years a huge literature has developed starting from the celebrated paper by Floer and Weinstein [10] (see the monograph [2] by Ambrosetti and Malchiodi and references therein). Concerning the soliton (or, equivalently, point-particle) dynamics, that is the study of the qualitative behaviour of the solutions of this equation by choosing as initial datum a suitably rescaled ground state solution of an associated elliptic problem, we refer e.g. to the works [4,11,13,17] and to the recent monograph [5] (see also e.g. [15,16] for works in the mathematical physics community). Very recently, in [3], Benci, Ghimenti and Micheletti provided the first result on the soliton dynamics with uniform global estimates in time.

Class II. If $m = 1$, $\beta_j = \omega_{ij} = \gamma_{ij} = 0$ and $\alpha_j = 1$, one finds:

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon - \dfrac{1}{2}\left(\dfrac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2\zeta_\varepsilon - V(x)\zeta_\varepsilon + |\zeta_\varepsilon|^{2p}\zeta_\varepsilon = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ \zeta_\varepsilon(x,0) = \zeta_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

This is the *Schrödinger equation with a time-independent external magnetic field*. For general facts about this equation, we refer again to [6] and to the references therein. For the semi-classical analysis of standing wave solutions, we refer the reader to the recent work [7] and to the various references included. For the full (soliton) dynamics, we refer to the recent papers [23,25] which, to our knowledge, are the first contributions for this equation. In [25], the concentration centre is precisely the one predicted by the WKB theory.

Class III. If $m = 1$, $A = 0$ and $\alpha_j = \gamma_{ij} = \omega_{ij} = 0$, one finds:

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\zeta_\varepsilon - V(x)\zeta_\varepsilon + \dfrac{\beta}{\varepsilon^N}\Phi * |\zeta_\varepsilon|^2\zeta_\varepsilon = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ \zeta_\varepsilon(x,0) = \zeta_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

This is the *Hartree or Newton–Schrödinger type equation*. For basic facts about this equation, we refer again to [6] and references therein. For the study of standing waves in the semi-classical regime, we refer to [26] and the references included. The physical motivations for these equations were detected by Penrose who derived the Schrödinger–Newton equation by coupling the linear 3D Schrödinger equation with the Newton law of gravitation, yielding

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\zeta_\varepsilon - V(x)\zeta_\varepsilon + \Psi_\varepsilon * \zeta_\varepsilon = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ -\varepsilon^2\Delta\Psi_\varepsilon = \mu|\zeta_\varepsilon|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where μ is a positive constant. Of course, this system is equivalent to the nonlocal equation

$$\mathrm{i}\varepsilon\partial_t\zeta_\varepsilon + \frac{\varepsilon^2}{2}\Delta\zeta_\varepsilon - V(x)\zeta_\varepsilon + \Psi_\varepsilon * |\zeta_\varepsilon|^2\zeta_\varepsilon = 0 \quad \text{in } \mathbb{R}^3\times(0,\infty), \qquad \Psi_\varepsilon(x) = \frac{\mu}{4\pi\varepsilon^2}\frac{1}{|x|}.$$

For the study of point-particle dynamics for this equation with smooth nonlocal potentials, we refer the reader to [12], where the authors follow an approach different from that used in [4,17].

Class IV. If $m=1$ and $\alpha_j = \gamma_{ij} = \omega_{ij} = 0$, one finds:

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon - \dfrac{1}{2}\left(\dfrac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2\zeta_\varepsilon - V(x)\zeta_\varepsilon + \dfrac{\beta}{\varepsilon^N}\Phi * |\zeta_\varepsilon|^2\zeta_\varepsilon = 0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ \zeta_\varepsilon(x,0) = \zeta_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

This is the *Hartree type equation with magnetic field*. As for the previous cases, concerning the basic facts about this equation, we refer to [6]. With respect to the semi-classical analysis of standing waves we are not aware of any paper. The soliton dynamics behaviour is contained in the present paper for smooth potentials.

Class V. If $m=2$, $A=0$ and $\beta_j = \omega_{ij} = 0$, one finds:

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon^1 + \dfrac{\varepsilon^2}{2}\Delta\zeta_\varepsilon^1 - V(x)\zeta_\varepsilon^1 + \alpha_1\left|\zeta_\varepsilon^1\right|^{2p}\zeta_\varepsilon^1 + \gamma_{12}\left|\zeta_\varepsilon^2\right|^{p+1}\left|\zeta_\varepsilon^1\right|^{p-1} = 0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ \mathrm{i}\varepsilon\partial_t\zeta_\varepsilon^2 + \dfrac{\varepsilon^2}{2}\Delta\zeta_\varepsilon^2 - V(x)\zeta_\varepsilon^2 + \alpha_2\left|\zeta_\varepsilon^2\right|^{2p}\zeta_\varepsilon^2 + \gamma_{12}\left|\zeta_\varepsilon^1\right|^{p+1}\left|\zeta_\varepsilon^2\right|^{p-1} = 0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ \zeta_\varepsilon(x,0) = \zeta_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

This is the *weakly coupled Schrödinger system with two components*. With respect to the semi-classical analysis of standing waves, in the last few years the interest for this systems has considerably increased. We refer for instance to [1,18,20,24] for the study of the structure of the associated ground states solutions (vector versus scalar ground states depending upon the strength of the interaction $\gamma_{12} > 0$). For the behaviour in the semi-classical limit, we refer the reader to [8,20]. The soliton dynamics behaviour is contained in [21,22], essentially in the 1D case.

1.2. The main result

In this section we shall provide the suitable background allowing us to formulate the statement of the main theorem of the paper.

1.2.1. Framework and main ingredients

Throughout this paper we denote by $H_{A,\varepsilon}$ the Hilbert space defined as the closure of $C_c^\infty(\mathbb{R}^N;\mathbb{C}^m)$ under the scalar product

$$(u,v)_{H_{A,\varepsilon}} = \Re\int\left(Du\cdot\overline{Dv} + V(\varepsilon x)u\bar{v}\right)dx,$$

where $Du = (D_1u,\ldots,D_Nu)$ and $D_j = \mathrm{i}^{-1}\partial_j - A_j(\varepsilon x)$, with induced norm

$$\|u\|^2_{H_{A,\varepsilon}} = \int\left|\frac{1}{\mathrm{i}}\nabla u - A(\varepsilon x)u\right|^2 dx + \int V(\varepsilon x)|u|^2\,dx < \infty.$$

The dual space of $H_{A,\varepsilon}$ is denoted by $H'_{A,\varepsilon}$, while the space $H^2_{A,\varepsilon}$ is the set of u such that

$$\|u\|^2_{H^2_{A,\varepsilon}} = \|u\|^2_{L^2} + \left\|\left(\frac{1}{\mathrm{i}}\nabla - A(\varepsilon x)\right)^2 u\right\|^2_{L^2} < \infty.$$

Finally, $H^1(\mathbb{R}^N;\mathbb{C}^m)$ is equipped with the standard norm $\|\phi\|^2_{H^1} = \|\nabla\phi\|^2_{L^2} + \|\phi\|^2_{L^2}$. We study problem (P) for an initial datum $\phi_0:\mathbb{R}^N\to\mathbb{C}^m$ given by

$$\phi_0^j(x) = r_j\left(x - x_\varepsilon(0)\right)e^{\mathrm{i}[A(\varepsilon x_\varepsilon(0))\cdot(x-x_\varepsilon(0)) + x\cdot\xi_\varepsilon(0)]}, \quad j=1,\ldots,m, \tag{I}$$

where x_0/ε and ξ_0 are the initial position and the initial velocity in $\mathbb{R}^N$ of the following first order differential system

$$\begin{cases} \dot{x}_\varepsilon(t) = \xi_\varepsilon(t),\\ \dot{\xi}_\varepsilon(t) = -\varepsilon\nabla V\left(\varepsilon x_\varepsilon(t)\right) - \varepsilon\xi_\varepsilon(t)\times B\left(\varepsilon x_\varepsilon(t)\right),\\ x_\varepsilon(0) = \dfrac{x_0}{\varepsilon},\\ \xi_\varepsilon(0) = \xi_0, \end{cases} \tag{D}$$

with $B = \nabla \times A$. Notice that, for the solution of (D), we have

$$x_\varepsilon(t) = \frac{x(\varepsilon t)}{\varepsilon}, \qquad \xi_\varepsilon(t) = \xi(\varepsilon t), \qquad \begin{cases} \dot{x}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V\big(x(t)\big) - \xi(t) \times B\big(x(t)\big), \\ x(0) = x_0, \\ \xi(0) = \xi_0. \end{cases} \tag{1.3}$$

The rescaled components $(x(t), \xi(t))$ of system (1.3) might appear in the proofs of some result. Notice that the initial datum referred to the original problem (S) reads as

$$\zeta_0^j(x) = \phi_0^j\left(\frac{x}{\varepsilon}\right) = r_j\left(\frac{x - x_0}{\varepsilon}\right)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}, \quad x \in \mathbb{R}^N,\ j = 1, \dots, m.$$

This is the usual formula for the (soliton) initial datum considered in [4,17] when $A = 0$ and in [23,25] when $A \neq 0$. Furthermore, we assume that $r = (r_1, \dots, r_m) \in H^1(\mathbb{R}^N, \mathbb{R}^m)$ is (up to translation) a real ground state solution of the elliptic system

$$\begin{cases} -\dfrac{1}{2}\Delta r_j + r_j = |r|_j^{2p} r_j \quad \text{in } \mathbb{R}^N, \\ j = 1, \dots, m, \end{cases} \tag{S}$$

with respect to the notation of $|\cdot|_j$ previously introduced. We also set

$$m_j := \|r_j\|_{L^2}^2, \quad j = 1, \dots, m, \qquad M := \sum_{j=1}^m m_j. \tag{1.4}$$

Notice that, setting for all $t \in \mathbb{R}^+$

$$\mathcal{H}(t) = \frac{1}{2}\big|\xi_\varepsilon(t)\big|^2 + V\big(\varepsilon x_\varepsilon(t)\big) + \mathcal{M}, \tag{1.5}$$

where

$$\mathcal{M} := -\frac{\Phi(0)}{2M}\left\{\sum_{j=1}^m \beta_j m_j^2 + \sum_{i\neq j}^m \omega_{ij} m_i m_j\right\},$$

it follows that $\mathcal{H}$ is a first integral associated with (D), namely

$$\mathcal{H}(t) = \mathcal{H}(0) = \frac{1}{2}|\xi_0|^2 + V(x_0) + \mathcal{M}, \quad \text{for all } t \in \mathbb{R}^+.$$

In turn, the function $\mathcal{H}$ is independent of both time and $\varepsilon > 0$.

1.2.2. Assumptions on the potentials

We first give the following

Definition 1.1. Consider the potentials $V : \mathbb{R}^N \to \mathbb{R}$, $A : \mathbb{R}^N \to \mathbb{R}^N$ and $\Phi : \mathbb{R}^N \to \mathbb{R}$ and a ground state solution r of (S) which is chosen to build up the initial datum (I). We say that (V, A, Φ, r) is an admissible string for the point-particle dynamics of problem (P) if r_j is radially symmetric, $x_i r_j \in L^2(\mathbb{R}^N)$ for all $i = 1, \dots, N$ and $j = 1, \dots, m$ and the following Properties 1.2 (well-posedness) and 1.3 (non-degeneracy/energy convexity inequality) hold true.

Property 1.2 *(Well-posedness).* Assume that $0 < p < 2/N$. Then, for all $\varepsilon > 0$ and $\phi_0 \in H_{A,\varepsilon}$, there exists a unique global solution

$$\phi_\varepsilon \in C\big(\mathbb{R}^+, H_{A,\varepsilon}\big) \cap C^1\big(\mathbb{R}^+, H'_{A,\varepsilon}\big),$$

of problem (P) with $\sup_{t\in\mathbb{R}^+} \|\phi_\varepsilon(t)\|_{H_{A,\varepsilon}} < \infty$. Furthermore, the mass $\mathcal{N}_\varepsilon^j$ associated with $\phi_\varepsilon^j(t)$,

$$\mathcal{N}_\varepsilon^j(t) := \int \big|\phi_\varepsilon^j(t)\big|^2\,dx, \quad t \in \mathbb{R}^+,\ j = 1, \dots, m,$$

and the total energy E_ε,

$$E_\varepsilon(t) := \frac{1}{2}\int\left|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon(x) - A(\varepsilon x)\phi_\varepsilon\right|^2 dx + \int V(\varepsilon x)\big|\phi_\varepsilon(x)\big|^2\,dx - \frac{1}{p+1}\sum_{j=1}^m \alpha_j \int \big|\phi_\varepsilon^j(x)\big|^{2p+2}\,dx$$

$$-\frac{1}{p+1}\sum_{i,j,\,i\neq j}^{m}\gamma_{ij}\int\left|\phi_\varepsilon^i(x)\right|^{p+1}\left|\phi_\varepsilon^j(x)\right|^{p+1}dx-\frac{1}{2}\sum_{j=1}^{m}\beta_j\iint\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^j(x)\right|^2\left|\phi_\varepsilon^j(y)\right|^2dx\,dy$$
$$-\frac{1}{2}\sum_{i,j,\,i\neq j}^{m}\omega_{ij}\iint\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^i(x)\right|^2\left|\phi_\varepsilon^j(y)\right|^2dx\,dy,\quad t\in\mathbb{R}^+,$$

are conserved in time, namely

$$\mathcal{N}_\varepsilon^j(t)=\mathcal{N}_\varepsilon^j(0)\quad\text{and}\quad E_\varepsilon(t)=E_\varepsilon(0),\quad\text{for all } t\in\mathbb{R}^+,\ j=1,\dots,m.$$

Finally if $\phi_0\in H^2_{A,\varepsilon}$, then $\phi_\varepsilon\in C(\mathbb{R}^+,H^2_{A,\varepsilon})\cap C^1(\mathbb{R}^+,L^2(\mathbb{R}^N;\mathbb{C}^m))$.

We also consider the functional $\mathcal{E}:H^1(\mathbb{R}^N;\mathbb{R}^m)\to\mathbb{R}$ associated with system (S)

$$\mathcal{E}(u)=\frac{1}{2}\int\left|\nabla u(x)\right|^2dx-\sum_{j=1}^{m}\frac{\alpha_j}{p+1}\int\left|u_j(x)\right|^{2p+2}dx-\sum_{i,j,\,i\neq j}^{m}\frac{\gamma_{ij}}{p+1}\int\left|u_i(x)\right|^{p+1}\left|u_j(x)\right|^{p+1}dx.$$

In a large range of relevant situations, a ground state solution r of (S) satisfies the characterisation

$$\mathcal{E}(r)=\min\left\{\mathcal{E}(u):\ u\in H^1\big(\mathbb{R}^N,\mathbb{R}^m\big),\ \|u\|_{L^2}=\|r\|_{L^2}\right\}. \tag{1.6}$$

For $m=1$ this is a classical fact. For $m=2$ see e.g. [19].

We consider now the following

Property 1.3 *(Non-degeneracy/energy convexity inequality).* There exist two positive constants C and C' such that the following condition holds: if $U\in H^1(\mathbb{R}^N;\mathbb{C}^m)$ is such that $\|U\|_{L^2}=\|r\|_{L^2}$, where r is a ground state solution of (S), then

$$\Gamma_U\leqslant C\big(\mathcal{E}(U)-\mathcal{E}(r)\big), \tag{1.7}$$

where

$$\Gamma_U=\inf_{\substack{y\in\mathbb{R}^N\\ \theta_1,\dots,\theta_m\in[0,2\pi)}}\left\|U(\cdot)-\big(e^{\mathrm{i}\theta_1}r_1(\cdot+y),\dots,e^{\mathrm{i}\theta_m}r_m(\cdot+y)\big)\right\|^2_{H^1}, \tag{1.8}$$

provided that $\Gamma_U<C'$.

The energy convexity inequality is essentially a feature of a ground state solution r. It is generally a quite delicate issue to consider, based upon nontrivial spectral estimates and the fact that the kernel of the linearized operator is N-dimensional and spanned by the partial derivatives $\partial_j r$ of r. Let us point out which is the current knowledge of particular cases, within our framework, where this assumption is indeed satisfied. For the Schrödinger equation with or without magnetic field, Property 1.3 is satisfied, since the (unique) ground state solution of $-\frac{1}{2}\Delta r+r=r^{2p+1}$ is non-degenerate and satisfies suitable spectral estimates (see the striking works of Weinstein [27,28]). For systems, already in the case of two components, the situation is still very far from being completely understood. On the other hand, very recently Dancer and Wei have proved in [8] the existence of non-degenerate ground state solutions in some particular cases, providing an important tool in connection with Property 1.3. In the one-dimensional case, Property 1.3 has been verified in [22] for two-components weakly coupled nonlinear Schrödinger system. The main obstacle in dealing with the higher-dimensional case is the smoothness of the energy functional $\mathcal{E}$ which is not of class C^2 due to the presence of the coupling terms $\int|\phi^i|^{p+1}|\phi^j|^{p+1}$, being $p<2/N<1$.

1.2.3. Statement of the result

On the external potentials V and A, on the nonlocal term Φ and on the ground state solution r of (S) which is chosen to build the initial datum (I), we assume that they are admissible for the point-particle dynamics in the sense indicated above and that the following conditions hold:

(**V**) $V\in C^3(\mathbb{R}^N)$ is positive and $\|V\|_{C^3}<\infty$;
(**A**) $A\in C^3(\mathbb{R}^N;\mathbb{R}^N)$ with $\|A\|_{C^3}<\infty$;
($\boldsymbol{\Phi}$) $\Phi\in C^3(\mathbb{R}^N)$ positive with $\|\Phi\|_{C^3}<\infty$.

We shall think Φ as a smooth function decaying at infinity as $|x|^{-\rho}$ for some $\rho>0$ (for instance, in $\mathbb{R}^N$ with $N\geqslant 3$, decaying as the Coulomb potential $|x|^{2-N}$) having a maximum point at the origin.

Under the previous assumptions, we can state the main result of this paper.

Theorem 1.4. *Assume that $\Phi = 0$ in the vectorial case $m > 1$. Let ϕ_ε be the family of solutions to problem (P) corresponding to the initial datum (I) modelled on a ground state r of (S) and let $(x_\varepsilon(t), \xi_\varepsilon(t))$ be the solution of (D). Then there exist $\delta > 0$, $\varepsilon_0 > 0$ and shift functions $\theta_\varepsilon^1, \dots, \theta_\varepsilon^m : \mathbb{R}^+ \to [0, 2\pi)$ such that, if $\|A\|_{C^2} < \delta$, then*

$$\phi_\varepsilon^j(x,t) = e^{\mathrm{i}(\xi_\varepsilon(t)\cdot x + \theta_\varepsilon^j(t) + A(\varepsilon x_\varepsilon(t))\cdot(x - x_\varepsilon(t)))} r_j\big(x - x_\varepsilon(t)\big) + \omega_\varepsilon^j(x,t),$$

where $\|\omega_\varepsilon^j(t)\|_{H^1} \leqslant \mathcal{O}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_0)$ and $j = 1, \dots, m$, locally uniformly in time with the time scale ε^{-1}. Furthermore, without restrictions on $\|A\|_{C^2}$, there exists $\varepsilon_0 > 0$ such that

$$\big|\phi_\varepsilon^j(x,t)\big| = r_j\big(x - x_\varepsilon(t)\big) + \hat{\omega}_\varepsilon^j(x,t), \tag{1.9}$$

where $\|\hat{\omega}_\varepsilon^j\|_{H^1} \leqslant \mathcal{O}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_0)$ and $j = 1, \dots, m$, locally uniformly in time with the time scale ε^{-1}.

This kind of results has the origin in some works in linear geometric asymptotics which go back to the 70's (see [14]). We stress that, in the vectorial case $m > 1$, we are not aware of any physically reasonable model including the nonlocal coupling terms. Hence, for $m > 1$, we consider systems of coupled Schrödinger equations with local terms, which are being extensively studied in the literature of recent years.

Remark 1.5. Rescaling back to problem (S), the approximated representation formula reads as

$$\zeta_\varepsilon^j(x,t) = e^{\frac{\mathrm{i}}{\varepsilon}(\xi(t)\cdot x + \vartheta_\varepsilon^j(t) + A(x(t))\cdot(x - x(t)))} r_j\left(\frac{x - x(t)}{\varepsilon}\right) + \Xi_\varepsilon^j(x,t),$$

locally uniformly in time, where we have set $\vartheta_\varepsilon^j(t) = \varepsilon\theta_\varepsilon^j(t/\varepsilon)$ and $\Xi_\varepsilon^j(x,t) = \omega_\varepsilon^j(x/\varepsilon, t/\varepsilon)$, which reads as in [25] and in the previously cited papers in the particular cases $m = 1$, $A = 0$ and $\Phi = 0$.

Plan of the paper. In Section 2, we prove various preliminary lemmas, particularly focused on the asymptotic behaviour of the energy, for ε small. In Section 3, we prove some lemmas, focused on the asymptotic behaviour of the density and of the momentum associated with the solution, for ε small. In Section 4, we prove a result yielding a precise control on the norm of the error function ω_ε^j which appears in Theorem 1.4. Finally, in Section 5, we conclude the proof of the main result, Theorem 1.4.

Notations.

(1) The imaginary unit is denoted by i.
(2) The conjugate of any $z \in \mathbb{C}$ is denoted by $\bar{z}$, the real and imaginary parts by $\Re z$ and $\Im z$.
(3) The symbol $\mathbb{R}^+$ means the positive real line $[0, \infty)$.
(4) The ordinary inner product between two vectors $a, b \in \mathbb{R}^N$ is denoted by $a \cdot b$.
(5) The standard L^p norm, $1 < p \leqslant \infty$ of a function u is denoted by $\|u\|_{L^p}$.
(6) The symbols ∂_t and ∂_j mean $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_j}$, respectively. Δ means $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$.
(7) The symbol $C^k(\mathbb{R}^N; \mathbb{C}^m)$, for $k \in \mathbb{N}$, denotes the space of functions with continuous derivatives up to the order k. Sometimes $C^k(\mathbb{R}^N; \mathbb{C}^m)$ is endowed with the norm

$$\|\phi\|_{C^k} = \sum_{|\alpha| \leqslant k} \big\|D^\alpha \phi\big\|_{L^\infty} < \infty.$$

(8) The symbol $\int f(x)\,dx$ stands for the integral of f over $\mathbb{R}^N$ with the Lebesgue measure.
(9) The symbol C^{2*} denotes the dual space of C^2. The norm of a ν in C^{2*} is

$$\|\nu\|_{C^{2*}} = \sup\left\{\left|\int \phi(\varepsilon x)\nu\,dx\right| : \phi \in C^2\big(\mathbb{R}^N\big),\ \|\phi\|_{C^2} \leqslant 1\right\}.$$

Clearly, C^{2*} contains the space of bounded Radon measures.
(10) C denotes a generic positive constant, which may vary inside a chain of inequalities.
(11) $\mathcal{O}(\varepsilon)$ is a generic function such that the lim sup of $\varepsilon^{-1}\mathcal{O}(\varepsilon)$ is finite, as $\varepsilon \to 0$.

2. Some preliminary stuff

Observe that, from Property 1.2, due to the choice of the initial datum (I), the masses $\mathcal{N}_\varepsilon^j(t)$ are also *independent* of ε. Indeed, via the mass conservation law, by the form of the initial datum and (1.4), we have

$$\mathcal{N}_\varepsilon^j(t) = \mathcal{N}_\varepsilon^j(0) = \int \big|\phi_\varepsilon^j(x,0)\big|^2\,dx = \int \big|r_j\big(x - x_\varepsilon(0)\big)\big|^2\,dx = \|r_j\|_{L^2}^2 = m_j, \tag{2.1}$$

for all $\varepsilon > 0$, $t \in \mathbb{R}^+$ and $j = 1, \dots, m$.

We now recall a useful identity (see e.g. [17, Lemma 3.3]).

Lemma 2.1. *Assume that $g:\mathbb{R}^N\to\mathbb{R}$ is a function of class $C^2(\mathbb{R}^N)$, $\|g\|_{C^2}<\infty$, and that r is a ground state solution of (S). Then, as ε goes to zero, for any $i=1,\dots,m$ it holds*

$$\int g(\varepsilon x+y)r_i^2(x)\,dx=\int g(y)r_i^2(x)\,dx+\mathcal{O}\big(\varepsilon^2\big),$$

for every $y\in\mathbb{R}^N$.

In a similar fashion, we have the following counterpart to be used for the nonlocal term.

Lemma 2.2. *Assume that $g:\mathbb{R}^N\to\mathbb{R}$ is a function of class $C^2(\mathbb{R}^N)$, $\|g\|_{C^2}<\infty$, and that r is a ground state solution of (S). Then, as ε goes to zero, for any $i,j=1,\dots,m$ it holds*

$$\iint g\big(\varepsilon(x-y)\big)r_i^2(x)r_j^2(y)\,dx\,dy=m_im_jg(0)+\mathcal{O}\big(\varepsilon^2\big).$$

Proof. By Taylor expansion, for some point ξ of the form $\xi=\varepsilon\tau(x-y)$ with $\tau\in(0,1)$, we have

$$\begin{aligned}\iint g\big(\varepsilon(x-y)\big)r_i^2(x)r_j^2(y)\,dx\,dy&=g(0)\iint r_i^2(x)r_j^2(y)\,dx\,dy+\varepsilon\sum_{h=1}^N D_hg(0)\cdot\iint(x_h-y_h)r_i^2(x)r_j^2(y)\,dx\,dy\\&\quad+\frac{\varepsilon^2}{2}\sum_{h,k=1}^N\iint D_{hk}^2g(\xi)(x_h-y_h)(x_k-y_k)r_i^2(x)r_j^2(y)\,dx\,dy\\&=m_im_jg(0)+\varepsilon\sum_{h=1}^N D_hg(0)\int x_hr_i^2(x)\,dx\int r_j^2(y)\,dy\\&\quad-\varepsilon\sum_{h=1}^N D_hg(0)\int y_hr_j^2(y)\,dy\int r_i^2(x)\,dx\\&\quad+\frac{\varepsilon^2}{2}\sum_{h,k=1}^N\iint D_{hk}^2g(\xi)(x_h-y_h)(x_k-y_k)r_i^2(x)r_j^2(y)\,dx\,dy\\&=m_im_jg(0)+\frac{\varepsilon^2}{2}\sum_{h,k=1}^N\iint D_{hk}^2g(\xi)(x_h-y_h)(x_k-y_k)r_i^2(x)r_j^2(y)\,dx\,dy\\&=m_im_jg(0)+\mathcal{O}\big(\varepsilon^2\big).\end{aligned}$$

In the above computations we used the fact that $|D_{hk}^2g(\xi)|\leqslant\|g\|_{C^2}<\infty$, that, since r_i is radially symmetric, $\int z_hr_i^2(z)\,dz=0$ and, finally, that $z_hr_i\in L^2(\mathbb{R}^N)$ for any h and i (cf. Definition 1.1). □

In the next result we obtain an asymptotic formula for the energy, linking the functionals E_ε, $\mathcal{E}$ and $\mathcal{H}$, up to an error $\mathcal{O}(\varepsilon^2)$ (see also [25]).

Lemma 2.3. *For every $t\in\mathbb{R}^+$, as ε goes to zero, it holds*

$$E_\varepsilon(t)=\mathcal{E}(r)+M\mathcal{H}(t)+\mathcal{O}\big(\varepsilon^2\big).$$

Proof. Taking into account that, in view of Lemma 2.1, for all $j=1,\dots,m$ we have

$$\int r_j^2(x)\big|A(\varepsilon x+x_0)\big|^2\,dx=\big|A(x_0)\big|^2m_j+\mathcal{O}\big(\varepsilon^2\big),$$

$$\int r_j^2(x)A(\varepsilon x+x_0)\cdot\big(A(x_0)+\xi_0\big)\,dx=A(x_0)\cdot\big(A(x_0)+\xi_0\big)m_j+\mathcal{O}\big(\varepsilon^2\big),$$

as ε goes to zero, it is readily checked that, for any $j=1,\dots,m$, we get

$$\int\bigg|\bigg(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\bigg)\big(r_j\big(x-x_\varepsilon(0)\big)e^{\mathrm{i}[A(x_0)\cdot(x-x_\varepsilon(0))+x\cdot\xi_0]}\big)\bigg|^2\,dx=\int\big|\nabla r_j(x)\big|^2\,dx+m_j|\xi_0|^2+\mathcal{O}\big(\varepsilon^2\big).$$

In turn, by combining the conservation of energy (see Property 1.2) and the conservation of the function $\mathcal{H}$ (see definition (1.5)), taking into account Lemmas 2.1 and 2.2, as ε goes to zero, we get

$$\begin{aligned}
E_\varepsilon(t) &= E_\varepsilon(0) = E_\varepsilon\big(r\big(x - x_\varepsilon(0)\big)e^{\mathrm{i}[A(x_0)\cdot(x-x_\varepsilon(0))+x\cdot\xi_0]}\big) \\
&= \frac{1}{2}\sum_{j=1}^m \int \left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon x)\right)\big(r_j\big(x - x_\varepsilon(0)\big)e^{\mathrm{i}[A(x_0)\cdot(x-x_\varepsilon(0))+x\cdot\xi_0]}\big)\right|^2 dx \\
&\quad + \sum_{j=1}^m \int V(x_0+\varepsilon x) r_j^2(x)\,dx - \sum_{j=1}^m \frac{\alpha_j}{p+1}\int |r_j|^{2p+2}\,dx - \sum_{i,j=1,\,i\neq j}^m \frac{\gamma_{ij}}{p+1}\int |r_i|^{p+1}|r_j|^{p+1}\,dx \\
&\quad - \sum_{j=1}^m \frac{\beta_j}{2}\iint \Phi\big(\varepsilon(x-y)\big)\big|r_j(x)\big|^2\big|r_j(y)\big|^2\,dx\,dy - \sum_{i,j,\,i\neq j}^m \frac{\omega_{ij}}{2}\iint \Phi\big(\varepsilon(x-y)\big)\big|r_i(x)\big|^2\big|r_j(y)\big|^2\,dx\,dy \\
&= \mathcal{E}(r) + \sum_{j=1}^m \int V(x_0+\varepsilon x) r_j^2(x)\,dx + \frac{1}{2}\sum_{j=1}^m m_j|\xi_0|^2 + M\mathcal{M} + \mathcal{O}\big(\varepsilon^2\big) \\
&= \mathcal{E}(r) + \sum_{j=1}^m m_j V(x_0) + \frac{1}{2}\sum_{j=1}^m m_j|\xi_0|^2 + M\mathcal{M} + \mathcal{O}\big(\varepsilon^2\big) = \mathcal{E}(r) + M\mathcal{H}(t) + \mathcal{O}\big(\varepsilon^2\big). \qquad \square
\end{aligned}$$

The function $p_\varepsilon^A : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^{m+N}$ is the (magnetic) momentum of ϕ_ε, defined as

$$p_\varepsilon^A(x,t) := \Im\big(\bar{\phi}_\varepsilon(x,t)\big(\nabla\phi_\varepsilon(x,t) - \mathrm{i}A(\varepsilon x)\phi_\varepsilon(x,t)\big)\big), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+. \tag{2.2}$$

Then, we have the following

Lemma 2.4. *Let ϕ_ε be the solution to problem (P) corresponding to the initial datum (I). Then there exists a positive constant C such that*

$$\big\|\mathrm{i}^{-1}\nabla\phi_\varepsilon(\cdot,t) - A(\varepsilon x)\phi_\varepsilon(\cdot,t)\big\|_{L^2}^2 \leqslant C,$$

for all $t \in \mathbb{R}^+$ and any $\varepsilon \in (0,1]$. In particular,

$$\sup_{t\in\mathbb{R}^+}\left|\int p_\varepsilon^A(x,t)\,dx\right| < \infty.$$

Proof. By Property 1.2 the total energy E_ε is conserved and, in addition, can be bounded independently of ε (due to the choice of initial datum, see Lemma 2.3). Taking into account the positivity of V and the definition of E_ε, it follows that there exists a positive constant C such that

$$\begin{aligned}
\left\|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon(\cdot,t) - A(\varepsilon x)\phi_\varepsilon(\cdot,t)\right\|_{L^2}^2 &= \int\left|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon(x,t) - A(\varepsilon x)\phi_\varepsilon(x,t)\right|^2 dx \\
&= 2E_\varepsilon(t) - 2\int V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx + \frac{2}{p+1}\sum_{j=1}^m \alpha_j\int\big|\phi_\varepsilon^j(x,t)\big|^{2p+2}\,dx \\
&\quad + \frac{2}{p+1}\sum_{i,j,\,i\neq j}\gamma_{ij}\int\big|\phi_\varepsilon^i(x,t)\big|^{p+1}\big|\phi_\varepsilon^j(x,t)\big|^{p+1}\,dx \\
&\quad + \sum_{j=1}^m \beta_j\iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy \\
&\quad + \sum_{i,j,\,i\neq j}^m \omega_{ij}\iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy \\
&\leqslant C + \frac{2}{p+1}\sum_{j=1}^m \alpha_j\int\big|\phi_\varepsilon^j(x,t)\big|^{2p+2}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{p+1}\sum_{i,j,i\neq j}^{m}\gamma_{ij}\int\left|\phi_\varepsilon^i(x,t)\right|^{p+1}\left|\phi_\varepsilon^j(x,t)\right|^{p+1}dx\\
&+\sum_{j=1}^{m}\beta_j\iint\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^j(x)\right|^2\left|\phi_\varepsilon^j(y)\right|^2dx\,dy\\
&+\sum_{i,j,i\neq j}^{m}\omega_{ij}\iint\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^i(x,t)\right|^2\left|\phi_\varepsilon^j(y,t)\right|^2dx\,dy. \qquad (2.3)
\end{aligned}
$$

By combining the diamagnetic inequality (see e.g. [9] for a proof)

$$\left|\nabla\left|\phi_\varepsilon^j\right|\right|\leqslant\left|\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\right)\phi_\varepsilon^j\right|,\quad \text{a.e. in } \mathbb{R}^N$$

with the Gagliardo–Nirenberg inequality, setting $\vartheta=\frac{pN}{2p+2}\in(0,1)$, we obtain

$$\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^{2p+2}}\leqslant\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}^{1-\vartheta}\left\|\nabla\left|\phi_\varepsilon^j(\cdot,t)\right|\right\|_{L^2}^{\vartheta}\leqslant\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}^{1-\vartheta}\left\|\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\right)\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}^{\vartheta}$$

for any $j=1,\dots,m$. While, by the conservation of mass, we deduce that

$$\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}^2=\mathcal{N}_\varepsilon^j(t)=m_j,\quad j=1,\dots,m,$$

independently of ε (see formula (2.1)). Hence, for all $\varepsilon>0$, we get

$$\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^{2p+2}}^{2p+2}\leqslant m_j^{(1-\theta)(p+1)}\left\|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon^j(\cdot,t)-A(\varepsilon x)\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}^{pN}\leqslant C\big(\Upsilon_\varepsilon(t)\big)^{pN}, \qquad (2.4)$$

for any $j=1,\dots,m$ and for some positive constant C, where we have set, for $t>0$,

$$\Upsilon_\varepsilon(t)=\max_{j=1,\dots,m}\Upsilon_\varepsilon^j(t),\quad \Upsilon_\varepsilon^j(t)=\left\|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon^j(\cdot,t)-A(\varepsilon x)\phi_\varepsilon^j(\cdot,t)\right\|_{L^2}.$$

Observe also that, as Φ is uniformly bounded, for any $i,j=1,\dots,m$ we have

$$\iint\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^i(x,t)\right|^2\left|\phi_\varepsilon^j(y,t)\right|^2dx\,dy\leqslant C\int\left|\phi_\varepsilon^i(x,t)\right|^2dx\int\left|\phi_\varepsilon^j(y,t)\right|^2dy=Cm_im_j.$$

Finally, notice that, by Young inequality

$$\int\left|\phi_\varepsilon^i(x,t)\right|^{p+1}\left|\phi_\varepsilon^j(x,t)\right|^{p+1}dx\leqslant\frac{1}{2}\left\|\phi_\varepsilon^i(\cdot,t)\right\|_{L^{2p+2}}^{2p+2}+\frac{1}{2}\left\|\phi_\varepsilon^j(\cdot,t)\right\|_{L^{2p+2}}^{2p+2}, \qquad (2.5)$$

for any $j=1,\dots,m$. Putting now together all the previous inequalities from (2.3) to (2.5), we finally obtain $(\Upsilon_\varepsilon(t))^2\leqslant C+C(\Upsilon_\varepsilon(t))^{pN}$ for $t>0$. Taking into account that $pN<2$ by the assumption on p, if $\Upsilon_\varepsilon(t)$ was unbounded with respect to t or ε, the above inequality would yield a contradiction. Hence Υ_ε is uniformly bounded with respect to t and ε, so that the first assertion of Lemma 2.4 holds. In order to prove the final assertion observe that, taking into account the mass conservation law, by Hölder inequality we get

$$\left|\int p_\varepsilon^A(x,t)\,dx\right|\leqslant\int\left|p_\varepsilon^A(x,t)\right|dx\leqslant\left\|\phi_\varepsilon(\cdot,t)\right\|_{L^2}\left\|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon(\cdot,t)-A(\varepsilon x)\phi_\varepsilon(\cdot,t)\right\|_{L^2}\leqslant C,$$

for all $t\in\mathbb{R}^+$. The assertion follows by taking the supremum over t in $\mathbb{R}^+$. □

For the next lemma we need to introduce the total magnetic momentum q_ε^A defined as

$$q_\varepsilon^A(x,t)=\sum_{j=1}^{m}\big(p_\varepsilon^A\big)^j(x,t),\quad x\in\mathbb{R}^N,\ t>0.$$

Then, on a suitable function ψ_ε (related to the solution ϕ_ε), we have the following

Lemma 2.5. *Let ϕ_ε be the family of solutions to problem (P) corresponding to the initial datum (I). Let us set, for any $\varepsilon>0$, $t\in\mathbb{R}^+$ and $x\in\mathbb{R}^N$*

$$\psi_\varepsilon^j(x,t)=e^{-\mathrm{i}\xi_\varepsilon(t)\cdot[x+x_\varepsilon(t)]}e^{-\mathrm{i}A(\varepsilon x_\varepsilon(t))\cdot x}\phi_\varepsilon^j\big(x+x_\varepsilon(t),t\big),\quad j=1,\dots,m, \qquad (2.6)$$

where $(x_\varepsilon(t), \xi_\varepsilon(t))$ *is the solution of system* (D)*. Then, as* ε *goes to zero,*

$$\begin{aligned}\mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) = M\mathcal{H}(t) &- \int V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx + \frac{1}{2}M\big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \\ &- \big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\cdot\int q_\varepsilon^A(x,t)\,dx - (\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\cdot\int A(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx \\ &+ \frac{1}{2}\int\big|A(\varepsilon x)\big|^2\big|\phi_\varepsilon(x,t)\big|^2\,dx + \int A(\varepsilon x)\cdot q_\varepsilon^A(x,t)\,dx \\ &+ \frac{1}{2}\sum_{j=1}^m \beta_j\iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy \\ &+ \frac{1}{2}\sum_{i,j,\,i\neq j}^m \omega_{ij}\iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy + \mathcal{O}\big(\varepsilon^2\big).\end{aligned}$$

Proof. By a change of variable we see that $\|\psi_\varepsilon^j(t)\|_{L^2}^2 = m_j$ for $j = 1,\dots,m$. Hence the mass of $\psi_\varepsilon(t)$ is conserved through the motion. Let $p_\varepsilon^j(x,t) = \Im(\bar\phi_\varepsilon^j(x,t)\nabla\phi_\varepsilon^j(x,t))$ for $x\in\mathbb{R}^N$, $t\in\mathbb{R}^+$ and $j=1,\dots,m$ be the j-th magnetic-free momentum. A direct computation yields

$$\begin{aligned}\mathcal{E}\big(\psi_\varepsilon(t)\big) = \frac{1}{2}\int\big|\nabla\phi_\varepsilon(x,t)\big|^2\,dx &+ \frac{1}{2}\sum_{j=1}^m m_j\big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 - \sum_{j=1}^m\big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\cdot\int p_\varepsilon^j(x,t)\,dx \\ &- \frac{1}{p+1}\sum_{j=1}^m\alpha_j\int\big|\phi_\varepsilon^j(x,t)\big|^{2p+2}\,dx - \frac{1}{p+1}\sum_{i,j,\,i\neq j}^m\gamma_{ij}\int\big|\phi_\varepsilon^i(x,t)\big|^{p+1}\big|\phi_\varepsilon^j(x,t)\big|^{p+1}\,dx,\end{aligned}$$

so that we obtain

$$\begin{aligned}\mathcal{E}\big(\psi_\varepsilon(t)\big) = \frac{1}{2}\int\left|\frac{1}{\mathrm{i}}\nabla\phi_\varepsilon(x,t) - A(\varepsilon x)\phi_\varepsilon(x,t)\right|^2 dx &- \frac{1}{2}\int\big|A(\varepsilon x)\big|^2\big|\phi_\varepsilon(x,t)\big|^2\,dx + \sum_{j=1}^m\int A(\varepsilon x)\cdot p_\varepsilon^j(x,t)\,dx \\ &+ \frac{1}{2}\sum_{j=1}^m m_j\big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 - \sum_{j=1}^m\big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\cdot\int p_\varepsilon^j(x,t)\,dx \\ &- \frac{1}{p+1}\sum_{j=1}^m\alpha_j\int\big|\phi_\varepsilon^j(x,t)\big|^{2p+2}\,dx - \frac{1}{p+1}\sum_{i,j,\,i\neq j}^m\gamma_{ij}\int\big|\phi_\varepsilon^i(x,t)\big|^{p+1}\big|\phi_\varepsilon^j(x,t)\big|^{p+1}\,dx.\end{aligned}$$

Then, taking into account the definition of $E_\varepsilon(t)$ and of $\mathcal{H}$ and Lemma 2.3, we obtain

$$\begin{aligned}\mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) = M\mathcal{H}(t) &- \int V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx + \frac{1}{2}M\big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \\ &- \big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\cdot\int\sum_{j=1}^m p_\varepsilon^j(x,t)\,dx - \frac{1}{2}\int\big|A(\varepsilon x)\big|^2\big|\phi_\varepsilon(x,t)\big|^2\,dx \\ &+ \int A(\varepsilon x)\cdot\sum_{j=1}^m p_\varepsilon^j(x,t)\,dx + \frac{1}{2}\sum_{j=1}^m\beta_j\iint\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy \\ &+ \frac{1}{2}\sum_{i,j,\,i\neq j}^m\omega_{ij}\iint\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(x,t)\big|^2\big|\phi_\varepsilon^j(y,t)\big|^2\,dx\,dy + \mathcal{O}\big(\varepsilon^2\big),\end{aligned}$$

as ε goes to zero. Finally, since $p_\varepsilon^j(x,t) = (p_\varepsilon^A)^j(x,t) + A(\varepsilon x)|\phi_\varepsilon^j(x,t)|^2$ and recalling the definition of q_ε^A, we obtain the desired conclusion. □

Now let us introduce two functionals in the dual space of C^2

$$\int \Pi_\varepsilon^1(x,t)\cdot\varphi(x)\,dx = \int\varphi(\varepsilon x)\cdot q_\varepsilon^A(x,t)\,dx - M\varphi\big(\varepsilon x_\varepsilon(t)\big)\cdot\xi_\varepsilon(t), \quad \forall\varphi\in C^2\big(\mathbb{R}^N;\mathbb{R}^N\big), \tag{2.7}$$

$$\int \Pi_\varepsilon^2(x,t)\varphi(x)\,dx = \int\varphi(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx - M\varphi\big(\varepsilon x_\varepsilon(t)\big), \quad \forall\varphi\in C^2\big(\mathbb{R}^N;\mathbb{R}\big), \tag{2.8}$$

for all $t \in \mathbb{R}^+$, where M is given in formula (1.4). Moreover, define the function $\Omega_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ as $\Omega_\varepsilon(t) = \hat{\Omega}_\varepsilon(t) + \rho_\varepsilon^A(t)$, where

$$\hat{\Omega}_\varepsilon(t) := \left| \int \Pi_\varepsilon^1(x,t)\,dx \right| + \sup_{\|\varphi\|_{C^3} \leqslant 1} \left| \int \Pi_\varepsilon^2(x,t)\varphi(x)\,dx \right| + \left|\gamma_\varepsilon(t)\right|, \quad t \in \mathbb{R}^+,$$

$$\rho_\varepsilon^A(t) := \left| \int \Pi_\varepsilon^1(x,t) \cdot A(x)\,dx \right|, \quad t \in \mathbb{R}^+ \tag{2.9}$$

and

$$\gamma_\varepsilon(t) := M\varepsilon x_\varepsilon(t) - \int \varepsilon x \chi(\varepsilon x) \left|\phi_\varepsilon(x,t)\right|^2 dx, \quad t \in \mathbb{R}^+,$$

where $\chi \in C^\infty(\mathbb{R}^N)$ is such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 1$ in $B(0,\tilde{\rho})$ and $\chi(x) = 0$ in $\mathbb{R}^N \setminus B(0,2\tilde{\rho})$, for a suitable $\tilde{\rho} > 0$ that will be suitably chosen later.

Now we are able to prove an estimate on the energy of ψ_ε.

Lemma 2.6. *Assume that $\Phi = 0$ if $m > 1$ and let ψ_ε be the function defined in formula* (2.6)*. Then there exists a positive constant C independent of ε such that*

$$0 \leqslant \mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),$$

for all $t \in \mathbb{R}^+$ and any $\varepsilon > 0$.

Proof. We claim that $\Omega_\varepsilon(0) = \mathcal{O}(\varepsilon^2)$ as ε goes to zero. In fact, by definition of Ω_ε, we have

$$\Omega_\varepsilon(0) = \left| \int \Pi_\varepsilon^1(x,0)\,dx \right| + \sup_{\|\varphi\|_{C^3} \leqslant 1} \left| \int \Pi_\varepsilon^2(x,0)\varphi(x)\,dx \right| + \left|\gamma_\varepsilon(0)\right| + \rho_\varepsilon^A(0). \tag{2.10}$$

First of all, let us estimate the first term in the right-hand side of (2.10). Taking $\varphi \equiv 1$ in (2.7) and using (I), we get

$$\begin{aligned}
\int \Pi_\varepsilon^1(x,0)\,dx &= \int q_\varepsilon^A(x,0)\,dx - M\xi(0) \\
&= \sum_{j=1}^m \int \Im\big(\bar{\phi}_\varepsilon^j(x,0)\big(\nabla\phi_\varepsilon^j(x,0) - \mathrm{i}A(\varepsilon x)\phi_\varepsilon^j(x,0)\big)\big)\,dx - M\xi_0 \\
&= \sum_{j=1}^m \int r_j^2\big(x - x_\varepsilon(0)\big)\big[A(x_0) + \xi_0 - A(\varepsilon x)\big]\,dx - M\xi_0 \\
&= MA(x_0) - \sum_{j=1}^m \int r_j^2\big(x - x_\varepsilon(0)\big)A(\varepsilon x)\,dx \\
&= MA(x_0) - \sum_{j=1}^m \int r_j^2(x)A(\varepsilon x + x_0)\,dx = \mathcal{O}\big(\varepsilon^2\big),
\end{aligned}$$

as ε goes to zero, in light of Lemma 2.1. In a similar fashion, one gets $\rho_\varepsilon^A(0) = \mathcal{O}(\varepsilon^2)$. Now consider the second term in the right-hand side of (2.10). Let $\varphi \in C^3(\mathbb{R}^N)$ with $\|\varphi\|_{C^3} \leqslant 1$. Then,

$$\begin{aligned}
\int \Pi_\varepsilon^2(x,0)\varphi(x)\,dx &= \int \varphi(\varepsilon x)\left|\phi_\varepsilon(x,0)\right|^2 dx - M\varphi\big(x(0)\big) \\
&= \sum_{j=1}^m \int \varphi(\varepsilon x + x_0) r_j^2(x)\,dx - M\varphi(x_0) = \mathcal{O}\big(\varepsilon^2\big)
\end{aligned}$$

as ε goes to zero, again using Lemma 2.1. We finally estimate $\gamma_\varepsilon(0)$. As above we have

$$\begin{aligned}
\gamma_\varepsilon(0) &= Mx(0) - \varepsilon \int x\chi(\varepsilon x)\left|\phi_\varepsilon(x,0)\right|^2 dx \\
&= Mx_0 - \varepsilon \sum_{j=1}^m \int x\chi(\varepsilon x) r_j^2\big(x - x_\varepsilon(0)\big)\,dx = Mx_0 - \sum_{j=1}^m \int (\varepsilon x + x_0)\chi(\varepsilon x + x_0) r_j^2(x)\,dx \\
&= Mx_0 - \sum_{j=1}^m \int x_0 \chi(x_0) r_j^2(x)\,dx + \mathcal{O}\big(\varepsilon^2\big) = Mx_0\big(1 - \chi(x_0)\big) + \mathcal{O}\big(\varepsilon^2\big),
\end{aligned}$$

thanks to Lemma 2.1. Now, from [17, Lemmas 3.1–3.2] (where one has to use the δ_a at some point a is defined as $\langle \delta_a, \varphi\rangle = \varphi(\varepsilon a)$ for all $\varphi \in C^2(\mathbb{R}^N)$), we learn that there exist three positive constants K_0, K_1, K_2 such that, for all $y, z \in \mathbb{R}^N$, $K_1|\varepsilon y - \varepsilon z| \leqslant \|\delta_y - \delta_z\|_{C^{2*}} \leqslant K_2|\varepsilon y - \varepsilon z|$, provided that $\|\delta_y - \delta_z\|_{C^{2*}} \leqslant K_0$. Let then $\tilde{\rho} = K_1 \sup_{\varepsilon\in[0,1]} \sup_{t\in[0,T_0/\varepsilon]} |\varepsilon x_\varepsilon(t)| + K_0$, where $T_0 > 0$ is fixed (to be chosen later on, see Lemma 3.4). Then, in view of the definition of χ, we obtain that $\gamma_\varepsilon(0) = \mathcal{O}(\varepsilon^2)$ as ε goes to zero, since $|x_0| < \tilde{\rho}$. Hence the claim is proved.

Now we are ready to prove the assertion of Lemma 2.6. By using Lemma 2.5, the definition of $\mathcal{H}$, (2.7) and (2.8) we obtain

$$\begin{aligned}
\mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) &= \frac{1}{2}M\big|\xi_\varepsilon(t)\big|^2 + MV\big(\varepsilon x_\varepsilon(t)\big) + M\mathcal{M} \\
&\quad - \int V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx + \frac{1}{2}M\big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \\
&\quad - \int \Pi^1_\varepsilon(x,t)\big[\big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\big]\,dx - M\big[\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big]\cdot\xi_\varepsilon(t) \\
&\quad - \big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\cdot\bigg(\int \Pi^2_\varepsilon(x,t)A(x)\,dx + MA\big(\varepsilon x_\varepsilon(t)\big)\bigg) \\
&\quad + \frac{1}{2}\int \big|A(\varepsilon x)\big|^2\big|\phi_\varepsilon(x,t)\big|^2\,dx + \int \Pi^1_\varepsilon(x,t)A(x)\,dx + MA\big(\varepsilon x_\varepsilon(t)\big)\cdot\xi_\varepsilon(t) \\
&\quad + \frac{1}{2}\sum_{j=1}^m \beta_j \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi^j_\varepsilon(x,t)\big|^2\big|\phi^j_\varepsilon(y,t)\big|^2\,dx\,dy \\
&\quad + \frac{1}{2}\sum_{i,j,\,i\neq j}^m \omega_{ij} \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi^i_\varepsilon(x,t)\big|^2\big|\phi^j_\varepsilon(y,t)\big|^2\,dx\,dy + \mathcal{O}\big(\varepsilon^2\big).
\end{aligned}$$

Let us set (with the convention that $\omega_{ii} = \beta_i$)

$$\eta_\varepsilon(t) = \Bigg|\sum_{i,j=1}^m \frac{\omega_{ij}}{2}\iint \Phi\big(\varepsilon(x-y)\big)\big|\phi^i_\varepsilon(x,t)\big|^2\big|\phi^j_\varepsilon(y,t)\big|^2\,dx\,dy - \Phi(0)\sum_{i,j=1}^m \frac{\omega_{ij}}{2}m_i m_j\Bigg|.$$

In turn, using the definition of $\mathcal{M}$, we have

$$\begin{aligned}
\mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) &\leqslant \eta_\varepsilon(t) + MV\big(\varepsilon x_\varepsilon(t)\big) - \int \Pi^2_\varepsilon(x,t)V(x)\,dx - MV\big(\varepsilon x_\varepsilon(t)\big) + \frac{1}{2}M\big|A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \\
&\quad - \int \Pi^1_\varepsilon(x,t)\big[\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big]\,dx - \big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\int \Pi^2_\varepsilon(x,t)A(x)\,dx - M\big|A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \\
&\quad + \frac{1}{2}\int \Pi^2_\varepsilon(x,t)\big|A(x)\big|^2\,dx + \frac{1}{2}M\big|A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 + \int \Pi^1_\varepsilon(x,t)A(x)\,dx + \mathcal{O}\big(\varepsilon^2\big) \\
&= \eta_\varepsilon(t) - \int \Pi^2_\varepsilon(x,t)V(x)\,dx - \int \Pi^1_\varepsilon(x,t)\big[\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big]\,dx \\
&\quad - \big(\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big)\int \Pi^2_\varepsilon(x,t)A(x)\,dx \\
&\quad + \frac{1}{2}\int \Pi^2_\varepsilon(x,t)\big|A(x)\big|^2\,dx + \int \Pi^1_\varepsilon(x,t)A(x)\,dx + \mathcal{O}\big(\varepsilon^2\big) \\
&\leqslant \eta_\varepsilon(t) + C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),
\end{aligned}$$

for ε sufficiently small. If $m > 1$ we assume that $\Phi = 0$, and the assertion follows. If instead $m = 1$, observe first that from definition (2.8), by choosing $\varphi(x) = \Phi(x - \varepsilon y)$ and $\varphi(y) = \Phi(\varepsilon x_\varepsilon(t) - y)$ respectively, we have

$$\begin{aligned}
&\int \Phi(\varepsilon x - \varepsilon y)\big|\phi^1_\varepsilon(x,t)\big|^2\,dx = \int \Pi^2_\varepsilon(x,t)\Phi(x - \varepsilon y)\,dx + m_1\Phi\big(\varepsilon x_\varepsilon(t) - \varepsilon y\big), \\
&m_1\int \Phi\big(\varepsilon x_\varepsilon(t) - \varepsilon y\big)\big|\phi^1_\varepsilon(y,t)\big|^2\,dy = m_1\int \Pi^2_\varepsilon(y,t)\Phi\big(\varepsilon x_\varepsilon(t) - y\big)\,dy + \Phi(0)m_1^2.
\end{aligned}$$

In turn, we have

$$\eta_\varepsilon(t) \leqslant C\left|\int\left[\int \Pi_\varepsilon^2(x,t)\Phi(x-\varepsilon y)\,dx + m_1\Phi\big(\varepsilon x_\varepsilon(t)-\varepsilon y\big)\right]\left|\phi_\varepsilon^1(y,t)\right|^2 dy - \Phi(0)m_1^2\right|$$

$$= C\left|\int\left[\int \Pi_\varepsilon^2(x,t)\Phi(x-\varepsilon y)\,dx\right]\left|\phi_\varepsilon^1(y,t)\right|^2 dy + m_1\int \Phi\big(\varepsilon x_\varepsilon(t)-\varepsilon y\big)\left|\phi_\varepsilon^1(y,t)\right|^2 dy - \Phi(0)m_1^2\right|$$

$$\leqslant Cm_1 \sup_{\|\varphi\|_{C^3}\leqslant 1}\left|\int \Pi_\varepsilon^2(x,t)\varphi(x)\,dx\right| + Cm_1\left|\int \Pi_\varepsilon^2(y,t)\Phi\big(\varepsilon x_\varepsilon(t)-y\big)\,dy\right|$$

$$\leqslant C \sup_{\|\varphi\|_{C^3}\leqslant 1}\left|\int \Pi_\varepsilon^2(x,t)\varphi(x)\,dx\right| \leqslant C\hat{\Omega}_\varepsilon(t) \leqslant C\Omega_\varepsilon(t).$$

In turn, we conclude that

$$\mathcal{E}\big(\psi_\varepsilon(t)\big) - \mathcal{E}(r) \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big)$$

as ε goes to zero, for some positive constant C. Hence the proof of Lemma 2.6 is complete. □

Since the function $\{t \mapsto \Omega_\varepsilon(t)\}$ given in (2.9) is continuous and recalling that $\Omega_\varepsilon(0) = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$ (see the proof of Lemma 2.6), for any fixed $T_0 > 0$ and $\sigma_0 > 0$, we can define the time

$$T_\varepsilon^* := \sup\big\{t \in [0, T_0/\varepsilon]:\ \Omega_\varepsilon(s),\ \Gamma_{\psi_\varepsilon(s)} \leqslant \sigma_0,\ \text{for all } s \in (0,t)\big\} > 0, \tag{2.11}$$

for any $\varepsilon > 0$, where $\Gamma_{\psi_\varepsilon}$ is defined according to (1.8) and $\Gamma_{\psi_\varepsilon(0)} = 0$. Now we are able to provide the main result of this section, related to a representation formula for the solution ϕ_ε of problem (P). For the proof, it is enough to adapt the proof of [25, Theorem 4.2]. The fact:

Theorem 2.7. *Let ϕ_ε be the family of solutions to problem (P) corresponding to the initial datum (I) modelled on a ground state solution r of problem (S) and let $(x_\varepsilon(t), \xi_\varepsilon(t))$ be the global solution of (D). Then there exist positive constants ε_0 and C, locally bounded functions $\theta_\varepsilon^1, \dots, \theta_\varepsilon^m : \mathbb{R}^+ \to [0, 2\pi)$ and $y_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^N$ such that*

$$\phi_\varepsilon^j(x,t) = e^{\mathrm{i}(\xi_\varepsilon(t)\cdot x + \theta_\varepsilon^j(t) + A(\varepsilon x_\varepsilon(t))\cdot(x - x_\varepsilon(t)))} r_j\big(x - y_\varepsilon(t)\big) + \omega_\varepsilon^j(t),$$

where $\|\omega_\varepsilon^j(t)\|_{H^1} \leqslant C\sqrt{\Omega_\varepsilon(t)} + \mathcal{O}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T_\varepsilon^*)$ and $j = 1, \dots, m$.

3. Density and momentum identities

This section is devoted to some important identities involving the momentum p_ε^A and the total magnetic momentum q_ε^A related to problem (P).

Proposition 3.1. *Let ϕ_ε be the solution to problem (P) corresponding to the initial datum (I). Then the following identities hold true*

$$\frac{\partial|\phi_\varepsilon^j|^2}{\partial t}(x,t) = -\operatorname{div}_x\big(p_\varepsilon^A\big)^j(x,t), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+,\ j = 1, \dots, m, \tag{3.1}$$

$$\begin{aligned}\int \frac{\partial q_\varepsilon^A}{\partial t}(x,t)\,dx = {}& -\int q_\varepsilon^A(x,t) \times \varepsilon B(\varepsilon x)\,dx - \int \varepsilon\nabla V(\varepsilon x)\left|\phi_\varepsilon(x,t)\right|^2 dx \\ & + \sum_{j=1}^m \beta_j \iint \varepsilon\nabla\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^j(y)\right|^2\left|\phi_\varepsilon^j(x)\right|^2 dx\,dy \\ & + \sum_{i,j=1,\, i\neq j}^m \omega_{ij} \iint \varepsilon\nabla\Phi\big(\varepsilon(x-y)\big)\left|\phi_\varepsilon^i(y)\right|^2\left|\phi_\varepsilon^j(x)\right|^2 dx\,dy,\end{aligned} \tag{3.2}$$

for $t \in \mathbb{R}^+$, where $B = \nabla \times A$ is the magnetic field associated with A.

Proof. The proof follows the lines of the corresponding proof in [25] for the scalar case without the presence of nonlocal potentials. By formula (2.2), for any $j = 1, \dots, m$, $(p_\varepsilon^A)^j$ is the vector whose components, which we denote by $(p_\varepsilon^A)_\ell^j$, are given by $(p_\varepsilon^A)_\ell^j = \Im(\bar{\phi}_\varepsilon^j(x,t)(\partial_\ell\phi_\varepsilon^j(x,t) - \mathrm{i}A_\ell(\varepsilon x)\phi_\varepsilon^j(x,t)))$, for $\ell = 1, \dots, N$. Let us fix $j = 1, \dots, m$. Hence

$$
\begin{aligned}
-\operatorname{div}_x\big(p_\varepsilon^A\big)^j(x,t) &= -\sum_{\ell=1}^N \Im\big(\partial_\ell\bar\phi_\varepsilon^j(x,t)\big(\partial_\ell\phi_\varepsilon^j(x,t)-\mathrm{i}A_\ell(\varepsilon x)\phi_\varepsilon^j(x,t)\big)\big)\\
&\quad-\sum_{\ell=1}^N \Im\big(\bar\phi_\varepsilon^j(x,t)\big(\partial^2_{\ell\ell}\phi_\varepsilon^j(x,t)-\mathrm{i}\partial_\ell A_\ell(\varepsilon x)\phi_\varepsilon^j(x,t)-\mathrm{i}A_\ell(\varepsilon x)\partial_\ell\phi_\varepsilon^j(x,t)\big)\big)\\
&= 2A(\varepsilon x)\cdot\Re\big(\nabla\bar\phi_\varepsilon^j(x,t)\phi_\varepsilon^j(x,t)\big)-\Im\big(\bar\phi_\varepsilon^j(x,t)\Delta\phi_\varepsilon^j(x,t)\big)+\operatorname{div}_x A(\varepsilon x)\big|\phi_\varepsilon^j(x,t)\big|^2.
\end{aligned}
$$

Moreover, using (P) and taking into account the definition of L_A, we get

$$
\begin{aligned}
\frac{\partial|\phi_\varepsilon^j|^2}{\partial t}(x,t) &= 2\Im\big(\bar\phi_\varepsilon^j(x,t)\big(L_A\phi_\varepsilon^j(x,t)+V(\varepsilon x)\phi_\varepsilon^j(x,t)-\big|\phi_\varepsilon(x,t)\big|_j^{2p}\phi_\varepsilon^j(x,t)-\Phi(\varepsilon x)*|\phi_\varepsilon|_j^2\phi_\varepsilon^j(x,t)\big)\big)\\
&= -\Im\big(\bar\phi_\varepsilon^j(x,t)\Delta\phi_\varepsilon^j(x,t)\big)+2A(\varepsilon x)\cdot\Re\big(\phi_\varepsilon^j(x,t)\nabla\bar\phi_\varepsilon^j(x,t)\big)+\operatorname{div}_x A(\varepsilon x)\big|\phi_\varepsilon^j(x,t)\big|^2,
\end{aligned}
$$

so that identity (3.1) holds true. Now let us prove the second one. By definition of the total magnetic momentum q_ε^A, for any $\ell=1,\dots,N$, we have

$$
\begin{aligned}
\frac{\partial(q_\varepsilon^A)_\ell}{\partial t} &= \sum_{j=1}^m\frac{\partial(p_\varepsilon^A)_\ell^j}{\partial t} = \sum_{j=1}^m\big(\Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+\Im\big(\partial_\ell\big(\phi_\varepsilon^j\partial_t\phi_\varepsilon^j\big)\big)\big)-\sum_{j=1}^m\Im\big(\partial_\ell\bar\phi_\varepsilon^j\partial_t\phi_\varepsilon^j\big)-A_\ell(\varepsilon x)\sum_{j=1}^m\frac{\partial|\phi_\varepsilon^j|^2}{\partial t}\\
&= 2\sum_{j=1}^m\Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)-\sum_{j=1}^m\Im\big(\partial_\ell\big(\bar\phi_\varepsilon^j\partial_t\phi_\varepsilon^j\big)\big)-A_\ell(\varepsilon x)\sum_{j=1}^m\frac{\partial|\phi_\varepsilon^j|^2}{\partial t},
\end{aligned}
$$

and so, integrating over $\mathbb{R}^N$, it is easy to see that

$$
\int\frac{\partial(q_\varepsilon^A)_\ell}{\partial t}\,dx = 2\sum_{j=1}^m\int\Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx-\sum_{j=1}^m\int\Im\big(\partial_\ell\big(\bar\phi_\varepsilon^j\partial_t\phi_\varepsilon^j\big)\big)\,dx-\sum_{j=1}^m\int A_\ell(\varepsilon x)\frac{\partial|\phi_\varepsilon^j|^2}{\partial t}\,dx. \tag{3.3}
$$

Let us consider the first term in the right-hand side of (3.3). Conjugating the equation, multiplying it by $2i\partial_\ell\phi_\varepsilon^j$, $\ell=1,\dots,N$, and taking the imaginary part, we have

$$
\begin{aligned}
2\Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big) &= -\Re\big(\Delta\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+2A(\varepsilon x)\cdot\Im\big(\nabla\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+\big|A(\varepsilon x)\big|^2\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+\operatorname{div}_x A(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\\
&\quad+2V(\varepsilon x)\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)-2\Re\big(|\phi_\varepsilon|_j^{2p}\bar\phi_\varepsilon{}^j\partial_\ell\phi_\varepsilon^j\big)-2\Re\big(\big(\Phi(\varepsilon x)*|\phi_\varepsilon|_j^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\\
&= -\sum_{i=1}^m\Re\big(\partial_i\big(\partial_i\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\big)+\sum_{i=1}^m\partial_\ell\left(\frac{|\partial_i\phi_\varepsilon^j|^2}{2}\right)+2A(\varepsilon x)\cdot\Im\big(\nabla\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+\big|A(\varepsilon x)\big|^2\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\\
&\quad+\operatorname{div}_x A(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)+\partial_\ell\big(V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\big)-\varepsilon\partial_\ell V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\\
&\quad-\frac{\alpha_j}{p+1}\partial_\ell\big(\big|\phi_\varepsilon^j\big|^{2p+2}\big)-2\sum_{i=1,\,i\neq j}^m\gamma_{ij}\big|\phi_\varepsilon^i\big|^{p+1}\big|\phi_\varepsilon^j\big|^{p-1}\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\\
&\quad-2\beta_j\Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^j\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)-2\sum_{i=1,\,i\neq j}^m\omega_{ij}\Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^i\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big).
\end{aligned}
$$

Hence, integrating over $\mathbb{R}^N$ and using the H^2-regularity of the functions involved, for all $\ell=1,\dots,N$ we obtain the following identity

$$
\begin{aligned}
2\int\Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx &= 2\int A(\varepsilon x)\cdot\Im\big(\nabla\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx+\int\big|A(\varepsilon x)\big|^2\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx+\int\operatorname{div}_x A(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx\\
&\quad-\varepsilon\int\partial_\ell V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\,dx-2\sum_{i=1,\,i\neq j}^m\gamma_{ij}\int\big|\phi_\varepsilon^i\big|^{p+1}\big|\phi_\varepsilon^j\big|^{p-1}\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx\\
&\quad-2\beta_j\int\Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^j\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx-2\sum_{i=1,\,i\neq j}^m\omega_{ij}\int\Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^i\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx.
\end{aligned}
$$

Notice that

$$\int \operatorname{div}_x A(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j \partial_\ell \phi_\varepsilon^j\big)\,dx + 2\int A(\varepsilon x)\cdot \Im\big(\nabla\bar\phi_\varepsilon^j \partial_\ell \phi_\varepsilon^j\big)\,dx = \varepsilon \sum_{i=1}^m \int \partial_\ell A_i(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j \partial_i \phi_\varepsilon^j\big)\,dx.$$

Moreover, thanks to the regularity of ϕ_ε^j, we have

$$\begin{aligned}\sum_{i,j=1,\,i\neq j}^m \int \big|\phi_\varepsilon^i\big|^{p+1}\big|\phi_\varepsilon^j\big|^{p-1}\Re\big(\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx &= \sum_{i,j=1,\,i\neq j}^m \int \big|\phi_\varepsilon^i\big|^{p+1}\big|\phi_\varepsilon^j\big|^{p-1}\partial_\ell\bigg(\frac{|\phi_\varepsilon^j|^2}{2}\bigg)\,dx\\
&= \frac{1}{p+1}\sum_{i,j=1,\,i\neq j}^m \int \big|\phi_\varepsilon^i\big|^{p+1}\partial_\ell\big(\big|\phi_\varepsilon^j\big|^{p+1}\big)\,dx\\
&= \frac{1}{p+1}\sum_{i,j=1,\,i< j}^m \int \partial_\ell\big(\big|\phi_\varepsilon^i\big|^{p+1}\big|\phi_\varepsilon^j\big|^{p+1}\big)\,dx = 0,\end{aligned}$$

and

$$\begin{aligned}\int \Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^j\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx &= \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\Re\big(\bar\phi_\varepsilon^j(x)\partial_\ell\phi_\varepsilon^j(x)\big)\,dy\,dx\\
&= \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\partial_\ell\bigg(\frac{|\phi_\varepsilon^j(x)|^2}{2}\bigg)\,dy\,dx\\
&= -\frac12\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy\end{aligned}$$

for all $\ell = 1,\dots,N$. While, with the same arguments, we get

$$\sum_{i=1,\,i\neq j}^m \omega_{ij}\int \Re\big(\big(\Phi(\varepsilon x)*\big|\phi_\varepsilon^i\big|^2\big)\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx = -\frac12\sum_{i=1,\,i\neq j}^m \omega_{ij}\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy.$$

Hence, it is easy to see that

$$\begin{aligned}2\int \Im\big(\partial_t\bar\phi_\varepsilon^j\partial_\ell\phi_\varepsilon^j\big)\,dx &= \sum_{i=1}^m\int \varepsilon\partial_\ell A_i(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j\partial_i\phi_\varepsilon^j\big)\,dx + \int\big|A(\varepsilon x)\big|^2\partial_\ell\bigg(\frac{|\phi_\varepsilon^j|^2}{2}\bigg)\,dx\\
&\quad - \int \varepsilon\partial_\ell V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\,dx - \beta_j\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy\\
&\quad - \sum_{i=1,\,i\neq j}^m \omega_{ij}\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy\\
&= \sum_{i=1}^m\int \varepsilon\partial_\ell A_i(\varepsilon x)\Im\big(\bar\phi_\varepsilon^j\partial_i\phi_\varepsilon^j\big)\,dx + \sum_{i=1}^m\int \varepsilon A_i(\varepsilon x)\partial_\ell A_i(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\,dx\\
&\quad - \int \varepsilon\partial_\ell V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\,dx - \beta_j\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy\\
&\quad - \sum_{i=1,\,i\neq j}^m \omega_{ij}\iint \varepsilon\partial_\ell\Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy \qquad (3.4)\end{aligned}$$

for all $\ell = 1,\dots,N$. As for the second term in (3.3), using again the regularity of ϕ_ε^j, we get $\int \Im(\partial_\ell(\bar\phi_\varepsilon^j\partial_t\phi_\varepsilon^j))\,dx = 0$, for any $\ell = 1,\dots,N$. Finally, as for the third term in the right-hand side of (3.3), by (3.1) we get

$$\begin{aligned}\int A_\ell(\varepsilon x)\frac{\partial|\phi_\varepsilon^j|^2}{\partial t}(x,t)\,dx &= -\int A_\ell(\varepsilon x)\operatorname{div}_x\big(p_\varepsilon^A\big)^j(x,t)\,dx\\
&= \sum_{i=1}^m\int \varepsilon\partial_i A_\ell(\varepsilon x)\big(p_\varepsilon^A\big)_i^j(x,t)\,dx\end{aligned}$$

$$= \sum_{i=1}^{m} \int \varepsilon \partial_i A_\ell(\varepsilon x) \Im\big(\bar{\phi}_\varepsilon^j(x,t)\big(\partial_i \phi_\varepsilon^j(x,t) - i A_i(\varepsilon x)\phi_\varepsilon^j(x,t)\big)\big)$$

$$= \int \sum_{i=1}^{m} \varepsilon \partial_i A_\ell(\varepsilon x) \Im\big(\bar{\phi}_\varepsilon^j(x,t)\partial_i \phi_\varepsilon^j(x,t)\big)\,dx - \int \sum_{i=1}^{m} \varepsilon A_i(\varepsilon x)\partial_i A_\ell(\varepsilon x)\big|\phi_\varepsilon^j(x,t)\big|^2\,dx \tag{3.5}$$

for any $\ell = 1,\dots,N$. Then (3.3)–(3.5) yield

$$\begin{aligned}
\int \frac{\partial (q_\varepsilon^A)_\ell}{\partial t}(x,t)\,dx &= \sum_{i,j=1}^{m} \int \varepsilon\big(\partial_\ell A_i(\varepsilon x) - \partial_i A_\ell(\varepsilon x)\big)\Im\big(\bar{\phi}_\varepsilon^j \partial_i \phi_\varepsilon^j\big)\,dx \\
&\quad + \sum_{i,j=1}^{m} \int \varepsilon A_i(\varepsilon x)\big(\partial_\ell A_i(\varepsilon x) - \partial_i A_\ell(\varepsilon x)\big)\big|\phi_\varepsilon^j\big|^2\,dx \\
&\quad - \sum_{j=1}^{m} \int \varepsilon \partial_\ell V(\varepsilon x)\big|\phi_\varepsilon^j\big|^2\,dx - \sum_{j=1}^{m} \beta_j \iint \varepsilon \partial_\ell \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy \\
&\quad - \sum_{i=1,\,i\neq j}^{m} \omega_{ij} \iint \varepsilon \partial_\ell \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy \\
&= -\int \big(q_\varepsilon^A(x,t) \times \varepsilon B(\varepsilon x)\big)_\ell\,dx - \int \varepsilon \partial_\ell V(\varepsilon x)\big|\phi_\varepsilon\big|^2\,dx \\
&\quad - \sum_{j=1}^{m} \beta_j \iint \varepsilon \partial_\ell \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^j(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy \\
&\quad - \sum_{i=1,\,i\neq j}^{m} \omega_{ij} \iint \varepsilon \partial_\ell \Phi\big(\varepsilon(x-y)\big)\big|\phi_\varepsilon^i(y)\big|^2\big|\phi_\varepsilon^j(x)\big|^2\,dx\,dy
\end{aligned}$$

for any $\ell = 1,\dots,N$, so that (3.2) is proved. □

Remark 3.2. Taking into account the definition of q_ε^A, by (3.1) easily follows

$$\frac{\partial |\phi_\varepsilon|^2}{\partial t}(x,t) = -\operatorname{div}_x q_\varepsilon^A(x,t), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+,$$

which is consistent with the conservation's laws for the nonlinear Schrödinger equation.

We now give some estimates on the momentum p_ε^A and the total magnetic momentum q_ε^A related to problem (P).

Lemma 3.3. *Let ϕ_ε be the solution of problem (P) corresponding to the initial datum (I) and let $(x_\varepsilon(t), \xi_\varepsilon(t))$ be the global solution to (D). Then, in the notational framework of Theorem 2.7, there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\big\|\big|\phi_\varepsilon^j(x,t)\big|^2\,dx - m_j \delta_{y_\varepsilon(t)}\big\|_{C^{2*}} + \big\|q_\varepsilon^{A(\varepsilon x_\varepsilon(t))}(x,t)\,dx - M\xi_\varepsilon(t)\delta_{y_\varepsilon(t)}\big\|_{C^{2*}} \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),$$

for every $t \in [0, T_\varepsilon^*)$ and $\varepsilon \in (0, \varepsilon_0)$ and for all $j = 1,\dots,m$, where T_ε^* is given in (2.11).

Proof. For any $v \in H^1(\mathbb{R}^N)$, we have the formula $|\nabla|v||^2 = |\nabla v|^2 - \frac{|\Im(\bar{v}\nabla v)|^2}{|v|^2}$. Then, by virtue of Lemma 2.6, it follows that

$$0 \leqslant \mathcal{E}\big(|\psi_\varepsilon|\big) - \mathcal{E}(r) + \frac{1}{2}\sum_{j=1}^{m} \int \frac{|\Im(\bar{\psi}_\varepsilon^j \nabla \psi_\varepsilon^j)|^2}{|\psi_\varepsilon^j|^2}\,dx \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),$$

for all $t \in \mathbb{R}^+$ and $\varepsilon > 0$. Moreover, since $\||\psi_\varepsilon^j|\|_{L^2} = \|r_j\|_{L^2}$ for all $j = 1,\dots,m$ and $\mathcal{E}(|\psi_\varepsilon|) \geqslant \mathcal{E}(r)$ by means of (1.6), we have

$$\int \frac{|\Im(\bar{\psi}_\varepsilon^j \nabla \psi_\varepsilon^j)|^2}{|\psi_\varepsilon^j|^2}\,dx \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big), \tag{3.6}$$

for every $t \in \mathbb{R}^+$ and $\varepsilon > 0$ and for all $j = 1,\dots,m$. Following the blueprint of [25, Lemma 6.1], we get the assertion (see also [21]). □

Lemma 3.3 allows us to prove the following result on the distance between the point $y_\varepsilon(t)$ found out in Theorem 2.7 and the trajectory $x_\varepsilon(t)$. For the proof, follow the blueprint of [25, Lemma 6.3].

Lemma 3.4. *In the notational framework of Theorem 2.7 there exist $\varepsilon_0 > 0$ and $T_0 > 0$ (cf. the definition of $T^*_\varepsilon = T^*_\varepsilon(T_0)$) such that*

$$\|\delta_{x_\varepsilon(t)} - \delta_{y_\varepsilon(t)}\|_{C^{2*}} \leqslant C\big|\varepsilon x_\varepsilon(t) - \varepsilon y_\varepsilon(t)\big| \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),$$

for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T^*_\varepsilon)$, where T^*_ε defined as in (2.11).

Next, we state a strengthened version of Lemma 3.3, obtained thanks to Lemma 3.4. Follow the blueprint of [25, Lemma 6.4] for a proof.

Lemma 3.5. *Let T_0 be as in Lemma 3.4. Let ϕ_ε be the family of solutions to problem (P) with initial datum (I) and let $(x_\varepsilon(t), \xi_\varepsilon(t))$ be the global solution of (D). Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\big\|\big|\phi^j_\varepsilon(x,t)\big|^2\,dx - m_j\delta_{x_\varepsilon(t)}\big\|_{C^{2*}} + \big\|q^A_\varepsilon(x,t)\,dx - M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\big\|_{C^{2*}} \leqslant C\Omega_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big),$$

for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T^*_\varepsilon)$.

In particular, by the definition of Ω_ε, there exists $\delta > 0$ with

$$\big\|\big|\phi^j_\varepsilon(x,t)\big|^2\,dx - m_j\delta_{x_\varepsilon(t)}\big\|_{C^{2*}} + \big\|q^A_\varepsilon(x,t)\,dx - M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\big\|_{C^{2*}} \leqslant C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big), \tag{3.7}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$, provided that $\|A\|_{C^2} < \delta$.

Remark 3.6. In Lemma 3.5, while the C^{2*}-norm control holds on $\Pi^j_\varepsilon = |\phi^j_\varepsilon(x,t)|^2\,dx - m_j\delta_{x_\varepsilon(t)}$ for each $j = 1, \dots, m$, the control on the momentum holds for the total magnetic momentum $q^A_\varepsilon(x,t)$. This is in fact natural, since looking at the second identity in Proposition 3.1, it is clear that it cannot hold for each individual $(p^A_\varepsilon)^j$, unless some other (disturbing) integral terms are added to the formula.

4. Uniform estimation of $\boldsymbol{\Omega_\varepsilon}$

Before proving the main result we give an estimate showing that the quantity $\Omega_\varepsilon(t)$ can be made small at the order $\mathcal{O}(\varepsilon^2)$, uniformly on finite time intervals, as ε goes to zero.

Lemma 4.1. *Let T_0 be as in Lemma 3.4 and ε_0, δ as in Lemma 3.5. Then there exists $C > 0$ such that $\hat{\Omega}_\varepsilon(t) \leqslant C\varepsilon^2$, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$.*

In addition, if we assume that $\|A\|_{C^2} < \delta$ for $\delta > 0$ sufficiently enough, then $\Omega_\varepsilon(t) \leqslant C\varepsilon^2$, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$.

Proof. By the definition of Π^1_ε, Lemma 3.5, Proposition 3.1 and system (D), we obtain

$$\begin{aligned}
\left|\int \frac{d}{dt}\Pi^1_\varepsilon(x,t)\,dx\right| &= \left|\int \frac{\partial q^A_\varepsilon}{\partial t}(x,t)\,dx - M\dot{\xi}_\varepsilon(t)\right| \\
&= \left|\int q^A_\varepsilon(x,t) \times \varepsilon B(\varepsilon x)\,dx + \int \varepsilon\nabla V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2\,dx\right. \\
&\quad + \sum_{j=1}^m \beta_j \iint \varepsilon\nabla\Phi\big(\varepsilon(x-y)\big)\big|\phi^j_\varepsilon(y)\big|^2\big|\phi^j_\varepsilon(x)\big|^2\,dx\,dy \\
&\quad + \sum_{i,j=1,\, i\neq j}^m \omega_{ij} \iint \varepsilon\nabla\Phi\big(\varepsilon(x-y)\big)\big|\phi^i_\varepsilon(y)\big|^2\big|\phi^j_\varepsilon(x)\big|^2\,dx\,dy \\
&\quad \left. - M\varepsilon\nabla V\big(\varepsilon x_\varepsilon(t)\big) - M\varepsilon\xi_\varepsilon(t) \times B\big(\varepsilon x_\varepsilon(t)\big)\right|.
\end{aligned}$$

If $m > 1$, we do not have to manage the nonlocal terms, since $\Phi \equiv 0$. If instead $m = 1$, recalling that $\nabla\Phi(0) = 0$, by Lemma 3.5 and arguing as at the end of the proof of Lemma 2.6, we get

$$\left|\iint \varepsilon\nabla\Phi\big(\varepsilon(x-y)\big)\big|\phi^1_\varepsilon(y)\big|^2\big|\phi^1_\varepsilon(x)\big|^2\,dx\,dy\right| \leqslant \varepsilon\big[C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big)\big], \tag{4.1}$$

for some positive constant C, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$. In turn, it holds

$$\begin{aligned}\left|\int \frac{d}{dt}\Pi_\varepsilon^1(x,t)\,dx\right| &\leqslant \left|\int q_\varepsilon^A(x,t)\times\varepsilon B(\varepsilon x)\,dx+\int \varepsilon\nabla V(\varepsilon x)\left|\phi_\varepsilon(x,t)\right|^2dx\right.\\ &\quad\left.-\int M\varepsilon\nabla V(\varepsilon x)\delta_{x_\varepsilon(t)}\,dx-\int M\varepsilon\xi_\varepsilon(t)\times B(\varepsilon x)\delta_{x_\varepsilon(t)}\,dx\right|+\varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right]\\ &\leqslant \varepsilon\left|\int\left(q_\varepsilon^A(x,t)-M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\right)\times B(\varepsilon x)\,dx\right|\\ &\quad+\varepsilon\left|\int\nabla V(\varepsilon x)\left(\left|\phi_\varepsilon(x,t)\right|^2-M\delta_{x_\varepsilon(t)}\right)dx\right|+\varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right]\\ &\leqslant C\varepsilon\left\|q_\varepsilon^A(x,t)\,dx-M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\right\|_{C^{2*}}+C\varepsilon\left\|\left|\phi_\varepsilon(x,t)\right|^2dx-M\delta_{x_\varepsilon(t)}\right\|_{C^{2*}}+\varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right]\\ &\leqslant \varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right],\end{aligned}\tag{4.2}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $t\in[0,T_\varepsilon^*)$. Hence, recalling that $\Omega_\varepsilon(0)=\mathcal{O}(\varepsilon^2)$ as ε goes to zero,

$$\left|\int \Pi_\varepsilon^1(x,t)\,dx\right|\leqslant\left|\int \Pi_\varepsilon^1(x,0)\,dx\right|+\int_0^t\left|\int\frac{d}{dt}\Pi_\varepsilon^1(x,\tau)\,dx\right|d\tau\leqslant C\varepsilon^2(1+\varepsilon t)+C\varepsilon\int_0^t\hat{\Omega}_\varepsilon(\tau)\,d\tau,\tag{4.3}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $t\in[0,T_\varepsilon^*)$. Now, let $\varphi\in C^3(\mathbb{R}^N)$ such that $\|\varphi\|_{C^3}\leqslant 1$. Again in light of Proposition 3.1 and Lemma 3.5, we have

$$\begin{aligned}\left|\int\frac{d}{dt}\Pi_\varepsilon^2(x,t)\varphi(x)\,dx\right| &= \left|\int\varphi(\varepsilon x)\frac{\partial}{\partial t}\left|\phi_\varepsilon(x,t)\right|^2dx-M\varepsilon\nabla\varphi\left(\varepsilon x_\varepsilon(t)\right)\cdot\xi_\varepsilon(t)\right|\\ &=\left|-\int\varphi(\varepsilon x)\operatorname{div}_x q_\varepsilon^A(x,t)\,dx-M\varepsilon\nabla\varphi\left(\varepsilon x_\varepsilon(t)\right)\cdot\xi_\varepsilon(t)\right|\\ &=\left|\int\varepsilon\nabla\varphi(\varepsilon x)\cdot q_\varepsilon^A(x,t)\,dx-\int M\varepsilon\nabla\varphi(\varepsilon x)\cdot\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\,dx\right|\\ &=\left|\int\varepsilon\nabla\varphi(\varepsilon x)\cdot\left(q_\varepsilon^A(x,t)-M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\right)dx\right|\\ &\leqslant C\varepsilon\left\|q_\varepsilon^A(x,t)\,dx-M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\right\|_{C^{2*}}\leqslant\varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right],\end{aligned}\tag{4.4}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $t\in[0,T_\varepsilon^*)$. Thus, arguing as above, we get

$$\sup_{\|\varphi\|_{C^3}\leqslant 1}\left|\int\Pi_\varepsilon^2(x,t)\varphi(x)\,dx\right|\leqslant C\varepsilon^2(1+\varepsilon t)+C\varepsilon\int_0^t\hat{\Omega}_\varepsilon(\tau)\,d\tau,\tag{4.5}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $t\in[0,T_\varepsilon^*)$. Finally, again via Proposition 3.1 and Lemma 3.5, we have

$$\begin{aligned}\left|\dot{\gamma}_\varepsilon(t)\right| &=\left|M\varepsilon\xi_\varepsilon(t)+\int\varepsilon x\chi(\varepsilon x)\operatorname{div}_x q_\varepsilon^A(x,t)\,dx\right|\\ &=\left|M\varepsilon\xi_\varepsilon(t)-\int\nabla\left(\varepsilon x\chi(\varepsilon x)\right)\cdot q_\varepsilon^A(x,t)\,dx\right|\\ &=\varepsilon\left|\int\nabla\left(x\chi(\varepsilon x)\right)M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\,dx-\int\nabla\left(x\chi(\varepsilon x)\right)\cdot q_\varepsilon^A(x,t)\,dx\right|\\ &\leqslant\varepsilon C\left\|q_\varepsilon^A(x,t)\,dx-M\xi_\varepsilon(t)\delta_{x_\varepsilon(t)}\right\|_{C^{2*}}\leqslant\varepsilon\left[C\hat{\Omega}_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\right],\end{aligned}\tag{4.6}$$

which implies

$$\left|\gamma_\varepsilon(t)\right|\leqslant C\varepsilon^2(1+\varepsilon t)+C\varepsilon\int_0^t\hat{\Omega}_\varepsilon(\tau)\,d\tau,\tag{4.7}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $t\in[0,T_\varepsilon^*)$. Collecting the above inequalities, recalling the definition of $\hat{\Omega}_\varepsilon(t)$ and taking into account that, for $t<T_\varepsilon^*$, by the definition of T_ε^* it holds $\varepsilon t<\varepsilon T_\varepsilon^*\leqslant T_0$, we get

$$\hat{\Omega}_\varepsilon(t)\leqslant C\varepsilon^2(1+\varepsilon t)+C\varepsilon\int_0^t\hat{\Omega}_\varepsilon(\tau)\,d\tau\leqslant C\varepsilon^2+C\varepsilon\int_0^t\hat{\Omega}_\varepsilon(\tau)\,d\tau$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$. Hence, Gronwall Lemma yields

$$\hat{\Omega}_\varepsilon(t) \leqslant C\varepsilon^2 e^{\varepsilon t} \leqslant C\varepsilon^2,$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$, which gives the assertion. Finally, concerning the last assertion of the lemma, recalling again Lemma 3.5 and taking into account the definition of $\rho^A_\varepsilon(t)$, if $\|A\|_{C^2} < \delta$ for $\delta > 0$ small enough, we conclude the proof. □

5. Proof of Theorem 1.4 completed

5.1. First conclusion of Theorem 1.4

Let T_0 be as in Lemma 3.4 and ε_0, δ as in Lemma 3.5. By Lemma 4.1 and the definition (2.11) it follows that $T^*_\varepsilon = T_0/\varepsilon$, for all $\varepsilon \in (0, \varepsilon_0)$. Hence, $\Omega_\varepsilon(t) \leqslant C\varepsilon^2$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T_0/\varepsilon]$, in light of Lemma 4.1. Moreover, by Theorem 2.7 there exist functions $\theta^1_\varepsilon, \dots, \theta^m_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$ and $y_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^N$ such that

$$\phi^j_\varepsilon(x,t) = e^{\mathrm{i}(\xi_\varepsilon(t)\cdot x + \theta^j_\varepsilon(t) + A(\varepsilon x_\varepsilon(t))\cdot(x - x_\varepsilon(t)))} r_j\big(x - y_\varepsilon(t)\big) + \omega^j_\varepsilon(t),$$

where $\|\omega^j_\varepsilon(t)\|_{\mathbb{H}_\varepsilon} \leqslant C\sqrt{\Omega_\varepsilon(t)} + \mathcal{O}(\varepsilon)$, and hence, we have $\|\omega^j_\varepsilon(t)\|_{\mathbb{H}_\varepsilon} \leqslant \mathcal{O}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T_0/\varepsilon]$ and $j = 1, \dots, m$. Lemmas 3.4 and 4.1 also yield $|x_\varepsilon(t) - y_\varepsilon(t)| \leqslant \mathcal{O}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T_0/\varepsilon]$. Finally, using (**A**), (**V**) and (1.5), we get

$$\begin{aligned}
&\big\| e^{\mathrm{i}(\xi_\varepsilon(t)\cdot x + \theta^j_\varepsilon(t) + A(\varepsilon x_\varepsilon(t))\cdot(x - x_\varepsilon(t)))}\big(r_j\big(x - y_\varepsilon(t)\big) - r_j\big(x - x_\varepsilon(t)\big)\big)\big\|^2_{H^1} \\
&\quad\leqslant \int \big|\xi_\varepsilon(t) + A\big(\varepsilon x_\varepsilon(t)\big)\big|^2 \big|r_j\big(x - y_\varepsilon(t)\big) - r_j\big(x - x_\varepsilon(t)\big)\big|^2 dx + \int \big|\nabla r_j\big(x - y_\varepsilon(t)\big) - \nabla r_j\big(x - x_\varepsilon(t)\big)\big|^2 dx \\
&\qquad + \int \big|r_j\big(x - y_\varepsilon(t)\big) - r_j\big(x - x_\varepsilon(t)\big)\big|^2 dx \leqslant C\big|x_\varepsilon(t) - y_\varepsilon(t)\big|^2 \leqslant C\mathcal{O}\big(\varepsilon^2\big),
\end{aligned} \tag{5.1}$$

for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T_0/\varepsilon]$. Therefore, it follows that

$$\big\|\phi^j_\varepsilon(x,t) - e^{\mathrm{i}(\xi_\varepsilon(t)\cdot x + \theta^j_\varepsilon(t) + A(\varepsilon x_\varepsilon(t))\cdot(x - x_\varepsilon(t)))} r_j\big(x - x_\varepsilon(t)\big)\big\|^2_{H^1} \leqslant \mathcal{O}\big(\varepsilon^2\big), \tag{5.2}$$

for all $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T_0/\varepsilon]$ and $j = 1, \dots, m$. Hence, Theorem 1.4 holds true in $[0, T_0/\varepsilon]$. Now, let us take $x^\varepsilon_1 = x_\varepsilon(T_0/\varepsilon)$ and $\xi_1 = \xi_\varepsilon(T_0/\varepsilon)$ as new initial datum in system (D) and the functions

$$\phi^j_1(x) = r_j\big(x - x^\varepsilon_1\big) e^{\mathrm{i}[A(\varepsilon x^\varepsilon_1)\cdot(x - x^\varepsilon_1) + x\cdot\xi^\varepsilon_1]}, \quad x \in \mathbb{R}^N,\ j = 1, \dots, m,$$

as new initial datum for problem (P). Arguing as above, we can show that Theorem 1.4 holds true in $[T_0/\varepsilon, 2T_0/\varepsilon]$ and so, in any finite time interval $[0, T/\varepsilon]$, with $T > 0$. The proof of Theorem 1.4 is now complete under the assumption that $\|A\|_{C^2} < \delta$.

5.2. Second conclusion of Theorem 1.4

To prove the second part of Theorem 1.4, namely formula (1.9), we follow the argument of [23] (which is based upon the original paper by Bronski and Jerrard [4]). Let us give a brief sketch of the proof. Based upon the identity (see for instance [23, p. 2571]) holding for all $v \in H^1(\mathbb{R}^N)$

$$\left|\frac{\nabla v}{\mathrm{i}} - A(\varepsilon x)v\right|^2 = \frac{|p^{A(\varepsilon x)}(v)|^2}{|v|^2} + \big|\nabla|v|\big|^2, \quad p^A(v) := \Im\big(\bar{v}\big(\nabla v(x,t) - \mathrm{i}A(\varepsilon x)v(x,t)\big)\big),$$

the energy functional of the Schrödinger problem is rewritten as

$$E_\varepsilon(t) = E^{\mathrm{pot}}_\varepsilon(t) + E^{\mathrm{b}}_\varepsilon(t) + E^{\mathrm{k}}_\varepsilon(t) + E^{\mathrm{nl}}_\varepsilon(t),$$

where we have set

$$\begin{aligned}
E^{\mathrm{pot}}_\varepsilon(t) &:= \int V(\varepsilon x)\big|\phi_\varepsilon(x,t)\big|^2 dx, \\
E^{\mathrm{b}}_\varepsilon(t) &:= \frac{1}{2}\sum_{j=1}^m \int \big|\nabla\big|\phi^j_\varepsilon\big|(x,t)\big|^2 - \frac{1}{p+1}\sum_{j=1}^m \alpha_j \int \big|\phi^j_\varepsilon(x,t)\big|^{2p+2} dx \\
&\quad - \frac{1}{p+1}\sum_{i,j,\, i\neq j}^m \gamma_{ij}\int \big|\phi^i_\varepsilon(x,t)\big|^{p+1}\big|\phi^j_\varepsilon(x,t)\big|^{p+1} dx,
\end{aligned}$$

$$E^{\mathrm{k}}_\varepsilon(t) := \frac{1}{2}\sum_{j=1}^m \int \frac{|(p^{A(\varepsilon x)}(x,t))^j|^2}{|\phi^j_\varepsilon(x,t)|^2}\,dx,$$

$$E^{\mathrm{nl}}_\varepsilon(t) := -\frac{1}{2}\sum_{j=1}^m \beta_j \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi^j_\varepsilon(x,t)\big|^2\big|\phi^j_\varepsilon(y,t)\big|^2\,dx\,dy$$
$$-\frac{1}{2}\sum_{i,j,\,i\neq j}^m \omega_{ij} \iint \Phi\big(\varepsilon(x-y)\big)\big|\phi^i_\varepsilon(x,t)\big|^2\big|\phi^j_\varepsilon(y,t)\big|^2\,dx\,dy.$$

Notice that, with respect to our notations, we have $E^{\mathrm{b}}_\varepsilon(r_1,\dots,r_m) = \mathcal{E}(r_1,\dots,r_m)$ since r_i are real valued and positive functions. Moreover $E^{\mathrm{b}}_\varepsilon(|\psi^1_\varepsilon|,\dots,|\psi^m_\varepsilon|) = E^{\mathrm{b}}_\varepsilon(|\phi^1_\varepsilon|,\dots,|\phi^m_\varepsilon|) = \mathcal{E}(|\phi^1_\varepsilon|,\dots,|\phi^m_\varepsilon|)$. At this stage, keeping in mind that we possess Lemma 2.3, which expands the energy $E_\varepsilon(t)$ up to an error $\mathcal{O}(\varepsilon^2)$, by repeating the steps of the proof of [23, Lemma 3.5], it is readily seen that, as ε goes to zero,

$$0 \leqslant E^{\mathrm{b}}_\varepsilon\big(\big|\phi^1_\varepsilon\big|,\dots,\big|\phi^m_\varepsilon\big|\big) - E^{\mathrm{b}}_\varepsilon(r_1,\dots,r_m) \leqslant C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big).$$

This conclusion plays the role of Lemma 2.6 and, as a consequence, by the non-degeneracy/energy convexity property (applied with $U = (|\phi^1_\varepsilon|,\dots,|\phi^m_\varepsilon|)$, see e.g. [4, Proposition 1] for the scalar case), yields

$$\big\|\big(\big|\phi^1_\varepsilon\big|,\dots,\big|\phi^m_\varepsilon\big|\big) - \big(r_1\big(\cdot + y_\varepsilon(t)\big),\dots,r_m\big(\cdot + y_\varepsilon(t)\big)\big)\big\|^2_{H^1} \leqslant C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big), \tag{5.3}$$

for some $y_\varepsilon(t) \in \mathbb{R}^N$.

Moreover, again by the steps of the proof of [23, Lemma 3.5], we get

$$0 \leqslant E^{\mathrm{k}}_\varepsilon(t) - \frac{1}{2}\sum_{j=1}^m \frac{|\int (p^{A(\varepsilon x)}(x,t))^j|^2}{m_j} \leqslant C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big), \tag{5.4}$$

as ε goes to zero. To achieve this conclusion, one also needs to take into account the following elementary inequality (following from the standard Cauchy–Schwarz inequality)

$$\left|\int q^A_\varepsilon(x,t)\,dx\right|^2 \leqslant M\sum_{j=1}^m \frac{|\int (p^{A(\varepsilon x)}(x,t))^j\,dx|^2}{m_j}, \quad t \in \mathbb{R}^+.$$

Furthermore, for any $j = 1,\dots,m$ we have the inequality (see [23, inequality below formula (28)]; see also [4, formula (3.2)])

$$\frac{1}{2}\int\left|\frac{(p^{A(\varepsilon x)}(x,t))^j}{|\phi^j_\varepsilon(x)|} - \frac{(\int (p^{A(\varepsilon x)}(x,t))^j)}{m_j}\big|\phi^j_\varepsilon(x)\big|\right|^2 dx \leqslant \frac{1}{2}\int \frac{|(p^{A(\varepsilon x)}(x,t))^j|^2}{|\phi^j_\varepsilon(x)|^2}\,dx - \frac{1}{2}\frac{|\int (p^{A(\varepsilon x)}(x,t))^j|^2}{m_j}.$$

Summing over $j = 1,\dots,m$, we get

$$\frac{1}{2}\sum_{j=1}^m\int\left|\frac{(p^{A(\varepsilon x)}(x,t))^j}{|\phi^j_\varepsilon(x)|} - \frac{(\int (p^{A(\varepsilon x)}(x,t))^j)}{m_j}\big|\phi^j_\varepsilon(x)\big|\right|^2 dx \leqslant E^{\mathrm{k}}_\varepsilon(t) - \frac{1}{2}\sum_{j=1}^m \frac{|\int (p^{A(\varepsilon x)}(x,t))^j|^2}{m_j}.$$

In turn, in light of (5.4), we obtain

$$\int\left|\frac{(p^{A(\varepsilon x)}(x,t))^j}{|\phi^j_\varepsilon(x)|} - \frac{(\int (p^{A(\varepsilon x)}(x,t))^j)}{m_j}\big|\phi^j_\varepsilon(x)\big|\right|^2 dx \leqslant C\hat{\Omega}_\varepsilon(t) + \mathcal{O}\big(\varepsilon^2\big), \tag{5.5}$$

as ε goes to zero, for any $j = 1,\dots,m$. Inequalities (5.3) and (5.5) are precisely what is needed in order to prove (3.7) of Lemma 3.5 (see the proof of Lemma 6.1 in [25], in particular formula (6.5) therein; see also the proof of Lemma 4.3 in [21]). Once inequality (3.7) of Lemma 3.5 holds true the rest of the proof continues as before, yielding the assertion from inequality (5.3).

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