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A note on the homotopy analysis method

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ABSTRACT

The present work is devoted to using an analytic approach, namely the homotopy analysis method, to obtain convergent series solutions of strongly nonlinear problems. On the basis of the homotopy derivative concept described in Liao (2009) [3], a theorem is proved here which generalizes some lemmas and theorems provided in Liao (2009) [3] and Molabahrami and Khani (2007) [4]. Significant applicability of the theorem obtained here in some practical situations is demonstrated.

1. Introduction

The homotopy analysis method, introduced first by Liao [1], is a general approximate analytic approach used to obtain series solutions of nonlinear equations of various types, including algebraic equations, ordinary differential equations, partial differential equations, differential–integral equations, differential–difference equations, and coupled such equations. This method is valid no matter whether a nonlinear problem contains small/large physical parameters or not, which is essentially required in perturbation techniques. More importantly, unlike all perturbation and traditional non-perturbation methods, the homotopy analysis method provides us with both the freedom to choose proper base functions for approximating a nonlinear problem and a simple way to ensure the convergence of the solution series.

After the publication of Liao's book [2] on the homotopy analysis method, a number of researchers have successfully applied this method to various nonlinear problems in science and engineering; for an extensive monograph one can refer to [3,2]. As described therein, briefly speaking, by means of the homotopy analysis approach, one constructs a continuous mapping of an initial guessed approximation to the exact solution of the equations considered. An auxiliary linear operator is chosen for constructing such a continuous mapping, and an auxiliary parameter is used to ensure the convergence of the solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub-problems, which is the advantage of the method in this computer age.

To solve the nonlinear algebraic or differential equations by the homotopy method, the existing (linear and nonlinear) terms need to be differentiated using the homotopy derivative concept. Using the concept, derivatives of some frequently used familiar functions such as polynomials x^k (k is a natural number), simple natural exponential function e^x and simple trigonometric $\sin x$ and $\cos x$ functions were given as lemmas and theorems in the recent papers [3,4]. However, for a function of a more general type it is unknown as yet how to take a homotopy derivative, which places serious restrictions on the use of the powerful homotopy analysis method. With the aim of filling this gap, in this work we present a theorem which generalizes some of the theorems of [3,4], and use it later to solve several new nonlinear term problems not dealt with before.

The following strategy is pursued in the rest of this work. Section 2 outlines the definitions and the theorem. Applications of the theorem to several examples are presented in Section 3. Conclusions are eventually drawn in Section 4.

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2. The homotopy derivative of a general function

As mentioned in the Introduction, the so-called homotopy derivative is used to deduce the high-order deformation equations during the implementation of the homotopy analysis method while solving nonlinear equations of science and engineering. Here, we first outline some definitions as already given in [3] and then prove a rigorous theorem on evaluating the homotopy derivative of a general function that is manifested within a structure with linear or nonlinear terms in the equations.

Definition 2.1. Let x be a function of the *homotopy parameter* $p \in [0, 1]$, whose Maclaurin series is given by

$$x = \sum_{k=0}^{\infty} x_k p^k. \tag{2.1}$$

This series is called the *homotopy series* of x.

Definition 2.2. Let x be a function of the *homotopy parameter* p; then in view of the formula for $D(m, x)$

$$D(m, x) = \frac{1}{m!}\frac{\partial^m x}{\partial p^m}, \tag{2.2}$$

the expression for $D_m(x)$ obtained by setting Eq. (2.2) at $p = 0$, that is,

$$D_m(x) = D(m, x)|_{p=0}$$

which is called the mth-order *homotopy derivative* of x, where $m \geq 0$ is an integer.

Lemma 2.1. *It holds true that*

$$D_m(x) = x_m. \tag{2.3}$$

The proof is clear from the unique coefficient statement of the Taylor theorem.

Theorem 2.2. *Let $f(x)$ be an infinitely differentiable function of x, dependent on the homotopy parameter p. Together with the definitions*

$$D(0, f(x)) = f(x), \qquad D_0(f(x)) = D(0, f(x))|_{p=0} = f(x_0),$$

for the homotopy series of x given in Eq. (2.1) it holds that

$$\text{(a)} \quad D(k, f(x)) = \sum_{m=0}^{k-1}\left(1 - \frac{m}{k}\right) D(k - m, x)\frac{\partial}{\partial x} D(m, f(x)), \tag{2.4}$$

$$\text{(b)} \quad D_k(f(x)) = D(k, f(x))|_{p=0},$$

where $k \geq 1$ are integers.

Proof. Denoting $\frac{\partial}{\partial x}$ by a prime and also taking into account the Leibniz's rule for derivatives of a product, one obtains

$$\frac{1}{k!}\frac{\partial^k f(x)}{\partial p^k} = \frac{1}{k!}\frac{\partial^{k-1}\left(\frac{\partial x}{\partial p} f'(x)\right)}{\partial p^{k-1}} = \frac{1}{k}\sum_{m=0}^{k-1}\frac{1}{m!(k-1-m)!}\frac{\partial^{k-m} x}{\partial p^{k-m}}\frac{\partial^m f'(x)}{\partial p^m}$$

$$= \sum_{m=0}^{k-1}\frac{k-m}{k}\frac{1}{(k-m)!}\frac{\partial^{k-m} x}{\partial p^{k-m}}\frac{1}{m!}\frac{\partial^m f'(x)}{\partial p^m},$$

and a further use of (2.2) yields (a). Setting $p = 0$ in the above expression and making use of Definition 2.2 and Lemma 2.1, one easily gets (b) and the proof is complete. □

It should be remarked that Theorem 2.2 is a more general case involving any function of x, unlike the theorems corresponding to some particular cases of the function f as stated in [3,4].

3. Examples and discussion

To further demonstrate the practical applicability of the above presented theorem and the concept that it generalizes some specific theorems given in [3] (see Theorems 2.1, 2.3, 2.4 and 2.5 in [3]) and [4], we apply it here to some nonlinear algebraic and differential equations.

Example 1. As used in several papers (see for illustration certain boundary layer flows [5,6]), when $f(x) = x^2$ the resulting homotopy derivative is given by the recurrence relation:

$$D_k(f(x)) = \sum_{j=0}^{k} D_j(x)D_{k-j}(x), \tag{3.5}$$

for $k \geq 0$. In this case since

$$\frac{\partial}{\partial x}D(m, f(x))\bigg|_{p=0} = 2D_m(x).$$

Theorem 2.2 leads to

$$D_k(f(x)) = \sum_{m=0}^{k-1} 2\left(1 - \frac{m}{k}\right) D_{k-m}(x)D_m(x)$$

which is the same as Eq. (3.5). This example can be extended to any positive integer power of x, giving rise to a formula similar to (3.5), as was already stated as a lemma in [4]. Therefore, the lemma of [4] is only a special case of the Theorem 2.2.

Example 2. In some nonlinear problems, it is required to take the homotopy derivative of the natural exponential function $f(x) = e^x$; see for instance the Gelfand equation [7]. For this basic exponential function it is true that

$$\frac{\partial}{\partial x}D(m, e^x)\bigg|_{p=0} = D_m(e^x),$$

and thus the formulae in (2.4) lead us to

$$D_k(e^x) = \sum_{m=0}^{k-1} \left(1 - \frac{m}{k}\right) D_{k-m}(x)D_m(e^x),$$

which is the same statement as was provided in [3] (see Theorem 2.4 in [3]). In addition to this, employing the Euler identity $e^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$, the formulae given in Theorem 2.2 successfully generate the homotopy derivatives $D_k(\sin x)$ and $D_k(\cos x)$ (used, for instance, for the solution of the simple pendulum problem [8], the nonlinear eigenvalue problem of a beam with uniform cross-section acted on by an axial load [9] and the large deformation of a cantilever beam under a point load at the free tip problem [10]), which conforms exactly with Theorem 2.5 of [3].

In the light of explanatory Examples 1 and 2 above, it can be deduced that Theorem 2.2 stated here generalizes the homotopy derivative of some basic functions used in research in the literature. In particular, any non-integer powers of x can be differentiated at any order via the formulae (2.4) as shown in the next example.

Example 3. Let us consider now a nonlinear algebraic equation

$$f(x) = 0, \tag{3.6}$$

with the aim of finding the roots, where the function f is differentiable up to any desired order; see for instance [11]. The homotopy analysis method necessitates the construction of such a homotopy as

$$H[x; p] = (1 - p)F(x, x_0) + phf(x), \tag{3.7}$$

where $F(x, x_0)$ is any suitable function approximating the initial guess x_0 of x, h is an auxiliary parameter for speeding up the convergence and $p \in [0, 1]$ is called the homotopy parameter. For a more general homotopy see [2]. Although it is not necessary, for our purposes, $F(x, x_0)$ can be simply chosen as $f(x) - f(x_0)$ as in [3]. Obviously, at $p = 0$ and $p = 1$, one has

$$H[x; 0] = f(x) - f(x_0), \qquad H[x; 1] = f(x),$$

respectively. Thus, as p increases from 0 to 1, $H[x; p]$ varies continuously from $f(x) - f(x_0)$ to $f(x)$. Such continuous variation is called continuous deformation in topology. Enforcing $H[x; p] = 0$ in (3.7), as the homotopy parameter p increases from 0 to 1, the one-parameter family of solutions $x(p)$ varies (or deforms) from the initial guess x_0 to the solution x of $f(x) = 0$ in (3.6). Moreover, the Maclaurin series expansion of $x(p)$ gives the homotopy series (2.1) of x. If the homotopy series (2.1) is convergent at $p = 1$, a condition that is generally met with proper choices of the auxiliary parameter h, then using the relationship $x(p = 1) = x$, one has the so-called homotopy series solution x of (3.6), expressed in the form

$$x = x_0 + \sum_{k=1}^{\infty} x_k. \tag{3.8}$$

The kth order of approximation to the solution x is generally sufficient in practice, defined by

$$x = \sum_{m=0}^{k} x_m.$$

According to a rigorous proof addressed in [2], the convergent solution of the homotopy series (3.8) necessarily converges to the true solution of the relevant nonlinear equation (3.6). Therefore, there is no doubt that the converged solutions as shown in the figures below fully represent the solutions of the corresponding nonlinear equations.

Part 1. We consider the following algebraic nonlinear equation:

$$x^{1/3} - \mathrm{e}^{-x} = 0. \tag{3.9}$$

Selecting the initial solution to (3.9) as $x_0 = 1$, fixing $h = -0.8$ and taking into account the homotopy introduced in (3.7) within the framework of the homotopy derivative Theorem 2.2, we obtain the 14th-order homotopy result $x = 0.34997$ which is the exact solution to the desired accuracy.

Part 2. Now consider the following algebraic nonlinear equation involving the irrational powers of x and also an exponential function:

$$x^{\sqrt{2}} + 2x^{\sqrt{3}} - \pi^x = 0. \tag{3.10}$$

Assuming the initial approximation to the solution as $x_0 = 1$ together with $h = 1.1$, the fifth-order homotopy result is obtained as $x = 1.11697$. The other root of (3.10) can be reached by the assignment of the initial guess and auxiliary parameter, respectively, as $x_0 = 2$ and $h = 1.3$, which produces at the twelfth order the homotopy result $x = 1.79006$. Both results are accurate enough as compared to the exact roots. It should be clarified that to get the homotopy derivatives of the functions of type x^α and β^x (α and β are constants) involved in this example, Theorem 2.2 generates respectively the following:

$$x_0^\alpha, \alpha x_0^{-1+\alpha} x_1, \frac{1}{2}(-1+\alpha)\alpha x_0^{-2+\alpha} x_1^2 + \alpha x_0^{-1+\alpha} x_2, \ldots,$$

$$\beta^{x_0}, \beta^{x_0} x_1 \ln(\beta), \frac{1}{2}\beta^{x_0} x_1^2 \ln(\beta)^2 + \beta^{x_0} x_2 \ln(\beta), \ldots.$$

Part 3. Let's now consider a functionally involved non-algebraic and nonlinear equation:

$$\ln\left(\sin(\sqrt{x})\right) + \mathrm{e}^{x^2} = 0. \tag{3.11}$$

The exact solution is given by $x = 0.136437$. The homotopy analysis method results in the same value when the choices of $x_0 = 0.2$ and $h = 0.9$ are made at the seventh order of approximation. Theorem 2.2 in this case produces the homotopy derivatives for the natural logarithmic function $\ln(x)$ as follows:

$$\ln(x_0), \frac{x_1}{x_0}, -\frac{x_1^2}{2x_0^2} + \frac{x_2}{x_0}, \ldots.$$

Example 4. Finally, it should be recalled that the homotopy derivative as introduced in Theorem 2.2 can be further made use of in the solution of differential equations. For instance, [12] studied the strongly nonlinear Gelfand equation

$$u'' + \lambda \mathrm{e}^u = 0, \qquad u(0) = u(1) = 0. \tag{3.12}$$

The Gelfand equation (3.12) represents the steady state of diffusion and transfer of heat conduction of a thermal reaction process in combustion [13].

In order to make it solvable through the homotopy analysis method, Eq. (3.12) was first converted into another heavily involved nonlinear differential equation not containing the exponential term; see Eqs. (4), (5) of [12]. However, it is believed that such an operation requires much more computational effort and computational cost. In place of that, using the same homotopy as in [12] while conserving the original equation, Theorem 2.2 can be used for the computation of homotopy derivatives. Details have been omitted here for the sake of not duplicating the procedure. In this way, assuming an initial guess for the solution as $u_0 = 0$ naturally satisfying the boundary conditions and also taking the auxiliary parameter as $h = 1.3$, we get the results for $\lambda = 2$ that $u'(0) = 1.248217517$ and $u(1/2) = 0.328952421$ at the 18th order of approximation, which compare excellently with their exact counterparts.

It should be emphasized that with only a few orders of approximation the solutions that we obtained reveal excellent agreement with the exact numerical solutions. Addition of higher approximations from the homotopy technique would naturally yield more remarkable agreement. Finally, it should be addressed that the homotopy derivative as introduced in Theorem 2.2 can be further used for the approximate analytic solution of nonlinear differential equations governing nonlinear phenomena involving a general function of x, not just restricted to the functions studied in the literature. In the absence of Theorem 2.2, non-natural powers of x could not have been studied; for instance, see, amongst many other examples, the boundary layer flows of non-Newtonian fluids near a forward stagnation point [14] (Eq. (9)) and nanoboundary layer flows involving a nonlinear Navier boundary condition ([15], Eqs. (8), (9)), which can now be accessed and evaluated thanks to Theorem 2.2.

4. Concluding remarks

The homotopy analysis method, a powerful analytic approach for obtaining convergent series solutions of strongly nonlinear problems governing physical models in applied science and engineering, has been the main concern in this work. Using the homotopy derivative concept as described in [3], some lemmas and theorems provided in [3,4] and in other homotopy papers in the literature have been unified here via a new theorem.

The applicability of the theorem in some practical situations as regards the algebraic and differential nonlinear equations has been demonstrated. It has been shown that one of the earlier limitations on the use of the homotopy analysis method for problems involving strong nonlinearity is remedied via the theorem given here.

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