Eventual differentiability of functional differential equations in Banach spaces✩

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Received 1 August 2005
Available online 30 May 2006
Submitted by G. Chen

Abstract

This paper concerns the regularity of a functional differential equation in the form: \( \dot{u}(t) = Au(t) + B_1 u(t - r) + \int_{-r}^{0} a(s) B_2 u(t + s) \, ds \), \( t > 0 \), where \( A \) is the generator of an analytic semigroup on a Banach space \( X \), and \( B_1 \), \( B_2 \) are \((\gamma - A)^{\alpha}\)-bounded linear operator for \( 0 < \alpha < 1 \). By spectral analysis, it is shown that the associated solution semigroup of this equation is eventually differentiable.

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Keywords: \( C_0 \)-semigroup; Eventual differentiability; Functional differential equation

1. Introduction

The goal of this paper is to study the regularity property of the functional differential equation in the form:

\[
\dot{u}(t) = Au(t) + B_1 u(t - r) + \int_{-r}^{0} a(s) B_2 u(t + s) \, ds, \quad t > 0,
\]

✩ Supported by the National Natural Science Foundation of China under Grants 10501039 and 10571161, and by Ningbo Natural Science Foundation under Grant 2005A610005.
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0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.04.038
\[ u(0) = g^0, \]
\[ u(\theta) = g^1(\theta), \quad \theta \in [-r, 0), \]  \hspace{1cm} (1.1)

where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( e^{At} \) on a Banach space \( X \), and \( B_1, B_2 \) are linear operators in \( X \).

It is well known that the spectrum-determined growth condition is important because it gives a practical criterion for assessing stability of an evolution problem, since calculating the growth bound of the solution semigroup for the evolution problem from definition is a formidable task, but calculating the spectra is much easier. Generally, the spectrum-determined growth condition is not valid for infinite dimensional systems unless the associated solution semigroups have some other regularity properties.

In the case of \( B_1, B_2 \in L(X) \), where \( L(X) \) is the space of all bounded linear operators on \( X \), there are various conditions on Eq. (1.1) such that the associated solution semigroup has some regularity properties. For example, in [1,6], the authors proved that the solution semigroup of (1.1) is continuous in norm for \( t > r \) in the state space \( X \times L^p([-r, 0], X) \) if the semigroup \( e^{At} \) is continuous in norm for \( t > 0 \). In [10], Mátrai proved that the immediate compactness of \( e^{At} \) implies the eventual compactness of the solution semigroup of (1.1) in \( X \times L^p([-r, 0], X) \), and an analogous result has been obtained in [15]. Recently, Batty [3] showed that if \( A \) satisfies the condition

\[ \| R(i\omega, A) \| \leq c|\omega|^{-\beta}, \quad \omega \in \mathbb{R}, \quad |\omega| > b, \]  \hspace{1cm} (1.2)

for some \( \beta > 0, \ b > 0 \), the solution semigroup of (1.1) is eventually differentiable in \( C([-r, 0], X) \). And we know that condition (1.2) is slightly weaker than the condition that \( A \) is a generator of an analytic semigroup.

Additionally, many researchers have studied the regularity properties of (1.1) in the case that \( B_1, B_2 \) are unbounded operators. Let \( e^{At} \) be an analytic semigroup and \( Z = F \times L^2([-r, 0], D(A)) \), where \( F \) is a suitable intermediate space between \( D(A) \) and \( X \). When the discrete delay term in (1.1) vanishes and \( B_2 \) is \( A \)-bounded, different regularity properties of (1.1) can be obtained in the space \( Z \). For example, in [4], Di Blasio et al. proved that if the weight function \( a(\cdot) \) appearing in the distributed delay term belongs to \( W^{1,2}(-r, 0) \), the associated solution semigroup of (1.1) is differentiable for \( t > r \) in \( Z \). In [7], Jeong showed that if \( a(\cdot) \) is Hölder continuous and \( B_2 = A \), the solution semigroup of (1.1) is Hölder continuous in norm for \( t > 3r \) in \( Z \). If \( a \in L^2(-r, 0) \) and \( B_2 = A \), Mastinšek [9] proved that the solution semigroup is continuous in norm for \( t > r \) in \( Z \). However, when the discrete delay term appears in Eq. (1.1) and \( B_1, B_2 \) are \( A \)-bounded, the solution semigroup is not eventually continuous in norm in the space \( Z \), as shown in [7]. The regularity of (1.1) has also been discussed in [2,5,8,12] under other proper conditions.

Now, we assume \( e^{At} \) is an analytic semigroup on \( X \) and \( B_1, B_2 \) are \((\gamma - A)^\alpha\)-bounded operators, where \( 0 < \alpha < 1 \) and \( \gamma > \omega_0(A) \). In this paper, we will study the eventual differentiability of the solution semigroup of (1.1), in which both the discrete delay term and the distributed delay term appear. We will prove that the solution semigroup of (1.1) is differentiable for \( t > \frac{3r}{1-\alpha} \) in \( X \times L^p([-r, 0], D((\gamma - A)^\alpha)) \) for \( 1 < p < \frac{1}{\gamma} \).

2. Wellposedness and semigroup setting

Let \( X \) be a Banach space. We will use the following assumptions:

(I) \( A \) is a generator of an analytic semigroup \( e^{At} \) \((t \geq 0) \), which satisfies \( \|e^{At}\| \leq M e^{\mu t} \) on \( X \).
Now taking $\gamma > \mu$, we can define
\[ A_{\alpha}^{-1} = (\gamma - A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} e^{-(\gamma-A)t} \, dt, \quad 0 < \alpha < 1, \]
\[ A_{\alpha} = (\gamma - A)^{\alpha} = (A_{\alpha}^{-1})^{-1}. \]

Let $X_{\alpha} = D(A_{\alpha})$ with the norm $\|x\|_{\alpha} = \|A_{\alpha}x\|$ for $x \in X_{\alpha}$. Obviously, $X_{\alpha}$ is a Banach space.

(II) $B_1$ and $B_2$ are $A_{\alpha}$-bounded operators, i.e., $B_i \in L(X_{\alpha}, X)$.

**Remark 2.1.** We can see from [13, Lemma 2.3.5] that the Banach space $X_{\alpha}$ is independent of the choice of $\gamma$.

Given the following functional differential equation:
\[ \dot{u}(t) = Au(t) + B_1u(t-r) + \int_{-r}^{0} a(s)B_2u(t+s)ds, \quad t > 0, \]
\[ u(0) = g^0, \]
\[ u(\theta) = g^1(\theta), \quad \theta \in [-r, 0), \]
where $r > 0$, $g^0 \in X$, $g^1 \in L^p([-r, 0], X_{\alpha})$, $a(\cdot) \in L^q(-r, 0)$, $1/p + 1/q = 1$, we at first study its wellposedness. To this end, we introduce a Banach space
\[ M_p = X \times L^p([-r, 0], X_{\alpha}) \]
with the norm
\[ \|g\|_{M_p} = \left( \|g^0\|^p + \int_{-r}^{0} \|g^1(s)\|^p_{\alpha} ds \right)^{\frac{1}{p}} \]
for $g = \left( g^0 \ g^1 \right) \in M_p$

and the operator
\[ A_g = \left( Ag^0 + B_1 g^1(-r) + \int_{-r}^{0} a(s)B_2 g^1(s)ds \right) \frac{d g^1(s)}{ds} \]
for $g = \left( g^0 \ g^1 \right) \in D(A)$.

where $D(A) = \{(g^0, g^1) \in D(A) \times W^{1,p}([-r, 0], X_{\alpha}); \ g^1(0) = g^0 \}$. The following assumption is needed:

(III) $1 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$.

**Proposition 2.1.** Under the settings above, $A$ is a generator of a $C_0$-semigroup $\{ T(t), t \geq 0 \}$ on $M_p$.

**Proof.** This has be proved in [2] under the conditions (I)–(III).

Thus (2.1) can be reduced to the following abstract evolution equation in the state space $M_p$:
\[ \begin{cases} \frac{dU(t)}{dt} = AU(t), \\ U(0) = g. \end{cases} \]
Moreover, for any \( g = (g^0, g^1) \in D(A) \), the unique solution of (1.1) can be given by
\[
  u(t) = \begin{cases} 
    \pi_1(T(t)g), & t \geq 0, \\
    g^1(t), & t \in [-r, 0),
  \end{cases}
\]
where \( \pi_1 : M_p \to X \) is the projection of the product space \( M_p \) onto the space \( X \). □

3. Eventual differentiability

The following useful result can be found in [2].

**Lemma 3.1.** Let \( A \) be the infinitesimal generator of \( T(t) \). We define a dense linear operator in \( X \) as
\[
  \Delta(\lambda) = \lambda - A - e^{-\lambda r} B_1 - \int_{-r}^{0} e^{\lambda \theta} a(\theta) B_2 d\theta.
\]
Thus \( \lambda \in \rho(A) \) if and only if \( \Delta^{-1}(\lambda) \in L(X) \). Moreover, for any \( \lambda \in \rho(A) \), \((x, f(\cdot)) \in M_p\), the resolvent is given by
\[
  (\lambda - A)^{-1} (x, f(\cdot)) = (g(0), g(\cdot)),
\]
where
\[
  g(\theta) = e^{\lambda \theta} g(0) + \int_{\theta}^{0} e^{\lambda (\theta - s)} f(s) ds \quad \text{for } \theta \in [-h, 0],
\]
\[
  g(0) = \Delta^{-1}(\lambda) \left( x + \int_{-r}^{0} e^{-\lambda (\theta + r)} B_1 f(\theta) d\theta + \int_{-r}^{0} a(\theta) e^{\lambda \theta} \int_{\theta}^{0} e^{-\lambda s} B_2 f(s) ds d\theta \right).
\]

**Lemma 3.2.** Let \( 0 < \alpha < 1 \). Then there exists a constant \( K > 0 \), such that
\[
  \| A^\alpha (\lambda - A)^{-1} \| \leq K |\omega|^{\alpha - 1}, \quad 0 \neq \omega \in \mathbb{R},
\]
where \( \lambda = \tau + i\omega, \: \tau \geq \gamma \) and \( K \) is independent of \( \lambda \).

**Proof.** For \( x \in X \), we have
\[
  (\lambda - A)^{-1} x = \int_{-r}^{0} e^{-(\lambda - A)t} x dt,
\]
where \( \lambda = \tau + i\omega, \: \tau \geq \gamma \).

At first, we consider the case of \( \omega < 0 \). Suppose \( \gamma > \gamma' > \mu \). Since \( A \) generates an analytic semigroup, there exists \( \Delta_\eta \subset \mathbb{C} \), where
\[
  \Delta_\eta = \left\{ \zeta \in \mathbb{C}: |\arg \zeta| < \eta < \frac{\pi}{2} \right\},
\]
such that \( e^{-(\gamma' - A)t} \) has a bounded analytic extension in sector \( \Delta_\eta \).
Now we take \( \delta \in (0, \eta) \) such that \( 2 \tan \delta < \tan \eta \). Thus \( \Delta_\delta \subset \Delta_\eta \) and
\[
\left\{ \rho \left( \frac{1}{2} \cos \theta + i \sin \theta \right) : 0 < \rho < \infty, \ |\theta| \leq \delta \right\} \subset \Delta_\eta.
\]
Let \( \bar{\Delta}_\delta \) denote the closure of \( \Delta_\delta \). From the uniform boundedness of \( e^{-(\tau' - A)z} \) in \( \bar{\Delta}_\delta \), \( e^{-(\lambda - A)z} = e^{(\tau' - \lambda)z} e^{-(\tau' - A)z} \) uniformly converges to 0 as \( \rho \to \infty \) for \( z \in \{ z \in \mathbb{C} : 0 \leq \arg z \leq \delta \} \) and \( \lambda \in \{ \lambda = \tau + i \omega : \tau \geq \gamma, \ \omega < 0 \} \). Moreover, from the analyticity of \( e^{-(\lambda - A)z} \) in \( \Delta_\eta \), we can shift the path of integration in (3.4) from the positive real axis to the ray
\[
\Gamma: z = \rho e^{i \delta}, \quad 0 < \rho < \infty.
\]
Due to the closedness of \( A_\alpha \), we have
\[
A_\alpha (\lambda - A)^{-1} = A_\alpha \int_{\Gamma} e^{(\tau' - \lambda)z} e^{-(\tau' - A)z} \, dz
= e^{i \delta} \int_0^\infty e^{(\tau' - \lambda)\rho e^{i \delta}} A_\alpha e^{-(\tau' - A)\rho \left( \frac{1}{2} \cos \delta + i \sin \delta \right)} \, d\rho.
\]
By [11], there exists a constant \( M_\alpha > 0 \) such that
\[
\| A_\alpha e^{At} \| \leq M_\alpha e^{\mu t} t^{-\alpha}, \quad t > 0.
\]
Thus
\[
\| A_\alpha (\lambda - A)^{-1} \| \leq \frac{M_\alpha'}{\Gamma(\alpha) \sin(\pi \alpha)} |\omega|^{\alpha-1},
\]
where \( M_\alpha' \) is independent of \( \lambda \).

When \( \omega > 0 \), we can shift the path of integration in (3.4) from the positive real axis to the ray
\[
\Gamma: z = \rho e^{-i \delta}, \quad 0 < \rho < \infty,
\]
and obtain the similar estimation. Thus this lemma has been proved. \( \square \)

Now, we give the main result of this paper.

**Theorem 3.1.** The \( C_0 \)-semigroup \( T(t) \) is differentiable for \( t > \frac{3r}{1-\alpha} \).

**Proof.** Since \( T(t) \) is a \( C_0 \)-semigroup on \( M_\rho \) and \( e^{At} \) is an analytic semigroup on \( X \), there exist \( M_0 > 0, \tau_0 > \max\{\gamma, 0\} \), such that
\[
\| T(t) \| \leq M_0 e^{\tau_0 t}, \quad t \geq 0,
\]
and
\[
\| (\tau_0 + i \omega - A)^{-1} \| \leq M_0 |\omega|^{-1}, \quad \omega \neq 0.
\]

For any given \( b > 0 \), we can choose \( a_b \) such that
(1) \( a_b \geq \tau_0 \);
(2) if \(|\omega| \geq e^{\frac{a_b - \tau_0}{b}}\), \( M_0 b |\omega|^{-\epsilon} \ln |\omega| \leq \frac{1}{2} \).

Define
\[
\Sigma_b = \{ \lambda = \tau + i\omega: a_b - b \ln |\omega| \leq \tau \leq \tau_0 \}.
\]

For \( \lambda = \tau + i\omega \in \Sigma_b \), we have \(|\omega| \geq e^{(a_b - \tau_0)/b} \) and
\[
\| (\tau - \tau_0)(\tau_0 + i\omega - A)^{-1} \| \leq (\tau_0 - a_b + b \ln |\omega|)M_0 |\omega|^{-1}
\]
\[
\leq b \ln |\omega|M_0 |\omega|^{-1} \leq \frac{1}{2}.
\]

Thus
\[
(\lambda - A)^{-1} = (\tau_0 + i\omega - A)^{-1}(I + (\tau - \tau_0)(\tau_0 + i\omega - A)^{-1})^{-1} \in L(X),
\]
which implies \( \Sigma_b \subset \rho(A) \). Moreover, it follows from (3.3) and (3.5) that
\[
\| B_1(\lambda - A)^{-1} \| = \| B_1 A_\alpha^{-1} A_\alpha (\lambda - A)^{-1} \|
\]
\[
\leq \| B_1 A_\alpha^{-1} \| \| A_\alpha (\tau_0 + i\omega - A)^{-1}(I + (\tau - \tau_0)(\tau_0 + i\omega - A)^{-1})^{-1} \|
\]
\[
\leq 2M_1 K |\omega|^\alpha^{-1}
\]
and similarly, we have
\[
\| B_2(\lambda - A)^{-1} \| \leq 2M_2 K |\omega|^\alpha^{-1},
\]
where \( M_1 = \| B_1 A_\alpha^{-1} \| \) and \( M_2 = \| B_2 A_\alpha^{-1} \| \).

According to the dominated convergence theorem, there exists a constant \( N > 0 \) such that if \( n \geq N \), then
\[
\left( \int_{-r}^{0} e^{np\theta} d\theta \right)^{\frac{1}{p}} \leq \frac{1}{8} \frac{1}{M_2 K} \left( \int_{-r}^{0} |a(\theta)|^q d\theta \right)^{\frac{1}{q}}.
\]

Take \( b = b_r = \frac{1-a}{r} \) and define
\[
\Sigma = \{ \lambda = \tau + i\omega: a - b_r \ln |\omega| \leq \tau \leq \tau_0 \},
\]
where \( a = \max\{a_b, N, \frac{1}{r} \ln(8M_1 K)\} \geq 0 \). Obviously, \( \Sigma \subset \Sigma_{br} \subset \rho(A) \). Noting (3.6) and (3.7) and the choice of \( a \), we have that for any \( \lambda = \tau + i\omega \in \Sigma \),
\[
\| e^{\lambda r} B_1(\lambda - A)^{-1} \| \leq 2M_1 K |\omega|^\alpha^{-1} e^{-\tau r} \leq 2M_1 K e^{-ar} \leq \frac{1}{4}
\]
and
\[
\left\| \int_{-r}^{0} e^{\lambda \theta} a(\theta) B_2(\lambda - A)^{-1} d\theta \right\| \leq \int_{-r}^{0} e^{\tau \theta} |a(\theta)| d\theta \left\| B_2(\lambda - A)^{-1} \right\|
\]
\[
\leq 2M_2 K |\omega|^\alpha^{-1} \left( \int_{-r}^{0} |a(\theta)|^q d\theta \right)^{\frac{1}{q}} \left( \int_{-r}^{0} e^{\tau p\theta} d\theta \right)^{\frac{1}{p}}.
\]
\begin{equation}
\leq 2M_2K\left(\int_{-r}^{0} |a(\theta)|^q d\theta \right)^{\frac{1}{q}} \left(\int_{-r}^{0} e^{\alpha \theta} d\theta \right)^{\frac{1}{p}}
\leq \frac{1}{4},
\end{equation}

Therefore, by (3.8) and (3.9),

\[ \Delta^{-1}(\lambda) = (\lambda - A)^{-1}\left(I - e^{-\lambda r} B_1(\lambda - A)^{-1} - \int_{-r}^{0} e^{\lambda \theta} a(\theta) B_2(\lambda - A)^{-1} d\theta \right)^{-1} \in L(X). \]

It follows from Lemma 3.1 that \( \Sigma \subset \rho(A) \). Moreover,

\begin{equation}
\|\Delta^{-1}(\lambda)\| \leq 2\|\lambda - A\|^{-1}
= 2\| (\tau_0 + i\omega - A)^{-1} \left(I + (\tau - \tau_0)(\tau_0 + i\omega - A)^{-1} \right)^{-1} \|
\leq 4M_0|\omega|^{-1}, \quad \forall \lambda = \tau + i\omega \in \Sigma,
\end{equation}

and

\begin{equation}
\|A_\alpha \Delta^{-1}(\lambda)\| \leq 2\|A_\alpha (\lambda - A)^{-1}\|
\leq 2\|A_\alpha (\tau_0 + i\omega - A)^{-1} \left(I + (\tau - \tau_0)(\tau_0 + i\omega - A)^{-1} \right)^{-1} \|
\leq 4K|\omega|^\alpha^{-1}, \quad \forall \lambda = \tau + i\omega \in \Sigma.
\end{equation}

Let \( \lambda = \tau + i\omega \in \Sigma \subset \rho(A) \), \((x, f(\cdot)) \in M_p\). Suppose

\( (\lambda - A)^{-1}(x, f(\cdot)) = (g(0), g(\cdot)) \),

where \( g(\theta) \) and \( g(0) \) are defined by (3.1) and (3.2), respectively. Straightforward calculations show that

\begin{align*}
\| \int_{-r}^{0} e^{\lambda(-s)} f(s) ds \|_{L^p_{\mu}} &\leq r|\omega|^{1-\alpha} \| f \|_{L^p_{\mu}}, \\
\| \int_{-r}^{0} e^{-\lambda(\theta+r)} B_1 f(\theta) d\theta \| &\leq r^{\frac{1}{\alpha}} M_1|\omega|^{1-\alpha} \| f \|_{L^p_{\mu}}, \\
\| \int_{-r}^{0} a(\theta)e^{\lambda \theta} \int_{\theta}^{0} e^{-\lambda s} B_2 f(s) ds d\theta \| &\leq r M_2\| a \|_{\mu}|\omega|^{1-\alpha} \| f \|_{L^p_{\mu}},
\end{align*}

where \( \| \cdot \|_{L^p_{\mu}} \) denotes the norm of \( L^p([-r, 0], X_\mu) \). Then, combining (3.10) and (3.11), we have

\[ \| g(0) \| = \| \Delta^{-1}(\lambda) \left(x + \int_{-r}^{0} e^{\lambda(\theta+r)} B_1 f(\theta) d\theta + \int_{-r}^{0} a(\theta)e^{\lambda \theta} \int_{\theta}^{0} e^{-\lambda s} B_2 f(s) ds d\theta \right) \|
\leq 4M_0|\omega|^{-1} \left[ \| x \| + \left( r^{\frac{1}{\alpha}} M_1 + r M_2\| a \|_{\mu} \right) |\omega|^{1-\alpha} \| f \|_{L^p_{\mu}} \right]
\]

and
\[ \|g(0)\|_\alpha = \|A_\alpha g(0)\| \leq 4K|\omega|^{\alpha-1}|\|x\| + \left( r^\frac{1}{p} M_1 + r M_2 \|a\|_q \right) |\omega|^{1-\alpha} \|f\|_{L^p_\alpha}. \]  

Thus

\[ \|g\|_{L^p_\alpha} = \left\| e^{\lambda \cdot} g(0) + \int_0^\infty e^{\lambda(-s)} f(s) ds \right\|_{L^p_\alpha} \leq r^\frac{1}{p} |\omega|^{1-\alpha} \|g(0)\|_\alpha + r |\omega|^{1-\alpha} \|f\|_{L^p_\alpha} \leq 4K r^\frac{1}{p} |\|x\| + (4M_1 K r + 4M_2 K r^{1+\frac{1}{p}} \|a\|_q + r) |\omega|^{1-\alpha} \|f\|_{L^p_\alpha}. \]  

(3.13)

By (3.12) and (3.13), we draw a conclusion that there exists a constant \( C > 0 \) such that

\[ \| (\lambda - A)^{-1} \| \leq C |\omega|, \quad \lambda = \tau + i \omega \in \Sigma. \]

Thus by Theorem 2.4.7 of [11], the \( C_0 \)-semigroup \( T(t) \) \((t \geq 0)\) is differentiable for \( t > \frac{3r}{1-\alpha} \).

4. An application

Here we want to give an application of our result. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. We consider the following reaction–diffusion equation with delay:

\[ \frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \sum_{i=1}^n b_i \frac{\partial u(x,t-r)}{\partial x_i} + \int_{-r}^0 a(s) \sum_{i=1}^n c_i \frac{\partial u(x,t+s)}{\partial x_i} ds, \quad t > 0, \]

\[ u(x,t) = 0, \quad x \in \partial \Omega, \quad t > 0, \]

\[ u(x,t) = f(x,t), \quad (x,t) \in \Omega \times [-r,0]. \]  

(4.1)

where \( x = (x_1, x_2, \ldots, x_n) \in \Omega, c_i, b_i, i = 1, 2, \ldots, n, \) are nonzero constants and \( a \in L^\infty(-r,0) \). Let \( X = L^2(\Omega) \). Take the Dirichlet–Laplacian as

\[ A = \Delta \quad \text{with} \quad D(A) = H^1_0(\Omega) \cap H^2(\Omega). \]

It is easy to see \( D((-A)^{1/2}) = H^1_0(\Omega) \) and by formula (2.9) in [14], the following two operators:

\[ B_1 = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}, \quad B_2 = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} \quad \text{with} \quad D(B_1) = D(B_2) = H^1_0(\Omega) \]

are \((-A)^{1/2}\)-bounded. Thus all our assumptions are satisfied and we can rewrite this reaction–diffusion equation as an abstract delay equation in the form of (2.1). By Theorem 3.1, we conclude that the solution semigroup of (4.1) is differentiable for \( t > 6r \) in \( M_p \), where \( 1 < p < 2 \).

Acknowledgments

The author is very grateful to the referees for their valuable suggestions concerning the manuscript.

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