Soft constraint abstraction based on semiring homomorphism

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ABSTRACT

The semiring-based constraint satisfaction problems (semiring CSPs), proposed by Bistarelli, Montanari and Rossi [S. Bistarelli, U. Montanari, F. Rossi, Semiring-based constraints solving and optimization, Journal of the ACM 44 (2) (1997) 201–236], is a very general framework of soft constraints. In this paper we propose an abstraction scheme for soft constraints that uses semiring homomorphism. To find optimal solutions of the concrete problem, we first work in the abstract problem and find its optimal solutions, and then use them to solve the concrete problem.

In particular, we show that a mapping preserves optimal solutions if and only if it is an order-reflecting semiring homomorphism. Moreover, for a semiring homomorphism α and a problem P over S, if t is optimal in $\alpha(P)$, then there is an optimal solution $\bar{t}$ of P such that $\bar{t}$ has the same value as t in $\alpha(P)$.

1. Introduction

There has been a growing interest in soft constraint satisfaction in the recent years. Various extensions of the classical constraint satisfaction problems (CSPs) [9,8] have been introduced in the literature, e.g. Fuzzy CSP [10,4,11], Probabilistic CSP [5], Weighted CSP [14,6], Possibilistic CSP [12], and Valued CSP [13]. Roughly speaking, these extensions are just like classical CSPs except that each assignment of values to variables in the constraints is associated to an element taken from a semiring. Furthermore, nearly all of these extensions, as well as classical CSPs, can be cast by the semiring-based constraint solving framework, called SCSP (for *Semiring CSP*), proposed by Bistarelli, Montanari and Rossi [2].

Compared with classical CSPs, SCSPs are usually more difficult to process and to solve. This is mainly caused by the complexity of the underlying semiring structure. Thus, working on a simplified version of the given problem would be worthwhile. Given a concrete SCSP, the idea is to get an abstract one by changing the semiring values of the constraints without changing the structure of the problem. Once the abstracted version of a given problem is available, one can first process the abstracted version and then bring back the information obtained to the original problem. The main objective is to find an optimal solution, or a reasonable estimation of it, for the original problem.

The translation from a concrete problem to its abstracted version is established via a mapping between the two semirings. More concretely, suppose P is an SCSP over S, and $\widetilde{S}$ is another semiring (possibly simpler than S). Given a mapping $\alpha : S \to \widetilde{S}$, we can translate the concrete problem P to another problem, $\alpha(P)$, over $\widetilde{S}$ in a natural way. We then ask when is an optimal solution of the concrete problem P also optimal in the abstract problem $\alpha(P)$? and, given an optimal solution of $\alpha(P)$, when and how can we find a reasonable estimation for an optimal solution of P?

The answers to these questions will be helpful in deriving useful information on the abstract problem and then taking some useful information back to the concrete problem.

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These questions were first studied in Bistarelli, Codognet and Rossi [1], where they established a Galois insertion-based abstraction framework for soft constraint problems. In particular, they showed that [1, Theorem 27] if α is an *order-preserving* Galois insertion, then optimal solutions of the concrete problem are also optimal in the abstract problem. Moreover, they also described methods for computing bounds that approximate an optimal solution of the concrete problem [1, Theorem 29].

Following the intuition of Bistarelli et al. [1], this paper provides a more general abstraction framework for soft constraint problems, where the notion of semiring homomorphism plays an important role. More precisely, we show that (Theorem 4.1) a mapping preserves optimal solutions if and only if it is an order-reflecting semiring homomorphism, where a mapping $\alpha : S \rightarrow \widetilde{S}$ is *order-reflecting* if for any two $a, b \in S$, we have $a <_S b$ from $\alpha(a) <_{\widetilde{S}} \alpha(b)$. Moreover, for a semiring homomorphism α and a problem P over S, if t is optimal in $\alpha(P)$, then there is an optimal solution $\bar{t}$ of P such that $\bar{t}$ has the same value as t in $\alpha(P)$ (see Theorem 5.1).

This paper is organized as follows. In Section 2 we give a summary of the theory of soft constraints. The notion of α-*translation* of semiring CSPs is introduced in Section 3, where we show that α preserves problem ordering if and only if α is a semiring homomorphism. Section 4 discusses when a translation α preserves optimal solutions, i.e. when all optimal solutions of the concrete problem are also optimal in the abstract problem. In Section 5, we discuss, given an optimal solution of the abstract problem, what we can say about optimal solutions of the concrete problem. Section 6 discusses related works and the last section concludes the paper.

2. Semiring constraint satisfaction problem

In this section we introduce several basic notions used in this paper. In particular, we give a brief summary of the theory of c-semiring based constraint satisfaction problem raised in [2] (Bistarelli, Montanari and Rossi 1997). The notion of semiring homomorphism is also introduced.

2.1. c-semirings

Definition 2.1 (*Semirings and c-Semirings [1]*). A semiring is a tuple $S = \langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$ such that:

1. S is a set and $\mathbf{0}, \mathbf{1} \in S$;
2. $+$ is commutative, associative and $\mathbf{0}$ is its unit element;
3. $\times$ is associative, distributive over $+$, $\mathbf{1}$ is its unit element and $\mathbf{0}$ is its absorbing element.

We call $+$ and $\times$, the *sum* and the *product* operation, respectively. A c-semiring is a semiring $\langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$ such that:

4. $+$ is idempotent, $\mathbf{1}$ is its absorbing element, and $\times$ is commutative.

Consider the relation $\leq_S$ defined over S such that $a \leq_S b$ iff $a + b = b$. Then it is possible to prove that [2]:

- $\langle S, \leq_S \rangle$ is a lattice, $\mathbf{0}$ is its bottom and $\mathbf{1}$ its top;
- $+$ is the lub (lowest upper bound) operator $\vee$ in the lattice $\langle S, \leq_S \rangle$;
- $\times$ is monotonic on $\leq_S$;
- If $\times$ is idempotent, that is $a \times a = a$ for each $a \in S$, then $\langle S, \leq_S \rangle$ is a distributive lattice and $\times$ is its glb (greatest lower bound) $\wedge$.

Remark 2.1. The above definition of c-semiring differs from the one given in [2] simply in that a c-semiring, with the induced partial order, is not necessarily complete. For example, suppose $\mathbb{Q}$ is the set of rational number and $S = [0, 1] \cap \mathbb{Q}$ is the subalgebra of the fuzzy semiring $S_{FCSP} = \langle [0, 1], \vee, \wedge, 0, 1 \rangle$. Then S is a c-semiring but $\langle S, \leq_S \rangle$ is not a complete lattice, where $\leq_S$ is the partial order induced by the semiring S, which happens to be the usual total order on S.

2.2. Semiring homomorphism

Definition 2.2 (*Homomorphism*). A mapping ψ from semiring $\langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$ to semiring $\langle \widetilde{S}, \widetilde{+}, \widetilde{\times}, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \rangle$ is said to be a semiring *homomorphism* if for any $a, b \in S$

- $\psi(\mathbf{0}) = \widetilde{\mathbf{0}}, \psi(\mathbf{1}) = \widetilde{\mathbf{1}}$; and
- $\psi(a + b) = \psi(a) \widetilde{+} \psi(b)$; and
- $\psi(a \times b) = \psi(a) \widetilde{\times} \psi(b)$.

A semiring homomorphism ψ is said to be a semiring *isomorphism* if ψ is a bijection. Note that a semiring isomorphism is also an order isomorphism w.r.t. the induced partial orders.

We give some examples of semiring homomorphism.

Example 2.1. Let S and $\widetilde{S}$ be two c-semirings such that

   (i) both $\leq_S$ and $\leq_{\widetilde{S}}$ are totally ordered;
  (ii) both $\times$ and $\widetilde{\times}$ are idempotent, i.e. both are glb operators.

Then a monotonic mapping $\alpha : S \to \widetilde{S}$ is a homomorphism if and only if $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, and $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$.

Recall that a congruence relation $\sim$ over a semiring S is an equivalence relation that satisfies:

if $a \sim a'$ and $b \sim b'$, then $a + b \sim a' + b'$, and $a \times b \sim a' \times b'$.

We write $S/\sim$ for the resulted quotient structure.

Example 2.2 (*Natural Homomorphism*). Suppose S is a (c-)semiring and $\sim$ is a congruence relation over S. Then $S/\sim$ is also a (c-)semiring and the natural homomorphism $\nu : S \to S/\sim$ is a semiring homomorphism.

Example 2.3 (*Projection*). Let $S = \prod_{j \in J} S_j$ be the Cartesian product of a set of (c-)semirings. Clearly, S itself is also a (c-)semiring. For each $j \in J$, the j-th projection $p_j : S \to S_j$ is a semiring homomorphism.

2.3. Soft constraints

Definition 2.3 (*Constraint System [2]*). A *constraint system* is a tuple $CS = \langle S, D, V \rangle$, where S is a c-semiring, D is a finite set, and V is an (possibly infinite) ordered set of variables.

Definition 2.4 (*Type*). Given a constraint system $CS = \langle S, D, V \rangle$. A *type* is a finite ordered subset of V. We write $\mathfrak{T} = \{\tau \subseteq V : \tau \text{ is finite}\}$ for the set of types.

Definition 2.5 (*Constraints [2]*). Given a constraint system $CS = \langle S, D, V \rangle$, where $S = \langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$, a constraint over CS is a pair $\langle \mathrm{def}, \mathrm{con} \rangle$ where

- con is a finite subset of V, called the *type* of the constraint;
- $\mathrm{def} : D^k \to S$ is called the *value* of the constraint, where $k = |\mathrm{con}|$ is the cardinality of con.

In the above definition, if $\mathrm{def} : D^k \to S$ is the maximal constant function, namely $\mathrm{def}(t) = \mathbf{1}$ for each k-tuple t, we call $\langle \mathrm{def}, \mathrm{con} \rangle$ the *trivial* constraint with type con.

Definition 2.6 (*Constraint Ordering [2]*). For two constraints $c_1 = \langle \mathrm{def}_1, \mathrm{con} \rangle$ and $c_2 = \langle \mathrm{def}_2, \mathrm{con} \rangle$ with type con over $CS = \langle S, D, V \rangle$, we say c_1 is *constraint below* c_2, noted as $c_1 \sqsubseteq_S c_2$, if for all $|\mathrm{con}|$-tuples t, $\mathrm{def}_1(t) \leq_S \mathrm{def}_2(t)$.

This relation can be extended to sets of constraints in an obvious way. Given two (possibly infinite) sets of constraints C_1 and C_2, assuming that both contain no two constraints of the same type, we say C_1 is *constraint below* C_2, noted as $C_1 \sqsubseteq_S C_2$, if for each type con $\subseteq V$ one of the following two conditions holds:

(1) There exist two constraints c_1 and c_2 with type con in C_1 and C_2 respectively, such that $c_1 \sqsubseteq_S c_2$;
(2) C_2 contains no constraints of type con, or C_2 contains the trivial constraint of type con.

Two sets of constraints C_1 and C_2 are called (*constraint*) *equal*, if $C_1 \sqsubseteq_S C_2$ and $C_2 \sqsubseteq_S C_1$. In this case, we write $C_1 = C_2$. This definition is in accordance with the basic requirement that adding to a set of constraints C a trivial constraint should not change the meaning of C.

Definition 2.7 (*Soft Constraint Problem [2]*). Given a constraint system $CS = \langle S, D, V \rangle$, a soft constraint satisfaction problem (SCSP) over CS is a pair $\langle C, \mathrm{con} \rangle$, where C is a finite set of constraints, and con, the type of the problem, is a finite subset of V. We assume that no two constraints with the same type appear in C.

Naturally, given two SCSPs $P_1 = \langle C_1, \mathrm{con} \rangle$ and $P_2 = \langle C_2, \mathrm{con} \rangle$, we say P_1 is *constraint below* P_2, noted as $P_1 \sqsubseteq_S P_2$, if $C_1 \sqsubseteq_S C_2$. Also, P_1 and P_2 are said to be (*constraint*) *equal*, if C_1 and C_2 are constraint equal. In this case, we also write $P_1 = P_2$. We call this the *constraint ordering* on sets of SCSPs with type con over CS. Clearly, two SCSPs are constraint equal if and only if they differ only in trivial constraints.

To give a formal description of the solution of an SCSP, we need two additional concepts.

Definition 2.8 (*Combination [2]*). Given a finite set of constraints $C = \{\langle \mathrm{def}_i, \mathrm{con}_i \rangle : i = 1, \ldots, n\}$, their *combination* $\bigotimes C$ is the constraint $\langle \mathrm{def}, \mathrm{con} \rangle$ defined by $\mathrm{con} = \bigcup_{i=1}^{n} \mathrm{con}_i$ and $\mathrm{def}(t) = \prod_{i=1}^{n} \mathrm{def}_i(t|_{\mathrm{con}_i}^{\mathrm{con}})$, where by $t|_Y^X$ we mean the projection of tuple t, which is defined over the set of variables X, over the set of variables $Y \subseteq X$.

Definition 2.9 (*Projection [2]*). Given a constraint $c = \langle \mathrm{def}, \mathrm{con} \rangle$ and a subset I of V, the *projection* of c over I, denoted by $c \Downarrow_I$, is the constraint $\langle \mathrm{def}', \mathrm{con}' \rangle$ where $\mathrm{con}' = \mathrm{con} \cap I$ and $\mathrm{def}'(t') = \sum \{\mathrm{def}(t) : t|_{\mathrm{con} \cap I}^{\mathrm{con}} = t'\}$. Particularly, if $I = \varnothing$, then $c \Downarrow_\varnothing : \{\varepsilon\} \to S$ maps 0-tuple ε to $\sum \{\mathrm{def}(t) : t \text{ is a tuple with type con}\}$, which is the sum of the values associated to all $|\mathrm{con}|$-tuples.

Now the concept of solution can be defined as the projection of the combination of all constraints over the type of the problem.

Definition 2.10 (*Solution and Optimal Solution*). The *solution* of an SCSP $P = \langle C, \text{con} \rangle$ is a constraint of type con which is defined as:

$$Sol(P) = \left(c^* \times \bigotimes C \right) \Downarrow_{\text{con}} \tag{1}$$

where c^* is the maximal constraint with type con.

Write $Sol(P) = \langle \text{def}, \text{con} \rangle$, a $|\text{con}|$-tuple t is an *optimal solution* of P if $\text{def}(t)$ is maximal, that is to say there is no t' such that $\text{def}(t') >_S \text{def}(t)$. We write $Opt(P)$ for the set of optimal solutions of P. For any $|\text{con}|$-tuple t, we also write $Sol(P)(t)$ for $\text{def}(t)$.

3. Translation and semiring homomorphism

Let $S = \langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$ and $\widetilde{S} = \langle \widetilde{S}, \widetilde{+}, \widetilde{\times}, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \rangle$ be two c-semirings and let $\alpha : S \to \widetilde{S}$ be an arbitrary mapping from S to $\widetilde{S}$. Also let D be a nonempty finite set and let V be an ordered set of variables. Fix a type $\text{con} \subseteq V$. We now investigate the relation between problems over S and those over $\widetilde{S}$.

Definition 3.1 (*Translation*). Let $P = \langle C, \text{con} \rangle$ be an SCSP over S where $C = \{c_0, \ldots, c_n\}$, $c_i = \langle \text{def}_i, \text{con}_i \rangle$, and $\text{def}_i : D^{|\text{con}_i|} \to S$. By applying α to each constraints respectively, we get an SCSP $\langle \widetilde{C}, \text{con} \rangle$ over $\widetilde{S}$, called the *α-translated problem* of P, which is defined by $\widetilde{C} = \{\widetilde{c_1}, \ldots, \widetilde{c_n}\}$, $\widetilde{c_i} = \langle \widetilde{\text{def}_i}, \text{con}_i \rangle$, and $\widetilde{\text{def}_i} = \alpha \circ \text{def}_i : D^{|\text{con}_i|} \to \widetilde{S}$.

$$
\begin{array}{ccc}
D^{|\text{con}_i|} & \xrightarrow{\;\mathbf{def}_i\;} & S \\
\widetilde{\mathbf{def}_i} \downarrow & & \downarrow \alpha \\
\widetilde{S} & =\!=\!=\!= & \widetilde{S}
\end{array}
$$

We write $\alpha(P)$ for the α-translated problem of P.

Without loss of generality, in what follows we assume $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, and $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. We say α *preserves problem ordering*, if for any two SCSPs P, Q over S, we have

$$Sol(P) \sqsubseteq_S Sol(Q) \Rightarrow Sol(\alpha(P)) \sqsubseteq_{\widetilde{S}} Sol(\alpha(Q)). \tag{2}$$

The following theorem then characterizes when α preserves problem ordering.

Theorem 3.1. *Let α be a mapping from c-semiring S to c-semiring $\widetilde{S}$ such that $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. Suppose D contains more than two elements and $k = |\text{con}| > 0$. Then α preserves problem ordering if and only if α is a semiring homomorphism, that is, for all $a, b \in S$, $\alpha(a \times b) = \alpha(a) \widetilde{\times} \alpha(b)$, $\alpha(a + b) = \alpha(a) \widetilde{+} \alpha(b)$.*

Proof. Note that if α preserves $+$ and $\times$, then α commutes with operators $\prod$ and $\sum$. Clearly α is also monotonic. Hence, by definition of solution, α preserves problem ordering.

On the other hand, suppose α preserves problem ordering. We first prove $\alpha(a + b) = \alpha(a) \widetilde{+} \alpha(b)$ for $a, b \in S$. We show this by construction.

Soft Problem 1. Suppose $\text{con} = \{y_1, y_2, \ldots, y_k\}$. Take $c_i = \langle \text{def}_i, \text{con}_i \rangle$ with $\text{con}_i = \{x_1, x_2\}$ ($i = 1, 2$), where $x_2 \in \text{con}, x_1 \notin \text{con}$ and

$$
\begin{array}{llll}
\text{def}_1 : D^2 \to S & (x_1, x_2) & \mapsto & a \text{ if } x_1 = x_2, \\
& (x_1, x_2) & \mapsto & b \text{ if } x_1 \neq x_2, \\
\text{def}_2 : D^2 \to S & (x_1, x_2) & \mapsto & a + b.
\end{array}
$$

Set $P = \langle \{c_1\}, \text{con} \rangle$ and $Q = \langle \{c_2\}, \text{con} \rangle$. Then for each k-tuple $(y_1, \ldots, y_k)$, $Sol(P)(y_1, \ldots, y_k) = a + b = Sol(Q)(y_1, \ldots, y_k)$. By the assumption that α preserves problem ordering, we have

$$\alpha(a) \widetilde{+} \alpha(b) = Sol(\widetilde{P})(y_1, \ldots, y_k) = Sol(\widetilde{Q})(y_1, \ldots, y_k) = \alpha(a + b).$$

Next, we prove $\alpha(a \times b) = \alpha(a) \widetilde{\times} \alpha(b)$ for $a, b \in S$. We also show this by construction.

Soft Problem 2. Suppose $\text{con} = \{y_1, y_2, \ldots, y_k\}$. Take $c_1 = \langle \text{def}_1, \{x\} \rangle$, $c_2 = \langle \text{def}_2, \text{con} \rangle$ and $c_3 = \langle \text{def}_3, \text{con} \rangle$, where $x \notin \text{con}$ and

$$
\begin{array}{llll}
\text{def}_1 : D \to S & x & \mapsto & a, \\
\text{def}_2 : D^k \to S & (y_1, \ldots, y_k) & \mapsto & b, \\
\text{def}_3 : D^k \to S & (y_1, \ldots, y_k) & \mapsto & a \times b.
\end{array}
$$

Set $P = \langle \{c_1, c_2\}, \text{con} \rangle$ and $Q = \langle \{c_3\}, \text{con} \rangle$. Then for each k-tuple $(y_1, \ldots, y_k)$, $Sol(P)(y_1, \ldots, y_k) = a \times b = Sol(Q)(y_1, \ldots, y_k)$. By assumption, we have

$$\alpha(a) \widetilde{\times} \alpha(b) = Sol(\widetilde{P})(y_1, \ldots, y_k) = Sol(\widetilde{Q})(y_1, \ldots, y_k) = \alpha(a \times b).$$

This ends the proof. □

Thus if α is a semiring homomorphism, it preserves problem ordering. Note that semiring homomorphism also preserves constraint ordering, i.e. for any two SCSPs P, Q over S, we have

$$P \sqsubseteq_S Q \Rightarrow \alpha(P) \sqsubseteq_{\widetilde{S}} \alpha(Q). \tag{3}$$

4. Mappings preserving optimal solutions

In this section we discuss when a translation preserves optimal solutions, i.e. when all optimal solutions of the concrete problem are also optimal in the abstract problem.

Definition 4.1. Let $\alpha : S \to \widetilde{S}$ be a mapping between two c-semirings. We say α preserves optimal solutions if $Opt(P) \subseteq Opt(\alpha(P))$ holds for any SCSP P over S.

The following order-reflecting property plays a key role.

Definition 4.2. Let $(\mathcal{C}, \sqsubseteq)$ and $(\mathcal{A}, \leq)$ be two posets. A mapping $\alpha : \mathcal{C} \to \mathcal{A}$ is said to be *order-reflecting* if

$$(\forall a, b \in \mathcal{C}) \quad \alpha(a) < \alpha(b) \Rightarrow a \sqsubset b. \tag{4}$$

In the remainder of this section we show that α preserves optimal solutions if and only if α is an order-reflecting semiring homomorphism. To this end, we need several lemmas.

Recall that $+$ is idempotent and monotonic on $\leq_S$ for any c-semiring $S = \langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$. The following lemma then identifies a necessary and sufficient condition for α preserving optimal solutions.

Lemma 4.1. *Let α be a mapping from c-semiring S to c-semiring $\widetilde{S}$ such that $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. Then α preserves optimal solutions for all constraint systems if and only if the following condition holds for any two positive integers m, n:*

$$\widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}) <_{\widetilde{S}} \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij}) \Rightarrow \sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij} <_S \sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij}. \tag{5}$$

Proof. Suppose that α satisfies the above Eq. (5). Given an SCSP $P = \langle C, \text{con} \rangle$ over S with $C = \{c_i\}_{i=1}^{m}$ and $c_i = \langle \text{def}_i, \text{con}_i \rangle$. Take a tuple t that is optimal in P. We now show t is also optimal in $\alpha(P)$.

Set $\text{CON} = \text{con} \cup \bigcup_{k=1}^{m} \text{con}_k$. Take $T(t) = \{t' : t'|_{\text{con}}^{\text{CON}} = t\}$. Set $n = |T(t)|$ and write $T(t) = \{t_i : 1 \leq i \leq n\}$. For each $1 \leq i \leq n$ and each $1 \leq j \leq m$, set $u_{ij} = c_j(t_i|_{\text{con}_j}^{\text{CON}})$. Then

$$u = Sol(P)(t) = \sum_{t_i \in T(t)} \prod_{j=1}^{m} c_j(t_i|_{\text{con}_j}^{\text{CON}}) = \sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij},$$

and

$$\widetilde{u} = Sol(\alpha(P))(t) = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}).$$

Suppose t is not optimal in $\alpha(P)$. Then there exists some $\bar{t}$ that has value $\widetilde{v} >_{\widetilde{S}} \widetilde{u}$ in $\alpha(P)$. Notice that $T(\bar{t}) = \{t' : t'|_{\text{con}}^{\text{CON}} = \bar{t}\}$ also has $n = |T(t)|$ elements. Similarly we can write

$$v = \sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij}$$

for the value of $\bar{t}$ in P. Now since

$$\widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}) = \widetilde{u} <_{\widetilde{S}} \widetilde{v} = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij}),$$

entreating Eq. (5), we have $u <_S v$. This contradicts the assumption that t is optimal in P with value u.

On the other hand, suppose that α preserves optimal solutions. By contradiction, suppose Eq. (5) does not hold. That is, we have some $u = \sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij}$ and $v = \sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij}$ such that

$$u \not<_S v, \quad \widetilde{u} = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}) <_{\widetilde{S}} \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij}) = \widetilde{v}.$$

Our next example shows that this is impossible.

Soft Problem 3. Take $D = \{d_1, d_2, \ldots, d_n\}$, $V = \{x_0, x_1, \ldots, x_n\}$, and $\text{con} = \{x_0\}$. For $1 \leq j \leq m$, set $\text{con}_j = V - \{x_j\}$, and define $\text{def}_j : D^n \to S$ as follows:

$$\text{def}_j(x_0, y_2, \ldots, y_n) = \begin{cases} u_{ij}, & \text{if } x_0 = d_1 \text{ and } y_2 = \cdots = y_n = d_i, \\ v_{ij}, & \text{if } x_0 = d_2 \text{ and } y_2 = \cdots = y_n = d_i, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Set $C = \{\langle \text{def}_j, \text{con}_j \rangle\}_{j=1}^{m}$. Consider now the SCSP $P = \langle C, \text{con} \rangle$. Then the two 1-tuples $t = (d_1)$ and $t' = (d_2)$ have values $u = \sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij}$ and $v = \sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij}$ respectively in P. Applying α to P, we have an SCSP $\alpha(P)$ over $\widetilde{S}$. Recall $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$. In the new problem, t and t' have values $\widetilde{u} = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij})$ and $\widetilde{v} = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij})$ respectively. Since t is an optimal solution of P, by the assumption that α preserves optimal solutions, t is also an optimal solution of $\alpha(P)$. Recall that $Sol(\alpha(P))(t) = \widetilde{u} <_{\widetilde{S}} \widetilde{v} = Sol(\alpha(P))(t')$. t cannot be optimal in $\alpha(P)$. This is a contradiction.

As a result, α preserves optimal solutions only if it satisfies Eq. (5). $\quad \square$

It is easy to show that if α preserves optimal solutions, then α is order-reflecting.

Lemma 4.2. *Let α be a mapping from c-semiring S to c-semiring $\widetilde{S}$ such that $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. Suppose $\alpha : S \to \widetilde{S}$ preserves optimal solutions. Then α is order-reflecting, that is, for all $u, v \in S$, $\alpha(u) <_{\widetilde{S}} \alpha(v)$ holds only if $u <_S v$.*

Proof. By Lemma 4.1, we know α satisfies Eq. (5) of Lemma 4.1. Taking $m = n = 1$, we know α is order-reflecting. $\square$

The next lemma shows that α preserves optimal solutions only if it is a semiring homomorphism.

Lemma 4.3. *Let α be a mapping from c-semiring S to c-semiring $\widetilde{S}$ such that $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. Suppose $\alpha : S \to \widetilde{S}$ preserves optimal solutions. Then α is a semiring homomorphism.*

Proof. By Lemma 4.1, we know α satisfies Eq. (5). We first show that α is monotonic. Take $u, v \in S$, $u \leq_S v$. Suppose $\alpha(u) \not\leq_{\widetilde{S}} \alpha(v)$. Then $\alpha(v) \widetilde{+} \alpha(v) = \alpha(v) <_{\widetilde{S}} \alpha(u) \widetilde{+} \alpha(v)$. By Eq. (5), we have $v = v + v <_S u + v = v$. This is a contradiction, hence we have $\alpha(u) \leq_{\widetilde{S}} \alpha(v)$.

Next, for any $u, v \in S$, we show $\alpha(u + v) = \alpha(u) \widetilde{+} \alpha(v)$. Since α is monotonic, we have $\alpha(u + v) \geq_{\widetilde{S}} \alpha(u) \widetilde{+} \alpha(v)$. Suppose $\alpha(u + v) \widetilde{+} \alpha(u + v) = \alpha(u + v) >_{\widetilde{S}} \alpha(u) \widetilde{+} \alpha(v)$. By Eq. (5) again, we have $(u + v) + (u + v) >_S u + v$, also a contradiction.

Finally, for $u, v \in S$, we show $\alpha(u \times v) = \alpha(u) \widetilde{\times} \alpha(v)$. Suppose not and set $w = \alpha(u) \widetilde{\times} \alpha(v) \widetilde{+} \alpha(u \times v)$. Then we have either $\alpha(u) \widetilde{\times} \alpha(v) <_{\widetilde{S}} w$ or $\alpha(u \times v) <_{\widetilde{S}} w$. Since $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}, \alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$, these two inequalities can be rewritten respectively as

$$\alpha(u) \widetilde{\times} \alpha(v) + \alpha(\mathbf{1}) \widetilde{\times} \alpha(\mathbf{0}) <_{\widetilde{S}} \alpha(u) \widetilde{\times} \alpha(v) \widetilde{+} \alpha(u \times v) \widetilde{\times} \alpha(\widetilde{\mathbf{1}})$$

and

$$\alpha(\mathbf{1}) \widetilde{\times} \alpha(\mathbf{0}) + \alpha(u \times v) \widetilde{\times} \alpha(\mathbf{1}) <_{\widetilde{S}} \alpha(u) \widetilde{\times} \alpha(v) \widetilde{+} \alpha(u \times v) \widetilde{\times} \alpha(\widetilde{\mathbf{1}}).$$

By Eq. (5) again, we have either $u \times v + \mathbf{1} \times \mathbf{0} <_S u \times v + (u \times v) \times \mathbf{1}$ or $\mathbf{1} \times \mathbf{0} + (u \times v) \times \mathbf{1} <_S u \times v + (u \times v) \times \mathbf{1}$. Both give rise to a contradiction. This ends the proof. $\square$

We now achieve our main result:

Theorem 4.1. *Let α be a mapping from c-semiring S to c-semiring $\widetilde{S}$ such that $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$. Then α preserves optimal solutions for all constraint systems if and only if α is an order-reflecting semiring homomorphism.*

Proof. The necessity part of the theorem follows from Lemmas 4.2 and 4.3. As for the sufficiency part, we need only to show that, if α is an order-reflecting semiring homomorphism, then α satisfies Eq. (5). Suppose

$$\widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}) <_{\widetilde{S}} \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij}).$$

Clearly we have

$$\alpha \left(\sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij} \right) = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(u_{ij}) <_{\widetilde{S}} \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{m} \alpha(v_{ij}) = \alpha \left(\sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij} \right)$$

since α commutes with $\sum$ and $\prod$. By order-reflecting, we have immediately

$$\sum_{i=1}^{n} \prod_{j=1}^{m} u_{ij} <_S \sum_{i=1}^{n} \prod_{j=1}^{m} v_{ij}.$$

This ends the proof. $\square$

5. Computing concrete optimal solutions from abstract ones

In the above section, we investigated conditions under which *all* optimal solutions of concrete problem can be related *precisely* to those of abstract problem. There are often situations where it suffices to find *some* optimal solutions or simply a good approximation of the concrete optimal solutions. This section shows that, even without the order-reflecting condition, semiring homomorphism can be used to find some optimal solutions of concrete problem using abstract ones.

Theorem 5.1. *Let $\alpha : S \to \widetilde{S}$ be a semiring homomorphism. Given an SCSP P over S, suppose $t \in Opt(\alpha(P))$ has value v in P and value $\widetilde{v}$ in $\alpha(P)$. Then there exists $\bar{t} \in Opt(P) \cap Opt(\alpha(P))$ with value $\bar{v} \geq_S v$ in P and value $\widetilde{v}$ in $\alpha(P)$. Moreover, we have $\alpha(\bar{v}) = \alpha(v) = \widetilde{v}$.*

Proof. Suppose $P = \langle C, \text{con} \rangle$, $C = \{c_i\}_{i=1}^m$ and $c_i = \langle \text{def}_i, \text{con}_i \rangle$. Set $\text{CON} = \text{con} \cup \bigcup \{\text{con}_j\}_{j=1}^m$ and $k = |\text{CON}|$. Suppose t is an optimal solution of $\alpha(P)$, with semiring value $\widetilde{v}$ in $\alpha(P)$ and v in P. By definition of solution, we have

$$v = Sol(P)(t) = \sum_{t'|_{\text{con}}^{\text{CON}} = t} \prod_{j=1}^m \text{def}_j(t'|_{\text{con}_j}).$$

Denote

$$T(t) = \{t' : t' \text{ is a } |k|\text{-tuple with } t'|_{\text{con}}^{\text{CON}} = t\}.$$

Set $n = |T(t)|$, and write $T = \{t_1, \ldots, t_n\}$. For each $1 \leq i \leq n$ and each $1 \leq j \leq m$, set $v_{ij} = \text{def}_j(t_i|_{\text{con}_j})$. Then

$$v = \sum_{i=1}^n \prod_{j=1}^m v_{ij}, \qquad \widetilde{v} = \widetilde{\sum}_{i=1}^n \widetilde{\prod}_{j=1}^m \alpha(v_{ij}).$$

Since α preserves sums and products, we have

$$\alpha(v) = \alpha\left(\sum_{i=1}^n \prod_{j=1}^m v_{ij} \right) = \widetilde{\sum}_{i=1}^n \alpha\left(\prod_{j=1}^m v_{ij} \right) = \widetilde{\sum}_{i=1}^n \widetilde{\prod}_{j=1}^m \alpha(v_{ij}) = \widetilde{v}.$$

Notice that if t is also optimal in P, then we can choose $\bar{t} = t$. Suppose t is not optimal in P. Then there is a tuple $\bar{t}$ that is optimal in P, say with value $\bar{v} >_S v$. Denote

$$T(\bar{t}) = \{t' : t' \text{ is a } |k|\text{-tuple with } t'|_{\text{con}}^{\text{CON}} = \bar{t}\}.$$

Clearly $|T(\bar{t})| = |T(t)| = n$. Write $T(\bar{t}) = \{\bar{t}_1, \ldots, \bar{t}_n\}$. For each $1 \leq i \leq n$ and each $1 \leq j \leq m$, set $u_{ij} = \text{def}_j(\bar{t}_i|_{\text{con}_j})$. Then

$$\bar{v} = \sum_{i=1}^n \prod_{j=1}^m u_{ij}.$$

Now we show $\alpha(\bar{v}) \leq_{\widetilde{S}} \widetilde{v}$.

By $v <_S \bar{v}$, we have $\alpha(v) \leq_{\widetilde{S}} \alpha(\bar{v})$. Then

$$\begin{aligned}
\widetilde{v} &= \widetilde{\sum}_{i=1}^n \widetilde{\prod}_{j=1}^m \alpha(v_{ij}) \\
&= \alpha\left(\sum_{i=1}^n \prod_{j=1}^m v_{ij} \right) \\
&= \alpha(v) \leq_{\widetilde{S}} \alpha(\bar{v}) = \alpha\left(\sum_{i=1}^n \prod_{j=1}^m u_{ij} \right) = \widetilde{\sum}_{i=1}^n \widetilde{\prod}_{j=1}^m \alpha(u_{ij}) = \widetilde{\bar{v}}
\end{aligned}$$

where the last term, $\widetilde{\bar{v}}$, is the value of $\bar{t}$ in $\alpha(P)$. Now since t is optimal in $\alpha(P)$, we have $\widetilde{v} = \alpha(v) = \alpha(\bar{v}) = \widetilde{\bar{v}}$. That is, $\bar{t}$ is also optimal in $\alpha(P)$ with value $\widetilde{v}$. □

Remark 5.1. To find some (as opposed to all) optimal solutions of the concrete problem P, by Theorem 5.1 we could first find all optimal solutions of the abstract problem $\alpha(P)$, and then compute their values in P, tuples that have maximal values in P are optimal solutions of P. In this sense, this theorem is more desirable than Theorem 4.1 because we do not need the assumption that α is order-reflecting.

Theorem 5.1 can also be applied to find good approximations of the optimal solutions of P. Given an optimal solution $t \in Opt(\alpha(P))$ with value $\widetilde{v} \in \widetilde{S}$, then by Theorem 5.1 there is an optimal solution $\bar{t} \in Opt(P)$ with value in the set $\{u \in S : \alpha(u) = \widetilde{v}\}$.

Note that Theorem 5.1 requires α to be a semiring homomorphism. This condition is still a little restrictive. Take the probabilistic semiring $S_{prop} = \langle [0, 1], \max, \times, 0, 1 \rangle$ and the classical semiring $S_{CSP} = \langle \{T, F\}, \vee, \wedge, F, T \rangle$ as example, there are no nontrivial homomorphisms between S_{prop} and S_{CSP}. This is because $\alpha(a \times b) = \alpha(a) \wedge \alpha(b)$ requires $\alpha(a^n) = \alpha(a)$ for any $a \in [0, 1]$ and any positive integer n, which implies $(\forall a > 0)\ \alpha(a) = 1$ or $(\forall a < 1)\ \alpha(a) = 0$.

In the remainder of this section, we relax this condition.

5.1. Semiring quasi-homomorphism

Definition 5.1 (*Quasi-Homomorphism*). A mapping ψ from semiring $\langle S, +, \times, \mathbf{0}, \mathbf{1} \rangle$ to semiring $\langle \widetilde{S}, \widetilde{+}, \widetilde{\times}, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \rangle$ is said to be a *quasi-homomorphism* if for any $a, b \in S$

- $\psi(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\psi(\mathbf{1}) = \widetilde{\mathbf{1}}$; and
- $\psi(a + b) = \psi(a) \widetilde{+} \psi(b)$; and
- $\psi(a \times b) \leq_{\widetilde{S}} \psi(a) \widetilde{\times} \psi(b)$.
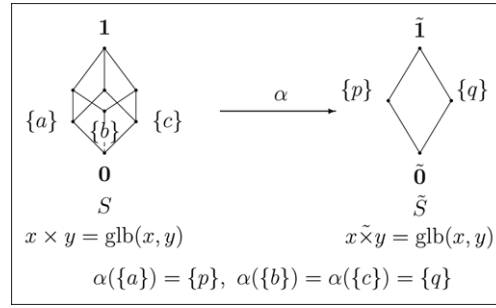
Fig. 1. A counter-example.

The last condition is exactly the *locally correctness* of $\tilde{\times}$ w.r.t. $\times$ [1]. Clearly, each monotonic surjective mapping between S_{prop} and S_{CSP} is a quasi-homomorphism.

The following theorem shows that a quasi-homomorphism is also useful.

Theorem 5.2. *Let $\alpha : S \to \tilde{S}$ be a semiring quasi-homomorphism. Given an SCSP P over S, suppose $t \in Opt(\alpha(P))$ has value v in P and value $\tilde{v}$ in $\alpha(P)$. Then there exists an optimal solution $\bar{t}$ of P, say with value $\bar{v} \geq_S v$ in P, such that $\alpha(\bar{v}) \not\succ_{\tilde{S}} \tilde{v}$.*

Proof. The proof is straightforward. $\square$

Note that if $\tilde{S}$ is totally ordered, then the above conclusion can be rephrased as $\alpha(\bar{v}) \leq_{\tilde{S}} \tilde{v}$. But the following example shows this is not always true.

Soft Problem 4. Take $D = \{d_1, d_2\}, X = \{a, b, c\}, Y = \{p, q\}$ and $V = \{x_1, x_2\}$. Then $S = \langle 2^X, \cup, \cap, \varnothing, X\rangle$ and $\tilde{S} = \langle 2^Y, \cup, \cap, \varnothing, Y\rangle$ are two c-semirings, see Fig. 1. Let $\alpha : S \to \tilde{S}$ be the mapping specified by $\alpha(\varnothing) = \varnothing, \alpha(\{a\}) = \{p\}, \alpha(\{b\}) = \alpha(\{c\}) = \alpha(\{b, c\}) = \{q\}$, and $\alpha(\{a, b\}) = \alpha(\{a, c\}) = \alpha(X) = Y$. Note that α preserves lubs. Moreover, since α is monotonic, we have $\alpha(U \cap W) \subseteq \alpha(U) \cap \alpha(W)$ for any $U, W \subseteq X$. Therefore α is a quasi-homomorphism.

Define $\mathsf{def}_i : D \to S$ ($i = 1, 2$) as follows:

$$\mathsf{def}_1(d_1) = \{a\}, \qquad \mathsf{def}_1(d_2) = \{b\};$$
$$\mathsf{def}_2(d_1) = \{a\}, \qquad \mathsf{def}_2(d_2) = \{c\};$$

Consider the SCSP $P = \langle C, V\rangle$ with $C = \{c_1, c_2\}$ and $c_i = \langle \mathsf{def}_i, \{x_i\}\rangle$ for $i = 1, 2$. Then

$$Sol(P)(d_1, d_1) = \{a\} \cap \{a\} = \{a\},$$
$$Sol(P)(d_1, d_2) = \{a\} \cap \{c\} = \varnothing$$
$$Sol(P)(d_2, d_1) = \{b\} \cap \{a\} = \varnothing,$$
$$Sol(P)(d_2, d_2) = \{b\} \cap \{c\} = \varnothing$$

and

$$Sol(\alpha(P))(d_1, d_1) = \{p\} \cap \{p\} = \{p\},$$
$$Sol(\alpha(P))(d_1, d_2) = \{p\} \cap \{q\} = \varnothing,$$
$$Sol(\alpha(P))(d_2, d_1) = \{q\} \cap \{p\} = \varnothing,$$
$$Sol(\alpha(P))(d_2, d_2) = \{q\} \cap \{q\} = \{q\}.$$

Set $t = (d_2, d_2)$. Clearly, t is an optimal solution of $\alpha(P)$ with value $\{q\}$ in $\alpha(P)$, and value $\varnothing$ in P. Notice that $\bar{t} = (d_1, d_1)$ is the unique optimal solution of P. Since $\alpha(\{a\}) = \{p\} \not\subseteq \{q\}$, there is no optimal solution $\hat{t}$ of P such that $\alpha(\hat{t}) \subseteq \{q\}$.

6. Related works

Our abstraction framework is closely related to the work of Bistarelli et al. [1] and that of de Givry et al. [3].

6.1. Galois insertion-based abstraction

Bistarelli et al. [1] proposed a Galois insertion-based abstraction scheme for soft constraints. The questions investigated in this paper were first studied in [1]. In particular, Theorems 27, 29, 31 of [1] correspond to our Theorems 4.1, 5.2 and 5.1, respectively.

We recall some basic notions concerning abstractions used in [1].

Definition 6.1 (*Galois Insertion [7]*). Let $(\mathcal{C}, \sqsubseteq)$ and $(\mathcal{A}, \leq)$ be two posets (the concrete and the abstract domain). A *Galois connection* $\langle\alpha, \gamma\rangle : (\mathcal{C}, \sqsubseteq) \rightleftarrows (\mathcal{A}, \leq)$ is a pair of monotonic mappings $\alpha : \mathcal{C} \to \mathcal{A}$ and $\gamma : \mathcal{A} \to \mathcal{C}$ such that

$$(\forall x \in \mathcal{C})(\forall y \in \mathcal{A}) \quad \alpha(x) \leq y \Leftrightarrow x \sqsubseteq \gamma(y). \tag{6}$$

In this case, we call γ the upper adjoint (of α), and α the lower adjoint (of γ). A Galois connection $\langle\alpha, \gamma\rangle : (\mathcal{C}, \sqsubseteq) \rightleftarrows (\mathcal{A}, \leq)$ is called a *Galois insertion* (of $\mathcal{A}$ in $\mathcal{C}$) if $\alpha \circ \gamma = id_{\mathcal{A}}$.

Definition 6.2 (*Abstraction*). A mapping $\alpha : S \to \widetilde{S}$ between two c-semirings is called an *abstraction* if

1. α has an upper adjoint γ such that $\langle\alpha, \gamma\rangle : S \rightleftarrows \widetilde{S}$ is a Galois insertion
2. $\widetilde{\times}$ is *locally correct* with respect to $\times$, i.e. $(\forall a, b \in S)\ \alpha(a \times b) \leq_{\widetilde{S}} \alpha(a)\widetilde{\times}\alpha(b)$.

Theorem 27 of [1] gives a sufficient condition for a Galois insertion preserving optimal solutions. This condition, called *order-preserving*, is defined as follows:

Definition 6.3 (*[1]*). Given a Galois insertion $\langle\alpha, \gamma\rangle : S \rightleftarrows \widetilde{S}$, α is said to be *order-preserving* if for any two sets I_1 and I_2, we have

$$\widetilde{\prod}_{x\in I_1}\alpha(x) \leq_{\widetilde{S}} \widetilde{\prod}_{x\in I_2}\alpha(x) \Rightarrow \prod_{x\in I_1} x \leq_S \prod_{x\in I_2} x. \tag{7}$$

This notion plays an important role in [1]. The next proposition shows that this property is too restrictive, since an order-preserving Galois insertion is indeed a semiring isomorphism.

Proposition 6.1. *Suppose $\langle\alpha, \gamma\rangle : S \rightleftarrows \widetilde{S}$ is a Galois insertion. Then α is order-preserving if and only if it is a semiring isomorphism.*

Proof. The sufficiency part is clear, and we now show the necessity part. Notice that α, as a Galois connection, is monotonic. On the other hand, given $x, y \in S$, suppose $\alpha(x) \leq_{\widetilde{S}} \alpha(y)$. By Eq. (7), we have $x \leq_S y$. That is to say, for any $x, y \in S$, $\alpha(x) \leq_{\widetilde{S}} \alpha(y)$ if and only if $x \leq_S y$. In particular, $\alpha(x) = \alpha(y)$ implies $x = y$. This means that α is injective. Moreover, by definition of Galois insertion, α is also surjective. Therefore α is an order isomorphism. As a consequence, it preserves sums.

We next show α preserves products. For $x, y \in S$, since α is surjective, we have some $z \in S$ with $\alpha(z) = \alpha(x)\widetilde{\times}\alpha(y)$. Applying the order-preserving property, we have $z = x \times y$, hence $\alpha(x \times y) = \alpha(z) = \alpha(x)\widetilde{\times}\alpha(y)$, i.e. α preserves products. In summary, α is a semiring isomorphism. $\square$

Theorem 29 of [1] concerns, given an optimal solution of the abstract problem, how to find a reasonable estimation for an optimal solution of the concrete problem. Let $\alpha : S \to \widetilde{S}$ be an abstraction. Given an SCSP P over S, suppose t is an optimal solution of $\alpha(P)$, with semiring value $\widetilde{v}$ in $\alpha(P)$ and v in P. Then [1, Theorem 29] asserts that there exists an optimal solution $\overline{t}$ of P, say with value $\overline{v}$, such that $v \leq \overline{v} \leq \gamma(\widetilde{v})$.

Our Soft Problem 4, however, shows that [1, Theorem 29] is only conditionally true. This is because the quasi-homomorphism α given there is also an abstraction. Since each abstraction is also a quasi-homomorphism, Theorem 5.2 holds for any abstraction.

Our Theorem 5.1 corresponds to Theorem 31 of [1], where the authors consider abstractions between totally ordered semirings with idempotent multiplicative operations. By Example 2.1, we know such an abstraction must be a homomorphism. Therefore our result is more general than [1, Theorem 31].

6.2. Aggregation compatible mapping

There is another abstraction scheme [3] for soft constraints that is closely related to ours, where *valued CSPs* [13] are abstracted in order to produce good lower bounds for the optimal solutions.

Definition 6.4 (*[3]*). A translation $\alpha : S \to \widetilde{S}$ between two totally ordered semirings is said to be *aggregation compatible* if

(1) α is monotonic and $\alpha(\mathbf{0}) = \widetilde{\mathbf{0}}$, $\alpha(\mathbf{1}) = \widetilde{\mathbf{1}}$; and
(2) For any two sets I_1 and I_2, we have[1]

$$\alpha\left(\prod_{x\in I_1} x\right) \leq_{\widetilde{S}} \alpha\left(\prod_{x\in I_2} x\right) \Rightarrow \widetilde{\prod}_{x\in I_1}\alpha(x) \leq_{\widetilde{S}} \widetilde{\prod}_{x\in I_2}\alpha(x). \tag{8}$$

The next theorem shows that an aggregate compatible mapping must be a semiring homomorphism.

Theorem 6.1. *Let $\alpha : S \to \widetilde{S}$ be a mapping between two totally ordered semirings. Then α is aggregate compatible if and only if α is a semiring homomorphism.*

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[1] Note that in Eq. (8) we replace the two $\geq$ in Definition 2 of [3] with $\leq$. This is because we should reverse the order of the valuation set S such that the aggregation operator $\circledast$ is a product operator.

Proof. A semiring homomorphism is clearly aggregate compatible. On the other hand, suppose α is aggregate compatible. Since it is monotonic, α preserves sums. Moreover, by Eq. (8), for any $a, b \in S$, taking $I_1 = \{a, b\}$, $I_2 = \{a \times b\}$, from $\alpha(a \times b) = \alpha(a \times b)$ we have $\alpha(a) \widetilde{\times} \alpha(b) = \alpha(a \times b)$. That is, α also preserves products. Hence α is a semiring homomorphism. $\square$

Therefore our framework is also a generalization of that of de Givry et al.. More importantly, results obtained in Sections 4 and 5 can be applied to valued CSPs.

We first note that any monotonic mapping from a totally ordered set is order-reflecting.

Lemma 6.1. *Let $(\mathcal{C}, \sqsubseteq)$ be a totally ordered set, and $(\mathcal{A}, \leq)$ a poset. Suppose $\alpha : \mathcal{C} \to \mathcal{A}$ is monotonic mapping. Then α is order-reflecting.*

Proof. By contradiction, suppose there are $a, b \in \mathcal{C}$ such that $\alpha(a) < \alpha(b)$ but $a \not\sqsubseteq b$. Then since $\sqsubseteq$ is a total order we know $b \sqsubseteq a$. But by the monotonicity of α, we have $\alpha(b) \leq \alpha(a)$. This contradicts the assumption that $\alpha(a) < \alpha(b)$. Therefore α is order-reflecting. $\square$

Now, we have the following corollary of Theorem 4.1, which was also obtained by de Givry et al. [3] for aggregation compatible mappings.

Corollary 6.1. *Let α be a semiring homomorphism between two c-semirings S and $\widetilde{S}$. Suppose S is a totally ordered c-semiring. Then α preserves optimal solutions.*

Proof. By Lemma 6.1, α is order-reflecting. The conclusion then follows directly from Theorem 4.1. $\square$

7. Conclusions

In this paper we proposed a homomorphism based abstraction scheme for soft constraints. The intuition is that, following Bistarelli et al. [1], we first work in the abstract problem, finding all optimal solutions, and then use them to find optimal solutions of the concrete problem. Surprisingly, our framework turns out to be a generalization of that of de Givry et al. [3], where they consider totally ordered sets.

More precisely, our Theorem 4.1 showed that a mapping preserves optimal solutions if and only if it is an order-reflecting semiring homomorphism; Theorem 5.1 showed that, for a semiring homomorphism α and a problem P over S, if t is an optimal solution of $\alpha(P)$, then there is an optimal solution of P, say $\bar{t}$, such that $\bar{t}$ is also optimal in $\alpha(P)$ and has the same value as t. These results greatly improve or generalize those obtained in [1].

Acknowledgements

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