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Discrete Applied Mathematics 78 (1997) 163–174

DISCRETE APPLIED MATHEMATICS

Optimal orientations of products of paths and cycles

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Received 13 February 1996; revised 5 December 1996

Abstract

For a graph G, let $\mathscr{D}(G)$ be the family of strong orientations of G, $\boldsymbol{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$ and $\rho(G) = \boldsymbol{d}(G) - d(G)$, where $d(G)$ and $d(D)$ are the diameters of G and D respectively. In this paper we show that $\rho(G) = 0$ if G is a cartesian product of (1) paths, and (2) paths and cycles, which satisfy some mild conditions.

Keywords: Path; Cycle; Bipartite graph; Diameter; Strong orientation

1. Introduction

Let G (resp., D) be a graph (resp., digraph) with vertex set $V(G)$ (resp., $V(D)$) and edge set $E(G)$ (resp., $E(D)$). For $v \in V(G)$, the *eccentricity* $e(v)$ of v is defined as $e(v) = \max\{d(v,x) \mid x \in V(G)\}$, where $d(v,x)$ denotes the distance from v to x. The notion $e(v)$ in D is similarly defined. The *diameter* of G (resp., D), denoted by $d(G)$ (resp., $d(D)$), is defined as $d(G) = \max\{e(v) \mid v \in V(G)\}$ (resp., $d(D) = \max\{e(v) \mid v \in V(D)\}$).

An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if every two vertices in D are mutually reachable in D. An edge e in a connected graph G is a *bridge* if $G - e$ is disconnected. Robbins' celebrated one-way street theorem [15] states that *a connected graph G has a strong orientation if and only if no edge of G is a bridge*. As a possible way of extending Robbins' theorem, Boesch and Tindell [1] introduced the notion $\rho(G)$ given below. For a connected graph G containing no bridges, let $\mathscr{D}(G)$ be the family

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PII S0166-218X(97)00017-6

of strong orientations of G. Define

$$\boldsymbol{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\} \quad \text{and} \quad \rho(G) = \boldsymbol{d}(G) - d(G).$$

The problem of evaluating $\rho(G)$ for an arbitrary connected graph G is very difficult. As a matter of fact, Chvátal and Thomassen [2] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

On the other hand, the parameter $\rho(G)$ has been studied in various classes of graphs including complete graphs [1, 11, 14], complete bipartite graphs [1, 3, 20], complete k-partite ($k \geqslant 3$) graphs [4, 6, 7, 13], and n-cubes [12, 20]. Let $G \times H$ denote the cartesian product of two graphs G and H (see Section 2 for the definition), and P_r, C_r and K_r, respectively, the path, cycle and complete graph of order r. Roberts and Xu [16–19], and independently Koh and Tan [5], evaluated the quantity $\rho(P_m \times P_n)$. Very recently, Koh and Tay have further determined the quantities $\rho(C_{2m} \times P_k)$ [8], $\rho(K_m \times P_k), \rho(K_m \times C_{2k+1})$ and $\rho(K_m \times K_n)$ [9] and $\rho(C_{2m} \times K_n)$ [10]. In this paper, we shall evaluate $\rho(G_1 \times G_2 \times \cdots \times G_m)$, where $m \geqslant 2$ and $\{G_i \mid 1 \leqslant i \leqslant m\}$ is a combination of paths and cycles.

2. Cartesian product of paths

The cartesian product of a family of graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1, G_2 \times \cdots \times G_n$ or $\prod_{i=1}^n G_i$, where $n \geqslant 2$, is the graph G having $V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ and two vertices $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists $r \in \{1, 2, \ldots, n\}$ such that $u_r v_r \in E(G_r)$ and $u_i = v_i$ for all $i = 1, 2, \ldots, n$ with $i \neq r$. In this section, we shall evaluate $\rho(G)$, where G is of the form $\prod_{i=1}^n P_{k_i}$ with $n \geqslant 2$ and $k_i \geqslant 2$ for each $i = 1, 2, \ldots, n$. For convenience, the vertices in the graph are labelled $(x_1, x_2, \ldots, x_n)$, where $1 \leqslant x_i \leqslant k_i$ for each $i = 1, 2, \ldots, n$, such that the vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent iff $|a_r - b_r| = 1$ for exactly one $r \in \{1, 2, \ldots, n\}$, and $a_i = b_i$ for all i with $i \neq r$.

Let D be a digraph. A dipath (resp., dicycle) in D is simply called a path (resp., cycle) in D. A path from u to v in D is simply called a u–v path. For $X \subseteq V(D)$, the subdigraph of D induced by X is denoted by $D[X]$. For $x, y \in V(D)$ and $A \subseteq V(D)$, we write '$x \to y$' if x is adjacent to y in D, and write '$x \to A$' (resp., '$A \to y$') if $x \to y$ for each $y \in A$ (resp., for each $x \in A$).

Our first main result is as follows:

Theorem 1. *$\rho(\prod_{i=1}^n P_{k_i}) = 0$, where $n \geqslant 2$, $k_1 \geqslant 3$, $k_2 \geqslant 6$ with $(k_1, k_2) \neq (3, 6)$.*

Let

$$G_n = \overbrace{P_2 \times P_2 \times \cdots \times P_2}^{n}$$

(i.e., the n-cube). In proving that $\rho(G_n) = 0$ for $n \geqslant 4$, McCanna [12] made use of the following subtle observation due to C. Thomassen.

Lemma 1. *If a bipartite graph G admits an orientation of diameter at most k, where $k \geqslant 3$, such that every vertex is in a cycle of length at most k, then the graph $G \times P_2$ admits an orientation of diameter at most $k+1$ such that every vertex is in a cycle of length at most k.*

We shall now extend Thomassen's observation from P_2 to $\prod_{i=1}^{n} P_{k_i}$, and shall make use of the extension to prove some of our main results in this paper.

Lemma 2. *If a bipartite graph G admits an orientation of diameter at most k, where $k \geqslant 3$, such that every vertex is in a cycle of length at most k, then the graph $G \times \prod_{i=1}^{n} P_{k_i}$, where $n \geqslant 1$, admits an orientation of diameter at most $k - n + \sum_{i=1}^{n} k_i$ such that every vertex is in a cycle of length at most k.*

Proof. Let V_1 and V_2 be the partite sets of G. Let $F \in \mathscr{D}(G)$ with $d(F) \leqslant k$ such that every vertex is in a cycle of length at most k in F. We shall now orient $G \times \prod_{i=1}^{n} P_{k_i}$ inductively as follows:

(i) In $G \times P_{k_1}$, for $1 \leqslant i \leqslant k_1 - 1$, orient $(x,i) \to (x,i+1)$ iff $x \in V_1$; and for $1 \leqslant i \leqslant k_1$, orient $(x,i) \to (y,i)$ iff $xy \in E(F)$.

(ii) Suppose $G \times \prod_{i=1}^{r} P_{k_i}$, where $1 \leqslant r \leqslant n-1$, has been oriented. Orient $G \times \prod_{i=1}^{r+1} P_{k_i}$ so that the orientation of $G \times \prod_{i=1}^{r} P_{k_i} \times \{j\}$ is isomorphic to that of $G \times \prod_{i=1}^{r} P_{k_i}$ for each $j = 1, 2, \ldots, k_{r+1}$, and for $1 \leqslant i \leqslant k_{r+1} - 1$, orient $(x, a_1, a_2, \ldots, a_r, i) \to (x, a_1, a_2, \ldots, a_r, i+1)$ iff $x \in V_1$.

Let F^* be the resulting orientation of $G \times \prod_{i=1}^{n} P_{k_i}$.

Claim. *$e(u) \leqslant k - n + \sum_{i=1}^{n} k_i$ for each vertex u in F^*.*

Let $u = (x, a_1, a_2, \ldots, a_n)$ and assume that $x \in V_1$, say. Take an arbitrary vertex $v = (y, b_1, b_2, \ldots, b_n)$ in F^*. As the cartesian product is commutative, we may assume that $a_i \leqslant b_i$ for $1 \leqslant i \leqslant m$ and $a_i > b_i$ for $m+1 \leqslant i \leqslant n$, where $m \leqslant n$.

(1) Let $w = (x, b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$. Observe that there is a u–w path of length at most $\sum_{i=1}^{m} k_i - m$ in F^*.

(2) If $x \neq y$, let $w' = (x', b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$, where x' is adjacent from x in an x–y path of length at most k in F. Then $w \to w'$ in F^*. (Note that $x' \in V_2$.) If $x = y$, take a cycle of length at most k containing x in F.

(3) There is a path of length at most $\sum_{i=m+1}^{n} k_i - (n-m)$ from w' to $(x', b_1, b_2, \ldots, b_n)$ in F^*.

(4) There is a path of length at most $k-1$ from $(x', b_1, b_2, \ldots, b_n)$ to v in F^*.

Combining (1)–(4), $d(u,v) \leqslant \sum_{i=1}^{m} k_i - m + 1 + \sum_{i=m+1}^{n} k_i - (n-m) + k - 1 = k - n + \sum_{i=1}^{n} k_i$. This proves the claim.

Thus $d(F^*) \leqslant k - n + \sum_{i=1}^{n} k_i$. The second part of Lemma 2 is obvious as each vertex in F^* is contained in a cycle of length at most k in F. $\square$

We need also the following result.

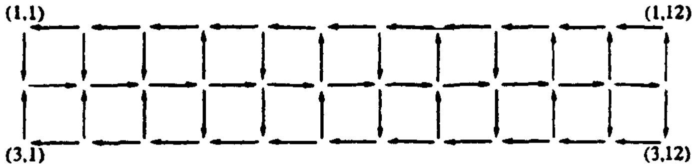

Fig. 1. Orientation of $P_3 \times P_{12}$.

Lemma 3. *For $m \geqslant 3$, $n \geqslant 6$ with $(m,n) \neq (3,6)$, there exists $F \in \mathscr{D}(P_m \times P_n)$ such that*

(i) *$d(F) = \boldsymbol{d}(P_m \times P_n) = m+n-2$ and*

(ii) *every vertex in $P_m \times P_n$ is in a cycle of length at most $m+n-2$ in F.*

Note. (1) $d(P_m \times P_n) = m+n-2$ for all $m, n \geqslant 1$.

(2) It was shown in [5] that $\boldsymbol{d}(P_3 \times P_6) = 8 (= m+n-1)$.

Proof of Lemma 3. Part (i) (except some isolated cases) was first obtained by Roberts and Xu [16–19]. Here, we shall use the orientations of $P_m \times P_n$ introduced by Koh and Tan [5] to prove part (ii). Following [5], we have seven cases to consider.

Case A: *$m=3$ and $n \equiv 0 \pmod 2$ with $n \geqslant 8$.* Define $F \in \mathscr{D}(P_m \times P_n)$ as follows (see Fig. 1):

(1) For $i=1,3$ and $j=1,2,\ldots,n-1$, orient $(i,j+1) \to (i,j)$;

(2) For $j=1,2,\ldots,n-1$, orient $(2,j) \to (2,j+1)$;

(3) For $j=1,2,3$, orient $\{(1,j),(3,j)\} \to (2,j)$;

(4) Orient $(2,4) \to \{(1,4),(3,4)\}$;

(5) Orient $(2,n) \to \{(1,n),(3,n)\}$ and $(2,n-1) \to \{(1,n-1),(3,n-1)\}$;

(6) For $j=5,6,\ldots,n-2$, orient $(3,j) \to (2,j) \to (1,j)$ if $j \equiv 0 \pmod 2$; and $(1,j) \to (2,j) \to (3,j)$ if $j \equiv 1 \pmod 2$.

Note that $d(F) = m+n-2$. Now, consider the following cycles (see also Fig. 1):

(A_1) $(1,1)(2,1)(2,2)(2,3)(2,4)(1,4)(1,3)(1,2)(1,1)$,

(A_2) $(3,1)(2,1)(2,2)(2,3)(2,4)(3,4)(3,3)(3,2)(3,1)$,

(A_3) $(3,5)(3,4)(3,3)(2,3)(2,4)(2,5)(3,5)$,

(A_4) $(1,n)(1,n-1)(1,n-2)(1,n-3)(2,n-3)(2,n-2)(2,n-1)(2,n)(1,n)$,

(A_5) $(3,n)(3,n-1)(3,n-2)(2,n-2)(2,n-1)(2,n)(3,n)$.

It can be checked that each of the above cycles is of length at most $m+n-2$, and that the cycles cover vertices $(3,5),(1,n-3),(2,n-3)$ and (i,j), where $i=1,2,3$ and $j=1,2,3,4,n-2,n-1,n$. On the other hand, each of the remaining vertices lies in a cycle of length 4 in F.

Case B: *$m=3$ and $n=7$.* Define $F \in \mathscr{D}(P_3 \times P_7)$ as shown in Fig. 2. It can be checked that $d(F) = 8 = m+n-2$ and that (ii) is satisfied as shown in Fig. 2.

Case C: *$m=3$ and $n \equiv 1 \pmod 2$ with $n \geqslant 9$.* Define $F \in \mathscr{D}(P_m \times P_n)$ as follows (see Fig. 3):

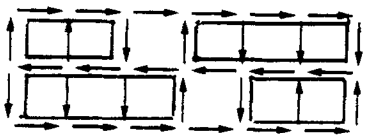

Fig. 2. Orientation of $P_3 \times P_7$.

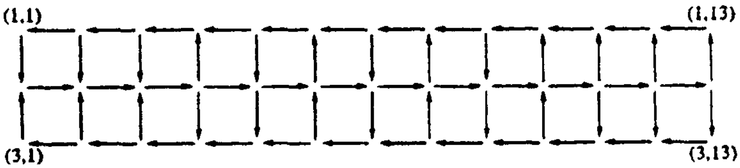

Fig. 3. Orientation of $P_3 \times P_{13}$.

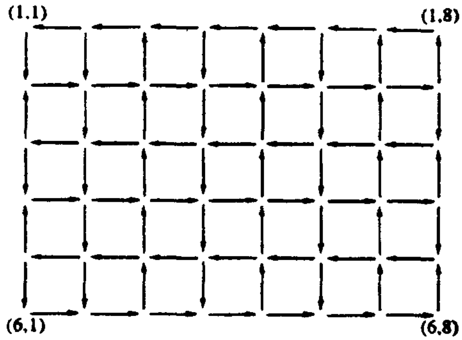

Fig. 4. Orientation of $P_6 \times P_8$.

(1) $F[P_m \times P_{n-1}]$ is identical with the orientation in Case A;
(2) Orient $(2,n) \to \{(1,n),(3,n)\}$;
(3) Orient $(1,n) \to (1,n-1), (2,n-1) \to (2,n)$, and $(3,n) \to (3,n-1)$.

Note that $d(F) = m+n-2$. Now, consider the following cycles (see also Fig. 3):

(C_1) $(1,1)(2,1)(2,2)(2,3)(2,4)(1,4)(1,3)(1,2)(1,1)$,

(C_2) $(3,1)(2,1)(2,2)(2,3)(2,4)(2,5)(3,5)(3,4)(3,3)(3,2)(3,1)$,

(C_3) $(1,n)(1,n-1)(1,n-2)(1,n-3)(1,n-4)(2,n-4)(2,n-3)(2,n-2)(2,n-1)$ $(2,n)(1,n)$,

(C_4) $(3,n)(3,n-1)(3,n-2)(3,n-3)(2,n-3)(2,n-2)(2,n-1)(2,n)(3,n)$.

Each of these cycles is of length at most $m+n-2$ and they cover vertices $(3,5)$, $(1,n-4),(2,n-4)$ and (i,j), where $i = 1,2,3$ and $j = 1,2,3,4,n-3,n-2,n-1,n$. On the other hand, each of the remaining vertices lies in a cycle of length 4 in F.

Case D: $m \equiv n \equiv 0 \pmod 2$ *with* $m \geqslant 4$ *and* $n \geqslant 6$. Define $F \in \mathscr{D}(P_m \times P_n)$ as follows (see Fig. 4):

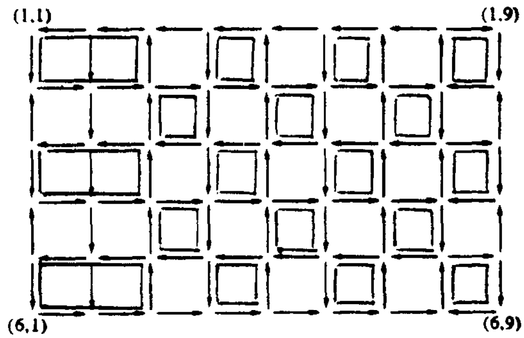

Fig. 5. Orientation of $P_6 \times P_9$.

(1) For $i=1,2,\dots,m$ and $j=1,2,\dots,n-1$, orient

$$(i,j) \to (i,j+1) \quad \text{if } i \equiv 0 \pmod 2,$$
$$(i,j+1) \to (i,j) \quad \text{if } i \equiv 1 \pmod 2;$$

(2) For $i=1,2,\dots,m-1$ and $j=2,3,\dots,n-1$, orient

$$(i,j) \to (i+1,j) \quad \text{if } j \equiv 0 \pmod 2,$$
$$(i+1,j) \to (i,j) \quad \text{if } j \equiv 1 \pmod 2;$$

(3) Orient $(1,1) \to (2,1)$ and $(i,1) \to \{(i-1,1),(i+1,1)\}$ for each $i=3,5,\dots,m-1$;
(4) Orient $(i,n) \to \{(i-1,n),(i+1,n)\}$ for each $i=2,4,\dots,m-2$; and
(5) Orient $(m,n) \to (m-1,n)$.
Note that $d(F)=m+n-2$. Now, consider the following cycles:
(D_1) For $i=1,3,\dots,m-1$,

$$(i,1)(i+1,1)(i+1,2)(i+1,3)(i,3)(i,2)(i,1);$$

(D_2) For $i=1,3,\dots,m-1$,

$$(i,n)(i,n-1)(i,n-2)(i+1,n-2)(i+1,n-1)(i+1,n)(i,n).$$

Each of these cycles is of length not exceeding $m+n-2$, and the cycles cover vertices (i,j), where $i=1,2,\dots,m$ and $j=1,2,3,n-2,n-1,n$. On the other hand, each of the remaining vertices lies in a cycle of length 4 in F.

Case E: $m \equiv 0 \pmod 2$ *and* $n \equiv 1 \pmod 2$ *with* $m \geqslant 4$ *and* $n \geqslant 7$. Define $F \in \mathscr{D}(P_m \times P_n)$ as follows (see Fig. 5):
(1) $F[P_m \times P_{n-1}]$ is identical with the orientation in Case D;
(2) For $i=1,3,\dots,m-1$, orient $(i,n-1) \to (i,n)$ and $(i+1,n) \to (i+1,n-1)$;
(3) For $i=3,5,\dots,m-1$, orient $(i,n) \to \{(i-1,n),(i+1,n)\}$;
(4) Orient $(1,n) \to (2,n)$.

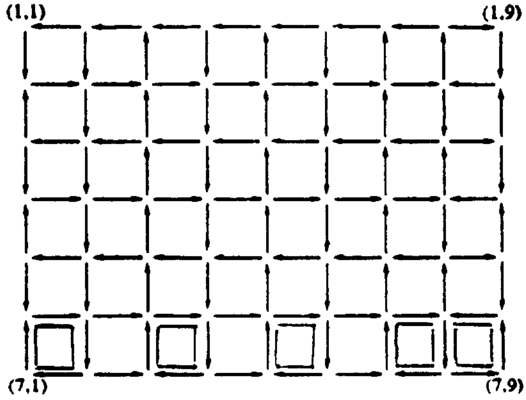

Fig. 6. Orientation of $P_7 \times P_9$.

Note that $d(F) = m + n - 2$. Also, it can be shown (see Fig. 5 as an illustration) that each vertex is in a cycle of length not exceeding $m + n - 2$ in F.

Case F: $m \equiv n \equiv 1 \pmod 2$ *with* $m \geqslant 5$ *and* $n \geqslant 7$. Define $F \in \mathscr{D}(P_m \times P_n)$ as follows (see Fig. 6):

(1) $F[P_{m-1} \times P_n]$ is identical with the orientation in Case E;

(2) For each $j = 2, 4, \ldots, n-1$, orient

$$(m,j) \rightarrow \{(m,j-1),(m,j+1)\},$$

$$(m-1,j) \rightarrow (m,j) \text{ and}$$

$$(m,j-1) \rightarrow (m-1,j-1);$$

(3) Orient $(m,n) \rightarrow (m-1,n)$.

Note that $d(F) = m + n - 2$. Also, it can be checked (see Fig. 6 as an illustration) that each vertex is in a cycle of length not exceeding $m + n - 2$ in F.

Finally, we consider the case when $m \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$ with $m \geqslant 5$ and $n \geqslant 6$. By symmetry and the result in Case E, we need only consider the following:

Case G: $m = 5$ *and* $n \equiv 0 \pmod 2$ *with* $n \geqslant 6$. Let $n = 2k$ and define $F \in \mathscr{D}(P_5 \times P_n)$ as follows (see Fig. 7):

(1) For $i = 1,2,4,5$ and $j = 1,2,\ldots,k-1$, orient $(i,j) \rightarrow (i,j+1)$ and $(3,j+1) \rightarrow (3,j)$;

(2) For $i = 1,2,4,5$ and $j = k+1, k+2, \ldots, 2k-1$, orient $(i,j+1) \rightarrow (i,j)$ and $(3,j) \rightarrow (3,j+1)$;

(3) For $j \neq k, k+1$, orient $(1,j) \leftarrow (2,j) \leftarrow (3,j) \rightarrow (4,j) \rightarrow (5,j)$;

(4) For $j = k, k+1$, orient $(1,j) \rightarrow (2,j) \rightarrow (3,j) \leftarrow (4,j) \leftarrow (5,j)$;

(5) Orient $(2,k) \rightarrow (2,k+1), (3,k) \leftarrow (3,k+1)$ and $(4,k) \rightarrow (4,k+1)$. The edges $(1,k)(1,k+1)$ and $(5,k)(5,k+1)$ may be arbitrarily oriented.

Note that $d(F) = n + 3$ and each vertex is in a cycle of length not exceeding $n + 3$ (see Fig. 7 as an illustration).

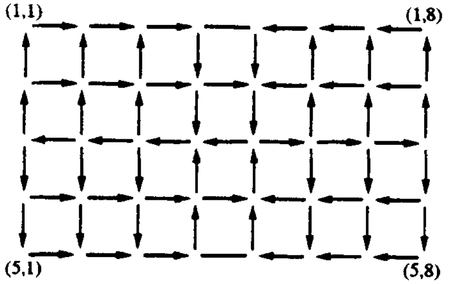

Fig. 7. Orientation of $P_5 \times P_8$.

The proof of Lemma 3 is now complete. □

Proof of Theorem 1. Let $G = P_{k_1} \times P_{k_2}$. By Lemma 3, the bipartite graph G admits an orientation F with $d(F) = k_1 + k_2 - 2$ such that each vertex in G is in a cycle of length at most $k_1 + k_2 - 2$ in F. By Lemma 2, the graph $\prod_{i=1}^{n} P_{k_i}$ admits an orientation F^* with

$$\begin{aligned} d(F^*) &\leqslant (k_1 + k_2 - 2) - (n - 2) + \sum_{i=3}^{n} k_i \\ &= \sum_{i=1}^{n} k_i - n \\ &= d\left(\prod_{i=1}^{n} P_{k_i}\right). \end{aligned}$$

The result thus follows. □

3. Cartesian product of paths and cycles

The main aim in this section is to prove the following results.

Theorem 2. (i) $\rho(C_{2n} \times \prod_{i=1}^{m} P_{k_i}) = 0$ *for* $m \geqslant 1$, $n \geqslant 2$ *and* $k_1 \geqslant 4$.
(ii) $\rho(\prod_{i=1}^{m} P_{k_i} \times \prod_{i=1}^{r} C_{n_i}) = 0$ *for* $m \geqslant 2$, $r \geqslant 0$, $k_1 \geqslant 3$ *and* $k_2 \geqslant 6$ *with* $(k_1, k_2) \neq (3, 6)$.
(iii) $\rho(C_{2n} \times \prod_{i=1}^{m} P_{k_i} \times \prod_{i=1}^{r} C_{n_i}) = 0$ *for* $m \geqslant 1$, $r \geqslant 0$, $n \geqslant 2$ *and* $k_1 \geqslant 4$.

Note that results (ii) and (iii) are overlapping with (ii) requiring stronger conditions on two paths whereas (iii) requiring a cycle to be even and the length of a path at least four.

In what follows, the vertices of $\prod_{i=1}^{r} C_{n_i}$ are labelled $(x_1, x_2, \ldots, x_r)$, where $1 \leqslant x_i \leqslant n_i$, $1 \leqslant i \leqslant r$ so that $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ are adjacent iff $|a_k - b_k| = 1 \pmod{n_k - 2}$

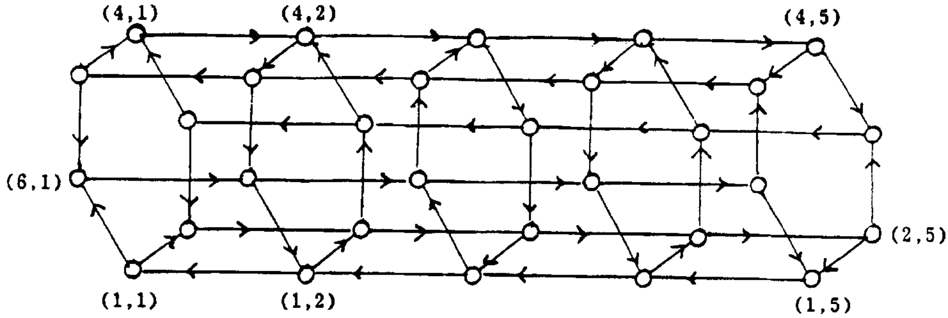

Fig. 8. Orientation of $C_6 \times P_5$.

for exactly one k, $1 \leqslant k \leqslant r$, and $a_i = b_i$ for all i with $i \neq k$. For a real x, we shall denote by $[x]$ the greatest integer not exceeding x.

To prove Theorem 2(i), we need the following result.

Lemma 4. *For $n \geqslant 2$ and $k \geqslant 4$, there exists $F \in \mathscr{D}(C_{2n} \times P_k)$ such that*
(i) *$d(F) = \boldsymbol{d}(C_{2n} \times P_k) = n + k - 1$ and*
(ii) *every vertex in $C_{2n} \times P_k$ is in a cycle of length at most $n + k - 1$ in F.*

Proof. In [8], the following orientation F of $C_{2n} \times P_k$ was introduced (see Fig. 8):
(i) For $i \equiv 1 \pmod 2$ and $1 \leqslant i \leqslant 2n - 1$, orient $(i, 1) \to \{(i + 1, 1), (i - 1, 1)\}$;
(ii) For $j \equiv 0 \pmod 2$, $2 \leqslant j \leqslant k - 1$, and $1 \leqslant i \leqslant 2n$, orient $(i, j) \to (i + 1, j)$;
(iii) For $j \equiv 1 \pmod 2$, $3 \leqslant j \leqslant k - 1$, and $1 \leqslant i \leqslant 2n$, orient $(i, j) \to (i - 1, j)$;
(iv) For $i \equiv 0 \pmod 2$ and $2 \leqslant i \leqslant 2n$, orient $(i, k) \to \{(i + 1, k), (i - 1, k)\}$;
(v) For $i \equiv 0 \pmod 2$, $2 \leqslant i \leqslant 2n$ and $1 \leqslant j \leqslant k - 1$, orient $(i, j) \to (i, j + 1)$;
(vi) For $i \equiv 1 \pmod 2$, $1 \leqslant i \leqslant 2n - 1$ and $2 \leqslant j \leqslant k$, orient $(i, j) \to (i, j - 1)$.

It was shown in [8] also that $d(F) = \boldsymbol{d}(C_{2n} \times P_k) = n + k - 1$ for $k \geqslant 4$ and $n \geqslant 2$. It remains to prove (ii).

Consider the following cycles (see also Fig. 8):
(A_1) For $j \equiv 0 \pmod 2$ with $2 \leqslant j \leqslant k - 3$ for odd k or $2 \leqslant j \leqslant k - 2$ for even k, and $i \equiv 1 \pmod 2$ with $1 \leqslant i \leqslant 2n - 1$, $(i, j)(i + 1, j)(i + 1, j + 1)(i, j + 1)(i, j)$;
(A_2) For $i \equiv 1 \pmod 2$, $1 \leqslant i \leqslant 2n - 1$, $(i, 1)(i - 1, 1)(i - 1, 2)(i, 2)(i, 1)$; and
(A_3) For $i \equiv 1 \pmod 2$, $1 \leqslant i \leqslant 2n - 1$, $(i + 1, k)(i, k)(i, k - 1)(i + 1, k - 1)(i + 1, k)$ if k is odd, or $(i - 1, k)(i, k)(i, k - 1)(i - 1, k - 1)(i - 1, k)$ if k is even.

Clearly, all the above cycles are of length 4 ($\leqslant n + k - 1$) and they cover $V(C_{2n} \times P_k)$. □

Proof of Theorem 2(i). Let $G = C_{2n} \times P_{k_1}$. By Lemma 4, the bipartite graph G admits an orientation F with $d(F) = n + k_1 - 1$ such that each vertex in G is in a cycle of length at most $n + k_1 - 1$ in F. By Lemma 2, the graph $H = C_{2n} \times \prod_{i=1}^{m} P_{k_i}$ admits an orientation F^* with

$$\begin{aligned} d(F^*) &\leqslant n + k_1 - 1 - (m-1) + \sum_{i=2}^{m} k_i \\ &= n - m + \sum_{i=1}^{m} k_i \\ &= d(H). \end{aligned}$$

This proves Theorem 2(i). □

To prove Theorem 2(ii), we shall extend Thomassen's observation from P_2 to $\prod_{i=1}^{n} C_{k_i}$.

Lemma 5. *If a bipartite graph G admits an orientation of diameter at most k, where $k \geqslant 3$, such that each vertex is in a cycle of length at most k, then the graph $G \times \prod_{i=1}^{n} C_{k_i}$ admits an orientation of diameter not exceeding $k + \sum_{i=1}^{n}[k_i/2]$ such that each vertex is in a cycle of length at most k.*

Proof. Let V_1 and V_2 be the partite sets of G. Let $F \in \mathscr{D}(G)$ with $d(F) \leqslant k$ such that every vertex is in a cycle of length at most k in F.

Orient $G \times \prod_{i=1}^{n} C_{k_i}$ inductively as follows:

(i) In $G \times C_{k_1}$, for $1 \leqslant i \leqslant k_1$, orient

$$(x,i) \to (x,i+1) \quad \text{iff } x \in V_1,$$
$$(x,i) \to (y,i) \quad \text{iff } xy \in E(F).$$

(Note that the second coordinate of $(x,i+1)$ is taken modulo k_1.)

(ii) Suppose $G \times \prod_{i=1}^{r} C_{k_i}$, where $1 \leqslant r \leqslant n-1$, has been oriented. Orient $G \times \prod_{i=1}^{r+1} C_{k_i}$ so that the orientation of $G \times \prod_{i=1}^{r} C_{k_i} \times \{j\}$ is isomorphic to that of $G \times \prod_{i=1}^{r} C_{k_i}$ for each $j = 1,2,\ldots,k_{r+1}$, and for $1 \leqslant i \leqslant k_{r+1}$, orient $(x,a_1,a_2,\ldots,a_r,i) \to (x,a_1,a_2,\ldots,a_r,i+1)$ iff $x \in V_1$ (note that the last coordinate is taken modulo k_{r+1}).

Let F^* be the resulting orientation of $G \times \prod_{i=1}^{n} C_{k_i}$. We shall now show that there is a path of length at most $k + \sum_{i=1}^{n}[k_i/2]$ from an arbitrary vertex u to any other vertex v in F^*. Let $u = (x,a_1,a_2,\ldots,a_n)$ and $v = (y,b_1,b_2,\ldots,b_n)$. We may assume that $x \in V_1$. As the cartesian product is commutative, we further assume that

$$0 \leqslant b_i - a_i \leqslant \left[\frac{k_i}{2}\right] \pmod{k_i} \quad \text{for } i = 1,2,\ldots,m$$

and

$$0 \leqslant a_i - b_i \leqslant \left[\frac{k_i}{2}\right] \pmod{k_i} \quad \text{for } i = m+1,\ldots,n.$$

(1) Let $w = (x,b_1,b_2,\ldots,b_m,a_{m+1},\ldots,a_n)$. Clearly, there is a $u-w$ path in F^* of length at most $\sum_{i=1}^{m}[k_i/2]$.

(2) If $x \neq y$, let $w' = (x', b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$, where x' is adjacent from x in an x–y path of length at most k in F. Observe that $x' \in V_2$. If $x = y$, take a cycle of length at most k containing x in F.

(3) Let $w^* = (x', b_1, b_2, \ldots, b_n)$. Clearly, there is a w'–w^* path of length at most $\sum_{i=m+1}^{n} [k_i/2]$ in F^*.

(4) There is a w^*–v path of length at most $k-1$ in F^*.

Combining (1)–(4), we have

$$d(u,v) \leqslant \sum_{i=1}^{m} \left[\frac{k_i}{2}\right] + 1 + \sum_{i=m+1}^{n} \left[\frac{k_i}{2}\right] + k - 1$$
$$= k + \sum_{i=1}^{n} \left[\frac{k_i}{2}\right].$$

This shows that $d(F^*) \leqslant k + \sum_{i=1}^{n} [k_i/2]$. The second part of Lemma 5 is obvious as each vertex in F^* is contained in a cycle of length at most k in F. □

Proof of Theorem 2(ii). Let $G = \prod_{i=1}^{m} P_{k_i}$. By Theorem 1 and Lemma 2, the bipartite graph G admits an orientation F with $d(F) = \sum_{i=1}^{m} k_i - m$, and every vertex in G lies in a cycle of length not exceeding $k_1 + k_2 - 2$ ($\leqslant \sum_{i=1}^{m} k_i - m$) in F. Thus by Lemma 5, the graph $H = G \times \prod_{i=1}^{r} C_{n_i}$ admits an orientation F^* with

$$d(F^*) \leqslant \sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} \left[\frac{n_i}{2}\right]$$
$$= d(H).$$

This proves Theorem 2(ii). □

Proof of Theorem 2(iii). Let $G = C_{2n} \times \prod_{i=1}^{m} P_{k_i}$. By Theorem 2(i) and Lemma 2, the bipartite graph G admits an orientation F with $d(F) = n + \sum_{i=1}^{m} k_i - m$, and every vertex in G lies in a cycle of length at most 4($\leqslant n + \sum_{i=1}^{m} k_i - m$) in F (see the proof of Lemma 4(ii)). Thus by Lemma 5, the graph $H = G \times \prod_{i=1}^{r} C_{n_i}$ admits an orientation F^* with

$$d(F^*) \leqslant n + \sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} \left[\frac{n_i}{2}\right]$$
$$= d(H).$$

This proves Theorem 2(iii). □

Acknowledgements

The authors would like to thank the referees for their helpful comments.

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