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Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey

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Abstract

In this brief survey special attention is paid to some recent procedures for constructing optimal interpolation processes, i.e., with Lebesgue constant having logarithmic behaviour. A new result on Lagrange interpolation based on the zeros of the associated Jacobi polynomials and on suitable additional nodes is given.

1. Introduction

The Lagrange interpolating polynomials are useful in several branches of numerical analysis and approximation theory. They are easily computable and can be useful tools to approximate functions in different metrics if the corresponding Lebesgue constants are optimal, i.e., have a logarithmic behaviour in the uniform norm or are uniformly bounded in the case of integral norms.

Until the end of the 1970s very few interpolatory processes were known having the above-mentioned properties. In the last decade the efforts of several authors were successfully devoted to the construction of large classes of interpolatory processes. Very recently, it was also proved that in some suitable function spaces Lagrange interpolation is equivalent to best approximation.

In this short survey we limit ourselves to the case of uniform metric. We present the main results and the recently used procedures for constructing optimal interpolation processes which are "really" computable. Moreover, we prove that, if we add a suitable number of knots near the endpoints of $[-1,1]$ to the zeros of the associated Jacobi polynomials, we obtain an interpolation process with

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optimal Lebesgue constants. This result is stated in Theorem 3.3, the proof of which is given in Section 4.

2. Trigonometric interpolation

We start with trigonometric interpolation for continuous 2π-periodic functions f ($f \in C_{2\pi}$), since this is a very simple and efficient process and for these reasons it is a natural starting point for algebraic interpolation.

Let

$$S_m(f,t) = \frac{a_0}{2} + \sum_{k=1}^{m} [a_k \cos(kt) + b_k \sin(kt)] = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin((2m+1)(x-t)/2)}{2\sin((x-t)/2)} f(x)\,dx, \tag{1}$$

$$f \in C_{2\pi}, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(kt)\,dt, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(kt)\,dt \tag{2}$$

be the mth Fourier sum of f. If we approximate the coefficients a_k and b_k by the quadrature sum

$$\int_0^{2\pi} g(t)\,dt \approx \frac{2\pi}{2m+1} \sum_{k=0}^{2m} g(t_k),$$

where $t_k = [2\pi/(2m+1)]k$, we obtain the unique trigonometric polynomial $L_m^*(f,t) \in \mathscr{T}_m$ interpolating the function f at the $2m+1$ knots $\{t_k\}_{k=0}^{2m}$. An expression for $L_m^*(f,t)$ is

$$L_m^*(f,t) = \frac{A_0}{2} + \sum_{k=1}^{m} [A_k \cos(kt) + B_k \sin(kt)],$$

where

$$A_k = \frac{2}{2m+1} \sum_{j=0}^{2m} f(t_j)\cos(kt_j), \qquad B_k = \frac{2}{2m+1} \sum_{j=0}^{2m} f(t_j)\sin(kt_j).$$

Define the mth Lebesgue constants of L_m^* and S_m by

$$\|L_m^*\| = \sup_{\|f\|=1} \|L_m^*(f)\|,$$

$$\|S_m\| = \sup_{\|f\|=1} \|S_m(f)\|,$$

where

$$\|g\| = \sup_{x \in [0,2\pi)} |g(x)|$$

is the sup-norm in $[0, 2\pi)$. It is well known that (see for instance [20,24]).

$$\|S_m\| \leqslant \|L_m^*\| \leqslant (1+\pi)\|S_m\| \tag{3}$$

and

$$\|S_m\| = \frac{4}{\pi^2} \log m + O(1).$$

Now we need some notations. By $\mathscr{C}$ we denote a positive constant which can be different in different formulas, and we shall write $\mathscr{C} \neq \mathscr{C}(a, b, \ldots)$ if $\mathscr{C}$ is independent of the parameters $a, b, \ldots$.

Moreover, if A and B are two expressions that depend on some variables, then we write

$$A \sim B \text{ iff } |AB^{-1}| \leqslant \mathscr{C}, \quad |A^{-1}B| \leqslant \mathscr{C}, \ \mathscr{C} \in \mathbb{R}$$

uniformly with respect to the variables in question.

As a consequence of (3), since S_m and L_m^* are projectors, error estimates are equivalent with respect to the convergence order, i.e.,

$$\|f - S_m\| \leqslant \mathscr{C} E_m^*(f) \log m$$

and

$$\|f - L_m^*\| \leqslant \mathscr{C} E_m^*(f) \log m, \tag{4}$$

where

$$E_m^*(f) = \min_{T \in \mathscr{T}_m} \|f - T\|$$

is the best uniform approximation error and $\mathscr{C}$ is an absolute constant.

Obviously, the construction of $L_m^*(f)$ is simpler than the construction of the mth Fourier partial sum $S_m(f)$. Moreover, $L_m^*(f)$ can be successfully applied to numerical differentiation.

Indeed, if $C_{2\pi}^k$ denotes the space of functions f such that $f^{(k)} \in C_{2\pi}$, the following theorem holds.

Theorem 2.1. *For any function $f \in C_{2\pi}^r$, we have*

$$\|f^{(k)} - (L_m^*(f))^{(k)}\| \leqslant \mathscr{C} E_m^*(f^{(k)}) \log m, \quad 0 < k \leqslant r, \tag{5}$$

where $\mathscr{C}$ is a positive constant independent of m and f.

The proof of this theorem will be given in Section 4.

The best uniform approximation error $E_m^*(f)$ can be estimated by Jackson's Theorem

$$E_m^*(f) \leqslant \mathscr{C} \omega^r\left(f, \frac{1}{m}\right), \quad 1 \leqslant r < m,$$

where $\omega^r(f, t)$ is the ordinary rth modulus of continuity. In particular, if $\|f^{(k)}\| < \infty$,

$$E_m^*(f) \leqslant \mathscr{C} \frac{\|f^{(k)}\|}{m^k}.$$

The following numerical test confirms the theoretical estimates given in (4) and (5).

Example.

$$f(x) = |\sin(x)|^3, \quad \|f^{(3)}\| \leqslant \mathscr{C}$$

m	$\|f - L_m^*(f)\|_\infty$	$\|f' - (L_m^*(f))'\|_\infty$
50	$1.86E-5$	$1.19E-4$
100	$2.39E-6$	$3.02E-5$
250	$1.55E-7$	$4.85E-6$

3. Algebraic interpolation

In order to consider algebraic interpolation, we recall some definitions and well-known results.

Let $\mathscr{X}=(q_m)_{m=1,2,\dots}$ be a sequence of polynomials such that, for any $m\geqslant 1$, q_m has exactly degree m ($q_m\in\mathbb{P}_m$) and m simple zeros $x_{m,1}<x_{m,2}<\cdots<x_{m,m}$ belonging to $[-1,1]$. If f is a continuous function in $[-1,1]$ ($f\in C^0$), we can associate with $\mathscr{X}$ the sequence of Lagrange polynomials $\{L_m(\mathscr{X},f)\}$ defined by

$$L_m(\mathscr{X},f,x)=\sum_{k=1}^{m} l_{m,k}(x)f(x_{m,k}),\quad l_{m,k}(x)=\frac{q_m(x)}{q_m'(x_{m,k})(x-x_{m,k})}\in\mathbb{P}_{m-1}.$$

Let us introduce in C^0 the sup-norm, i.e., $\|f\|=\max_{|x|\leqslant 1}|f(x)|$, and define in the usual way the Lebesgue constants as

$$\|L_m(\mathscr{X})\|=\sup_{\|f\|=1}\|L_m(\mathscr{X},f)\|,\quad m=1,2,\dots\ .$$

This sequence characterizes the quality of the interpolation process, since for $f\in C^0$,

$$\|f-L_m(\mathscr{X},f)\|\leqslant(1+\|L_m(\mathscr{X})\|)E_{m-1}(f),\tag{6}$$

where

$$E_{m-1}(f)=\min_{P\in\mathbb{P}_{m-1}}\|f-P\|$$

is the error of the best uniform approximation by algebraic polynomials.

In the numerical approximation of a function f it is often more useful to consider the following estimate:

$$\|f-L_m(\mathscr{X},f+\eta)\|\leqslant(1+\|L_m(\mathscr{X})\|)E_{m-1}(f)+\eta_m\|L_m(\mathscr{X})\|,\tag{7}$$

where η is a perturbation of the function f (due for instance to the evaluation of $f(x_{m,k})$) and $\eta_m=\max_{1\leqslant k\leqslant m}|\eta(x_{m,k})|$.

So, when the number of the interpolation knots increases, the first term on the right of (7) can be "small" while the second can take very "large" values. This happens, for example, if we use the interpolation points

$$\left\{-1+\frac{2}{m}k,\ k=0,1,\dots,m,\ m=2,3,\dots\right\},$$

since in this case $\|L_m(\mathscr{X})\|\sim \mathrm{e}^{m/2}$.

To estimate $E_m(f)$, we can use the algebraic version of the Jackson Theorem, using the φ-modulus of continuity [8]. That is,

$$E_m(f)\leqslant\mathscr{C}\omega_\varphi^r\left(f,\frac{1}{m}\right),\quad 1\leqslant r<m,\ \mathscr{C}\neq\mathscr{C}(m,f),$$

where

$$\omega_\varphi^r(f,t)=\sup_{0<h\leqslant t}\|\Delta_{h\varphi}^r f)\|_\infty$$

with

$$\varphi(x)=\sqrt{1-x^2}$$

and

$$\Delta^r_{h\varphi}f(x)=\sum_{i=0}^{r}(-1)^i\binom{r}{i}f\left(x+\frac{rh}{2}\varphi(x)-ih\varphi(x)\right).$$

In particular, if $\|f^{(k)}\varphi^k\|=\max_{|x|\leqslant 1}|f^{(k)}(x)|(\sqrt{1-x^2})^k<\infty$, then

$$E_m(f)\leqslant \mathscr{C}\frac{\|f^{(k)}\varphi^k\|}{m^k}.$$

Faber and Bernstein independently proved that for any $\mathscr{X}$, $\|L_m(\mathscr{X})\|>(2/\pi)\log m$. Hence, it is interesting to find sequences $\mathscr{X}$ with Lebesgue constants behaving like $\log m$, since in this case, if η_m is "small", (7) is numerically equivalent to (6).

In order to get a "good" $\mathscr{X}$, we mention two important ingredients, deriving from trigonometric interpolation. The first is the so-called "arc cosine distribution" of the zeros $x_{m,i}=\cos\theta_{m,i}$, $i=1,2,\dots,m$ of q_m, i.e.,

$$|\theta_{m,i}-\theta_{m,i+1}|\sim m^{-1},\quad i=0,1,\dots,m,\quad \theta_0=\pi,\quad \theta_{m+1}=0. \tag{8}$$

In fact, from a result in [39, p. 50], we get that, whenever, for some i and $m>m_0$, we have

$$|\theta_{m,i}-\theta_{m,i+1}|\sim\frac{1}{m^{1+\alpha}},\quad \alpha>0,$$

then

$$\|L_m(\mathscr{X})\|\geqslant\mathscr{C}m^{\alpha},\quad \mathscr{C}\neq\mathscr{C}(m).$$

The second ingredient is the uniform boundedness of the sequence $\mathscr{X}=(q_m)_m$, i.e., $\sup_m\|q_m\|<\infty$. In fact it holds the following:

Proposition A. *Let be* $\mathscr{X}=(q_m)_m$. *If for any* m *the zeros of* q_m *have an arc cosine distribution and, moreover,* $\sup_m\|q_m\|<\infty$, *then*

$$\|L_m(\mathscr{X})\|\geqslant\mathscr{C}\|q_m\|.$$

The previous proposition is an easy consequence of the Theorem 2.2 in [28] and a short proof is given in the Section 4.

We observe that the uniform boundedness of $(q_m)_m$ and the arc cosine distribution of its zeros are two independent conditions as is shown by the sequence of Chebyshev polynomials of the second kind $\{U_m(x)\}$ and $\{T_mT_{m+1}\}_m$. Indeed, the zeros of U_m satisfy (8) and $\|U_m\|=m$, while $\|T_mT_{m+1}\|=1$, but the zeros $x_i=\cos\theta_i$ of T_mT_{m+1} satisfy

$$\min_i|\theta_i-\theta_{i+1}|\sim\frac{1}{m^2}.$$

From the above reasoning we conclude that "candidates" for "good" interpolation processes are uniformly bounded sequences $\mathscr{X}$ with arc cosine distribution of their zeros.

The latter property is satisfied for many sequences of orthogonal polynomials in $[-1,1]$ but, unfortunately, only few of them are uniformly bounded. For this reason, at the end of the 1980s, we knew of only a few "good" interpolation processes.

On the other hand, the Lagrange interpolation polynomials are easily computable and, also for this reason, they are useful in the approximation of functions and their derivatives, in numerical integration and in projection method for the numerical treatment of functional equations. In view of these considerations, in this last 10 years several papers appeared that construct larger classes of optimal sequences $\mathscr{X}$.

To introduce some results available in the literature, we first state some notations and basic facts.

Setting $v^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta>-1$, $\{p_m(v^{\alpha,\beta})\}_m$ denotes the sequence of orthonormal Jacobi polynomials with positive leading coefficients and $L_m(v^{\alpha,\beta},f)\in\mathbb{P}_{m-1}$ denotes the Lagrange polynomial interpolating f at the zeros of $p_m(v^{\alpha,\beta})$.

A classical result of Szegö [40, Theorem 14.4, p. 335] assures that

$$\|L_m(v^{\alpha,\beta})\|\sim\begin{cases}\log m & \text{if } -1<\alpha,\beta\leqslant-\frac{1}{2},\\ m^{\gamma+1/2} & \gamma=\max(\alpha,\beta),\ \text{otherwise}.\end{cases}\tag{9}$$

According to estimate (9), and taking into account the estimate [32, (15), p. 673] (see also [40])

$$|p_m(v^{\alpha,\beta})|\leqslant\mathscr{C}\left(\sqrt{1-x}+\frac{1}{m}\right)^{-\alpha-(1/2)}\left(\sqrt{1+x}+\frac{1}{m}\right)^{-\beta-(1/2)},$$
$$|x|\leqslant 1,\quad \mathscr{C}\neq\mathscr{C}(m,x),\tag{10}$$

the reader can note that the sequence $\{p_m(v^{\alpha,\beta})\}$ is uniformly bounded in $[-1,1]$ for $-1<\alpha,\beta\leqslant-\frac{1}{2}$, and unbounded for $\alpha,\beta>-\frac{1}{2}$.

Moreover, in any segment $[a,b]\subset[-1,1]$, there holds $\sup_m\max_{x\in[a,b]}|p_m(v^{\alpha,\beta},x)|<\infty$, and in $[a,b]$ the Lebesgue constants behave like $\log m$ [40, Theorem 14.4, p. 335].

If we want to use the zeros of $p_m(v^{\alpha,\beta})$ with $\alpha,\beta>-\frac{1}{2}$, we may consider the following procedure, which is known as "*additional nodes method*".

Denote by $x_1<x_2<\cdots<x_m$ the zeros of $p_m(v^{\alpha,\beta})$ and define the nodes

$$y_j=-1+\frac{1+x_1}{1+s}j,\quad j=1,2,\ldots,s,$$

$$z_i=x_m+\frac{1-x_m}{1+r}i,\quad i=1,2,\ldots,r$$

and the polynomials

$$Y_s(x)=\prod_{j=1}^{s}(x-y_j),\quad Z_r(x)=\prod_{i=1}^{r}(x-z_i).\tag{11}$$

In case y_1 and z_r can be replaced by -1 and 1, respectively.

If $L_{m,r,s}(f)\in\mathbb{P}_{m+r+s-1}$ denotes the Lagrange polynomial interpolating f at the zeros of $Y_sZ_rp_m(v^{\alpha,\beta})$, we have the following theorem.

Theorem 3.1. *If the parameters* α, β, r, s *satisfy the relations*

$$\frac{\alpha}{2}+\frac{1}{4}\leqslant r<\frac{\alpha}{2}+\frac{5}{4},$$

$$\frac{\beta}{2}+\frac{1}{4}\leqslant s<\frac{\beta}{2}+\frac{5}{4},$$

then

$$\|L_{m,r,s}\| \sim \log m.$$

The proof of this theorem, which can be found in [33] (see also [25]), is essentially based on the following bounds:

$$|Y_s(x)Z_r(x)p_m(v^{\alpha,\beta},x)|\leqslant \mathscr{C}(\sqrt{1-x}+m^{-1})^{-\alpha-(1/2)+2r}(\sqrt{1+x}+m^{-1})^{-\beta-(1/2)+2s}, \quad |x|\leqslant 1$$

and

$$\sum_{i=1, i\neq d}^{m}\frac{v^{\gamma,\delta}(t_i)}{|x-t_i|}\Delta t_i\leqslant \mathscr{C}(\sqrt{1-x}+m^{-1})^{2\gamma}(\sqrt{1+x}+m^{-1})^{2\delta}\log m, \quad \Delta t_i=t_{i+1}-t_i,$$

which holds for any node system $-1<t_1<\cdots<t_m<1$ with arc cosine distribution, and where $|t_d-x|=\min_{1\leqslant k\leqslant m}|t_k-x|$ and $-1<\gamma,\delta\leqslant 0$.

Theoretically, Theorem 3.1 assures that it is possible to use the zeros of the Jacobi polynomials $\{p_m(v^{\alpha,\beta}\}$ also for $\alpha,\beta>-\frac{1}{2}$, to get the optimal order $\log m$.

Special case of this result $\{\cos(\pi/m)k,\ k=0,1,\ldots,m\}$ (the practical abscissas [3]) (and $\{\cos[2\pi/(2m+1)]k,\ k=0,1,\ldots,m\},$) are well known.

Theorem 3.1, by (6), implies the estimate

$$\|f-L_{m,r,s}(f)\|\leqslant \mathscr{C}E_{m-1}(f)\log m.$$

This estimate is the analogue of (4) for algebraic interpolation.

It is natural to ask if (5) can be extended to interpolation of smooth nonperiodic functions. A positive answer to this question would allow us to approximate the derivative f' of a function f by using only the values of f in some preassigned points.

Now, for any arbitrary sequence $\mathscr{X}$, we can obtain the estimates

$$\|[f-L_m(\mathscr{X},f)]'\|\leqslant \mathscr{C}m\|L_m(\mathscr{X})\|E_{m-2}(f'),$$

$$\|[f-L_m(\mathscr{X},f)]'\varphi\|\leqslant \mathscr{C}\|L_m(\mathscr{X})\|E_{m-2}(f')_{\varphi}, \tag{12}$$

where

$$E_{m-2}(f')_{\varphi}=\inf_{P\in\mathbb{P}_{m-2}}\|(f-P)'\varphi\|, \quad \varphi(x)=\sqrt{1-x^2}$$

and $\mathscr{C}\neq\mathscr{C}(m,f)$. Estimates (12) will be proved in the Section 4. But the (12) cannot be considered extensions of (5), even if $\|L_m(\mathscr{X})\|\sim\log m$.

On the other hand, by using the additional nodes method, we can state the following theorem, which, in some sense, extends (5) to the algebraic case.

Theorem 3.2. *With the same notations as in Theorem* 3.1, *for any function* $f \in C^q, q \geqslant 0$, *one has*

$$\|[f - L_{m,r,s}(f)]^{(i)}\| \leqslant \mathscr{C} \frac{E_{m-q}(f^{(q)})}{m^{q-i}} \log m, \quad i = 0, 1, \ldots, q, \ \mathscr{C} \neq \mathscr{C}(m, f),$$

if the parameters α, β, r, s, q *satisfy*

$$\tfrac{1}{2}(\alpha + q) + \tfrac{1}{4} \leqslant r < \tfrac{1}{2}(\alpha + q) + \tfrac{5}{4},$$

$$\tfrac{1}{2}(\beta + q) + \tfrac{1}{4} \leqslant s < \tfrac{1}{2}(\beta + q) + \tfrac{5}{4}.$$

The interested reader can find the proof of Theorem 3.2 in [33] or [25].

A brief history of the additional nodes method may be in order. According to our knowledge, in 1958 Egerváry and Turán [9] were the first to use the points ± 1. They proved that the sequence of Hermite–Fejér polynomials based on the Legendre zeros plus ± 1 was uniformly convergent (this result is false if we drop the points ± 1).

The first use of the additional points in Lagrange–Hermite interpolation is due to Szasz [39] (1959), while the use of the points ± 1 appeared in some papers by Freud [13,14] and Vértesi [41,42]. In 1987, Szabados [35,36] was the first who successfully used not only ± 1, but other additional points to minimize the norm of the derivatives of the Lagrange polynomials based on the Chebyshev zeros of the first kind. This problem was deeply investigated in some papers by Szabados and Vértesi [37], and by Halaśz [19]. In [33], and subsequently in [25], simultaneous interpolation processes based on the zeros of Jacobi polynomials were constructed. This procedure was then extensively used by several authors and in different contexts, and nowadays goes under the name of "Additional Nodes Method". For an exhaustive bibliography, the interested reader can consult [38, p. 279; 6] and the references therein.

We now prove a theorem on the existence of a new class of optimal interpolation processes.

If we associate with the Jacobi weight $v^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ the new weight function

$$w^{\alpha,\beta}(x) = \frac{v^{\alpha,\beta}(t)}{\pi^2 v^{2\alpha,2\beta}(t) + H^2(v^{\alpha,\beta}; t)},$$

where $H(g)$ is the finite Hilbert transform of the function g, i.e.,

$$H(g; t) = \int_{-1}^{1} - \frac{g(x)}{x - t}\, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{|x-t| \geqslant \varepsilon} \frac{g(x)}{x - t}\, \mathrm{d}x,$$

then $w^{\alpha,\beta}$ is not a classical weight and, at first sight, does not seem easy to handle. However, the corresponding orthogonal sequence $\{p_m(w^{\alpha,\beta})\}_m$ can be deduced from the corresponding system of Jacobi polynomials. In fact, let

$$p_m(v^{\alpha,\beta}, x) = \gamma_m(v^{\alpha,\beta}) x^m + \cdots, \quad \gamma_m(v^{\alpha,\beta}) > 0$$

and

$$x\, p_m(v^{\alpha,\beta}; x) = a_{m+1}\, p_{m+1}(v^{\alpha,\beta}; x) + b_m\, p_m(v^{\alpha,\beta}; x) + a_m\, p_{m-1}(v^{\alpha,\beta}; x),$$

where

$$p_{-1}(v^{\alpha,\beta};x)=0, \quad p_0(v^{\alpha,\beta};x)=\frac{1}{\sqrt{\int_{-1}^{1} v^{\alpha,\beta}(x)\,\mathrm{d}x}},$$

$$a_m=\frac{\gamma_{m-1}(v^{\alpha,\beta})}{\gamma_m(v^{\alpha,\beta})}, \quad \text{and} \quad b_m=\int_{-1}^{1} x p_m^2(v^{\alpha,\beta},x)v^{\alpha,\beta}(x)\,\mathrm{d}x$$

be the three-term recurrence relation for Jacobi polynomials. Then the sequence $\{p_m(w^{\alpha,\beta})\}$ satisfies the following relation:

$$x p_{m-1}(w^{\alpha,\beta};x)=a_{m+1}p_m(w^{\alpha,\beta};x)+b_m p_{m-1}(w^{\alpha,\beta};x)+a_m p_{m-2}(w^{\alpha,\beta};x), \quad m\geqslant 2,$$

where

$$p_{-1}(w^{\alpha,\beta},x)=0, \quad p_0(w^{\alpha,\beta},x)=\frac{1}{p_0^2(v^{\alpha,\beta})a_1}.$$

From this link, the zeros of $p_m(w^{\alpha,\beta})$, and the Christoffel numbers related to $w^{\alpha,\beta}$, can be computed solving the eigenvalue problem of the corresponding Jacobi matrix.

For more details on these polynomials, which are called “associated polynomials”, the interested reader may consult [1,2,4,18,22,31,34]. In addition, the zeros of $p_m(w^{\alpha,\beta})$ interlace with the zeros of $p_{m+1}(v^{\alpha,\beta};x)$.

Now, if we consider the sequence

$$\mathscr{X}=\{Y_s(x)Z_r(x)p_m(w^{\alpha,\beta};x)\}_{m=1,2,\ldots},$$

we can establish the following theorem.

Theorem 3.3. *The zeros of $p_m(w^{\alpha,\beta})$ have an arc cosine distribution. Moreover, if $L_{m,r,s}(w^{\alpha,\beta},f)$ denotes the Lagrange polynomial interpolating a given function f at the zeros of $Y_s(x)Z_r(x)p_m(w^{\alpha,\beta};x)$, we have*

$$\|L_{m,r,s}(w^{\alpha,\beta})\|_\infty \sim \log m \tag{13}$$

whenever the parameters α,β,r,s satisfy

$$\frac{|\alpha|}{2}+\frac{1}{4}\leqslant r<\frac{|\alpha|}{2}+\frac{5}{4},$$

$$\frac{|\beta|}{2}+\frac{1}{4}\leqslant s<\frac{|\beta|}{2}+\frac{5}{4}.$$

The proof of this theorem is given in Section 4.

We remark that, since $|\alpha|,|\beta|$ assume nonnegative values, the number of additional points r and s is greater than or equal to 1. The case $\alpha=\beta=0$ was separately considered in [26] and is now a special case of Theorem 3.3.

Now, we briefly want to mention the so-called “extended interpolation”. The underlying idea is to interpolate the function on the zeros of the sequence $\{q_M\}=\{p_m(v^{\alpha,\beta})p_n(v^{\gamma,\delta})Y_sZ_r\}$, where Y_s and Z_r are defined as in (11) and the parameters $\alpha,\beta,\gamma,\delta,r,s$ (and analogously m and n) are suitable related. Extended interpolation turns out to be useful for the numerical evaluation of the interpolation error based on zeros of orthogonal polynomials. More explicitly, let $L_m(w;f)$ be the polynomial interpolating f at the zeros of $p_m(w)$ and, say, $L_{2m}(w,u;f)$ the extended interpolation polynomial based on the zeros of $p_m(w)p_m(u)$. In practice, the difference $|L_n(w;f)-L_m(w;f)|$, $n>m$, is assumed to be the error of $L_m(w;f)$. So, if $n=m+1$, following this procedure, to compare $L_m(w;f)$ with $L_{m+1}(w;f)$, $2m+1$ evaluations of the function are needed. On the other hand, by using only $2m$ evaluations of the function f, we can compare $L_m(w;f)$ with $L_{2m}(w,u,f)$, which is more precise when both previous polynomials have the same order of convergence to f.

With regard to the convergence of this process, from what we have said before, it is necessary that the interpolation points have an arc cosine distribution and that $\sup_M \|q_M\| < \infty$. It is not difficult to satisfy the last condition. In fact, from the pointwise estimate for the Jacobi polynomials, it is possible to determine r and s (number of the additional nodes) in such a way that the sequence $\{q_M\}$ is uniformly bounded with respect to m.

The choice of parameters $\alpha,\beta,\gamma,\delta,r,s,m$ *and* n, *such that the zeros of* q_M *have an arc cosine distribution*, *is still an open problem.*

However, some important examples are known.

Consider the two sequences

$$\{q_{2m+2}\} = \{(1-x^2)p_m(v^{\alpha,-\alpha})p_m(v^{-\alpha,\alpha})\}$$

and

$$\{\tilde{q}_{2m+3}\} = \{(1-x^2)p_m(v^{\alpha,\beta})p_{m+1}(v^{-\alpha,-\beta})\}, \quad 0<\alpha,\beta<1, \alpha+\beta=1.$$

The zeros of q_{2m+2} and/or $\tilde{q}_{2m+3}$ are used in some quadrature methods and in the numerical treatment of singular integral equations (see for instance [27]).

If we denote by $L_{2m+2}f$ and $L_{2m+3}f$ the Lagrange polynomials based on the zeros of q_{2m+2} and $\tilde{q}_{2m+3}$, respectively, the following theorem was proved in [27]:

Theorem 3.4. *The zeros of* q_{2m+2} *and* $\tilde{q}_{2m+3}$ *have an arc cosine distribution and*, *moreover*,

$$\|L_{2m+2}\| \sim \log m \sim \|L_{2m+3}\|.$$

The following sequences are additional significant examples:

$$\{Q_{2m+1}\}_m = \{p_{m+1}(v^{\alpha,\beta})p_m(v^{\alpha+1,\beta+1})Y_s Z_r\},$$

$$\{\tilde{Q}_{2m}\}_m = \{p_m(v^{\alpha+1,\beta})p_m(v^{\alpha,\beta+1})Y_s Z_r\},$$

where Y_s and Z_r are defined as in (11), by replacing in those definitions x_1 and x_m with the first and the last zero of Q_{2m+1}, respectively (analogously for $\tilde{Q}_{2m}$). If we denote by $\{L_{2m+1}f\}_m$ and $\{\tilde{L}_{2m}f\}_m$ the sequences of the Lagrange polynomials related to these polynomial sequences, the following result holds.

Theorem 3.5. *The polynomials* Q_{2m+1} *and* $\tilde{Q}_{2m}$ *have simple zeros in* $[-1,1]$, *both having an arc cosine distribution. Moreover*,

$$\|L_{2m+1}\| \sim \log m \sim \|\tilde{L}_{2m}\|$$

holds if the parameters α, β, r, *and* s *satisfy the relations*

$$\alpha + 1 \leqslant r < \alpha + 2, \quad \beta + 1 \leqslant s < \beta + 2.$$

The first part of Theorem 3.5 is a consequence of a more general result proved in [5], while the second part can be found in [7].

We finally want to mention a result that recently appeared in [11]. In this paper the authors consider the sequence $\{R_{2m+1}\} = \{p_m E_{m+1}\}_{m \geqslant 1}$, where p_m is the mth Legendre polynomial and E_{m+1} is the $(m+1)$th Stieltjes polynomial defined by

$$\int_{-1}^{1} E_{m+1}(x) p_m(x) x^k \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, m, \ m \geqslant 1.$$

The zeros of R_{2m+1} have been used by Kronrod to construct the well-known extended quadrature formula, which was later extensively studied by several authors [10,12,15,16,29,30]. In [11] the authors study the distribution of the zeros of R_{2m+1} and the sequence $\{\mathscr{L}_{2m+1} f\}$ of Lagrange polynomials based on these zeros.

The result is as follows:

Theorem 3.6. *The zeros of* R_{2m+1} *have an arc cosine distribution and, moreover*,

$$\|\mathscr{L}_{2m+1}\| \sim \log m.$$

4. Proofs

Proof of Theorem 2.1. The theorem follows from the well-known relation $(S_m f)' = S_m f'$, from the Bernstein inequality $\|T_m'\| \leqslant m \|T_m\|$, which holds for any trigonometric polynomial of degree m, and from the Favard inequality

$$E_m^*(f) \leqslant \frac{\mathscr{C}}{m} E_m^*(f').$$

Indeed,

$$\begin{aligned} \|(f - L_m^*(f))'\| &\leqslant \|f' - S_m f'\| + \|(S_m f - L_m^*(f))'\| \\ &\leqslant \mathscr{C}[E_m^*(f') \log m + m \|S_m f - L_m^*(f)\|] \\ &\leqslant \mathscr{C}[E_m^*(f') \log m + m E_m^*(f) \log m] \leqslant \mathscr{C} E_m^*(f') \log m. \end{aligned} \tag{14}$$

The theorem now follows on induction over k. □

Proof of Proposition A. In [28] the authors, using a slightly changed notation, proved the following.

Theorem B. *Let* u, w *be weights and* $\mathscr{X} = (q_m)_m$. *Assume that every interval* $I \subset [-1, 1]$ *with* $\int_I w > 0$ *contains at least one root of* q_m, *whenever* m *is sufficiently large* ($\mathscr{X}$ w-*regular*). *Then for* *every* p, $0 < p \leqslant \infty$

$$\left(\int_{-1}^{1} |q_m(x)|^p u(x)\,\mathrm{d}x\right)^{1/p} \leqslant \mathscr{M}\left(\int_{-1}^{1} |q_m(x)| w(x)\,\mathrm{d}x\right) \times \sup_{\|g\|_\infty = 1}\left(\int_{-1}^{1} |L_m(\mathscr{X}, g, x)|^p u(x)\,\mathrm{d}x\right)^{1/p}$$

with a proper $\mathscr{M} > 0$.

Here $\mathscr{M} = \mathscr{M}(u, w, \mathscr{X}, p)$ does not depend on m. Further, if $0 < q_0 \leqslant p \leqslant \infty$ (q_0 fixed), then $\mathscr{M} > 0$ will not depend on p.

Proposition A follows from Theorem B by setting $u = w = 1$ and $p = \infty$. Since, for any m, the zeros of q_m have an arc cosine distribution, then $\mathscr{X}$ is 1-regular. Since $\sup_m \|q_m\| =: q < \infty$, and recalling that $\mathscr{M}$ for large value of p is independent of p, it follows

$$\|q_m\| \leqslant 2\mathscr{M}(1, 1, \mathscr{X}) q \|L_m(\mathscr{X})\|. \qquad \square$$

Proof of (12). From a result of Gopengauz–Telyakovskii [17, Theorem p. 113] it follows that for any function $f \in C^1([-1, 1])$, there exist an algebraic polynomial G_m of degree $m \geqslant 9$, such that

$$\|(f - G_m)^{(i)}\| \leqslant \frac{\mathscr{C}}{m^{1-i}} \omega\left(f', \frac{1}{m}\right), \quad i = 0, 1. \tag{15}$$

By using the previous inequality and the Markov polynomial inequality, for any $\mathscr{X}$ it follows

$$\begin{aligned} \|(f - L_m(\mathscr{X}, f))'\| &\leqslant \|f' - G_m'\| + \|(G_m - L_m(\mathscr{X}, f))'\| \\ &\leqslant \mathscr{C}[\|f'\| + m^2(\|f - G_m\| + \|f - L_m(\mathscr{X}, f)\|)]. \end{aligned} \tag{16}$$

Now,

$$\|f - G_m\| \leqslant \frac{\mathscr{C}}{m}\|f'\|$$

and

$$\|f - L_m(\mathscr{X}, f)\| \leqslant (1 + \|L_m(\mathscr{X})\|) E_{m-1}(f) \leqslant \frac{\mathscr{C}}{m}\|L_m(\mathscr{X})\|\|f'\|.$$

Then

$$\|(f - L_m(\mathscr{X}, f))'\| \leqslant \mathscr{C} m \|L_m(\mathscr{X})\|\|f'\|,$$

from which, replacing f with $g = f - \int_{-1}^{x} Q_{m-2}(t)\,\mathrm{d}t$, where $\|f' - Q_{m-2}\| = E_{m-2}(f')$, we get

$$\|(f - L_m(\mathscr{X}, f))'\| = \|(g - L_m(\mathscr{X}, g))'\| \leqslant \mathscr{C} m \|L_m(\mathscr{X})\| E_{m-2}(f')$$

with $\mathscr{C} \neq \mathscr{C}(m, f)$, i.e. the first of (12).

Moreover, for any polynomial $Q \in \mathbb{P}_m$ we have (see for instance [21, p. 662])

$$\|(f - Q)'\varphi\| \leqslant \mathscr{C} m \|f - Q\| + \mathscr{C} E_m(f')_\varphi, \quad \varphi(x) = \sqrt{1 - x^2}.$$

Then, using Favard inequality

$$E_m(f) \leqslant \frac{\mathscr{C}}{m} E_{m-1}(f')_\varphi,$$

we have

$$\begin{aligned}\|(f-L_m(\mathscr{X},f))'\varphi\| &\leqslant \mathscr{C}m\|f-L_m(\mathscr{X},f)\|+\mathscr{C}E_{m-1}(f')_\varphi\\ &\leqslant \mathscr{C}m\|L_m(\mathscr{X})\|E_{m-1}(f)+\mathscr{C}E_{m-1}(f')_\varphi\\ &\leqslant \mathscr{C}\|L_m(\mathscr{X})\|E_{m-2}(f')_\varphi.\end{aligned} \tag{17}$$

i.e. the second inequality of (12) follows. □

To prove Theorem 3.3, we first collect some preliminary results.

Denote by $\lambda_m(w^{\alpha,\beta},t)=[\sum_{k=0}^{m-1}p_k^2(w^{\alpha,\beta},t)]^{-1}$ the mth Christoffel function and by $x_{m,i}=\cos\theta_{m,i}$, $i=1,\dots,m$, the zeros of $p_m(w^{\alpha,\beta})$. The following lemma holds:

Lemma 4.1. *For $|t|\leqslant 1$ and m sufficiently large, we have*

$$\frac{1}{\mathscr{C}}\Delta_m(t)\leqslant\frac{\lambda_m(w^{\alpha,\beta},t)}{W_m^{\alpha,\beta}(t)}\leqslant\mathscr{C}\Delta_m(t),\quad \mathscr{C}\neq\mathscr{C}(m,t), \tag{18}$$

where

$$W_m^{\alpha,\beta}(t)=\frac{(\sqrt{1-t}+m^{-1})^{2|\alpha|}\sqrt{1+t}+m^{-1})^{2|\beta|}}{(\log^{\gamma_\alpha}e/(\sqrt{1-t}+m^{-1})+\log^{\gamma_\beta}e/(\sqrt{1+t}+m^{-1}))^2},$$

$$\Delta_m(t)=\frac{\sqrt{1-t^2}}{m}+\frac{1}{m^2},$$

$$\gamma_\alpha=\begin{cases}1 & \text{for } \alpha=0,\\ 0 & \text{for } \alpha\neq 0,\end{cases}$$

$$\gamma_\beta=\begin{cases}1 & \text{for } \beta=0,\\ 0 & \text{for } \beta\neq 0.\end{cases}$$

Moreover,

$$|\theta_{m,i}-\theta_{m,i+1}|\sim m^{-1}. \tag{19}$$

Proof. By Proposition 2.1 in [26] it follows that the weight $w^{\alpha,\beta}$ is equivalent to the weight

$$W^{\alpha,\beta}(t)=\frac{v^{|\alpha|,|\beta|}(t)}{(\log^{\gamma_\alpha}e/(1-t)+\log^{\gamma_\beta}e/(1+t))^2} \tag{20}$$

and therefore $\lambda_m(w^{\alpha,\beta},t)\sim\lambda_m(W^{\alpha,\beta},t)$. Now, $W^{\alpha,\beta}$ is a generalized Ditzian–Totik weight and, for this weights, (18) is true [23]. Estimate (19) easily follows from (18), by using the same arguments as in [26, p. 315]. □

A pointwise estimate of the polynomial $p_m(w^{\alpha,\beta})$ can be found in [4, Theorem 22]. The same estimate can be rewritten in the following form:

$$\begin{aligned}|p_m(w^{\alpha,\beta},t)| &\leqslant \mathscr{C}(\sqrt{1-t}+m^{-1})^{-|\alpha|-(1/2)}(\sqrt{1+t}+m^{-1})^{-|\beta|-(1/2)}\\ &\times\left[\log^{\gamma_\alpha}\frac{e}{\sqrt{1-t}+m^{-1}}+\log^{\gamma_\beta}\frac{e}{\sqrt{1+t}+m^{-1}}\right],\quad t\in[-1,1],\end{aligned} \tag{21}$$

where $\mathscr{C}$ is a positive constant independent of m,t.

To give a careful estimate of the kth fundamental Lagrange polynomial

$$l_{m,k}(w^{\alpha,\beta},t)=\frac{p_m(w^{\alpha,\beta},t)}{(t-x_{m,k})p'_m(w^{\alpha,\beta},x_{m,k})},$$

we need the following lemma, which is a variant of the Pollard transformation of the Christoffel–Darboux kernel defined as

$$K_m(w^{\alpha,\beta};x,t)=\sum_{j=0}^{m-1}p_j(w^{\alpha,\beta},x)p_j(w^{\alpha,\beta},t).$$

Lemma 4.2. *The following formula holds*:

$$K_m(w^{\alpha,\beta};x,t)=A_m\left\{\frac{p_m(w^{\alpha,\beta};x)q(t)-p_m(w^{\alpha,\beta};t)q(x)}{x-t}-C_m p_m(w^{\alpha,\beta};x)p_m(w^{\alpha,\beta};t)\right\},$$

where

$$q(t)=\frac{1}{\gamma_0}\int_{-1}^{1}\frac{(1-x^2)p_m(v^{\alpha+1,\beta+1};x)-(1-t^2)p_m(v^{\alpha+1,\beta+1};t)}{x-t}v^{\alpha,\beta}\,\mathrm{d}x,$$

$$A_m=\left(1+\frac{\gamma_m^2(v^{\alpha+1,\beta+1})}{\gamma_{m+1}^2(v^{\alpha,\beta})}\right)^{-1}\frac{\gamma_m(v^{\alpha+1,\beta+1})}{\gamma_{m+1}(v^{\alpha,\beta})},\quad C_m=\frac{\gamma_m(v^{\alpha+1,\beta+1})}{\gamma_{m+1}(v^{\alpha,\beta})}$$

and

$$\lim_m A_m=\tfrac{1}{2},\quad \lim_m C_m=1.$$

The proof of Lemma 4.2, mutatis mutandis, can be found in [26, p. 312]. Now, since

$$|l_{m,k}(w^{\alpha,\beta},t)|=\lambda_{m,k}(w^{\alpha,\beta})|K_m(w^{\alpha,\beta};t,x_{m,k})|$$

and by Lemma 4.2, it follows that

$$|K_m(w^{\alpha,\beta};t,x_{m,k})|\leqslant\mathscr{C}\frac{|p_m(w^{\alpha,\beta};t)q(x_{m,k})|}{|t-x_{m,k}|}.$$

Since

$$|q(t)|\leqslant\frac{1}{\gamma_0}\left|\int_{-1}^{1}\frac{(1-x^2)p_m(v^{\alpha+1,\beta+1};x)}{x-t}v^{\alpha,\beta}\,\mathrm{d}x\right|+(1-t^2)|p_m(v^{\alpha+1,\beta+1};t)|\left|\int_{-1}^{1}\frac{v^{\alpha,\beta}(x)}{x-t}\,\mathrm{d}x\right|,$$

by Criscuolo et al. [4, Theorem 2.1] and (10), it follows that

$$|q(x_{m,k})|\leqslant\mathscr{C}v^{1/4-(|\alpha|/2),1/4-(|\beta|/2)}(x_{m,k})\left[\log^{\gamma_\alpha}\frac{e}{1-x_{m,k}}+\log^{\gamma_\beta}\frac{e}{1+x_{m,k}}\right].\tag{22}$$

Then, by (21), (22), and Lemma 4.1, we have

$$|l_{m,k}(w^{\alpha,\beta},t)|\leqslant\frac{v^{-(|\alpha|/2)-1/4,-(|\beta|/2)-1/4}(t)}{v^{-(|\alpha|/2)-1/4,-(|\beta|/2)-1/4}(x_{m,k})}\frac{[\log^{\gamma_\alpha}e/(1-t)+\log^{\gamma_\beta}e/(1+t)]}{[\log^{\gamma_\alpha}e/(1-x_{m,k})+\log^{\gamma_\beta}e/(1+x_{m,k})]}\frac{\Delta x_{m,k}}{|t-x_{m,k}|},$$

$$|t|\leqslant 1-\frac{\mathscr{C}}{m^2},\quad \Delta x_{m,k}=x_{m,k+1}-x_{m,k}.\tag{23}$$

Proof of Theorem 3.3. Denote by $x_{m,1} < x_{m,2} < \cdots < x_{m,m}$ the zeros of $p_m(w^{\alpha,\beta})$, define the nodes

$$y_j = -1 + \frac{1 + x_{m,1}}{1 + s} j, \quad j = 1, 2, \ldots, s,$$

$$z_i = x_m + \frac{1 - x_{m,m}}{1 + r} i, \quad i = 1, 2, \ldots, r$$

and the polynomials

$$Y_s(t) = \prod_{j=1}^{s} (t - y_j), \quad Z_r(t) = \prod_{i=1}^{r} (t - z_i). \tag{24}$$

By Lemma 4.2 the Lagrange polynomial $L_{m,r,s}(w^{\alpha,\beta}, f)$ interpolating f at the zeros of $Y_s(t)$ $p_m(w^{\alpha,\beta}, t) Z_r(t)$ can be expressed in the following form:

$$L_{m,r,s}(w^{\alpha,\beta}, f, t) = Y_s(t) p_m(w^{\alpha,\beta}, t) \tilde{L}_r \left(\frac{f}{Y_s p_m(w^{\alpha,\beta})}, t \right) + \sum_{k=1}^{m} l_{m,k}(w^{\alpha,\beta}, t) f(x_{m,k}) \frac{Y_s(t) Z_r(t)}{Y_s(x_k) Z_r(x_k)}$$
$$+ Z_r(t) p_m(w^{\alpha,\beta}, t) \tilde{L}_s \left(\frac{f}{Z_s p_m(w^{\alpha,\beta})}, t \right), \tag{25}$$

where $\tilde{L}_s(g), \tilde{L}_r(g)$ denote the Lagrange polynomials interpolating the function g at the zeros of Y_s and Z_r, respectively.

For $\alpha = \beta = 0$ the theorem was proved in [26]. We do not give the details of the proof in the other cases, since we use very similar arguments. □

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