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# WHEN IS $R$-gr EQUIVALENT TO THE CATEGORY OF MODULES?*

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In the first part of this paper, we characterize graded rings $R = \bigoplus_{\sigma\in G} R_\sigma$ for which the category $R$-**gr** is equivalent with a category of modules over a certain ring.

In the second part, sufficient conditions are given for the following implication to hold: if $R$-**gr** is equivalent with $R_1$-**mod** (1 is the unit element of $G$), then $R$ is a strongly graded ring.

## Introduction

Let $G$ be a group, with identity element $e$, $R = \bigoplus_{\sigma\in G} R_\sigma$ a graded ring of type $G$. Consider the functor $R \otimes_{R_e} - : R_e$-**mod** $\to R$-**gr** given by $M \to R \otimes_{R_e} M$ where $M \in R_e$-**mod** and $R \otimes_{R_e} M$ is a graded left $R$-module by the grading $(R \otimes_{R_e} M)_\sigma = R_\sigma \otimes_{R_e} M$.

Dade's well-known result [3, Theorem 2.8] states that $R$ is strongly graded iff the functor $R \otimes_{R_e} -$ is an equivalence between $R_e$-**mod** and $R$-**gr**.

It is then natural to ask the following question:

What happens if $R_e$-**mod** and $R$-**gr** are just categorically equivalent? (1)

An easy example (see Example 3.5) shows that in this more general case $R$ is not strongly graded, even if the group $G$ is finite (and thus the statement of [2, Corollary 2.13] is slightly incorrect).

On the other hand, if the group $G$ is finite, then Cohen and Montgomery proved that $R$-**gr** is equivalent to $(R \# G^*)$-**mod** where $R \# G^*$ is the smash product of $R$ with $G$ (see [2, Theorem 2.2; 7]). Another proof of this fact can be found in [5] where it is shown that $R \# G^*$ is isomorphic to the ring of graded endomorphisms of ${}_RU = \bigoplus_{\sigma\in G} R(\sigma)$ which is clearly a projective and finitely

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generated generator of $R$-**gr**. Thus another question naturally arises:

What happens if $R$-**gr** is equivalent to $A$-**mod**, $A$ a ring? (2)

The main aim of this paper is to study questions (1) and (2).

For this purpose we proceed as follows. After recalling in Section 0 some definitions and results about modules over graded rings, in Section 1 we state the definition of graded equivalence: two graded rings $R$ and $S$ of type $G$ are called *left graded equivalent* if there is an equivalence $F: R\text{-}\mathbf{gr} \to S\text{-}\mathbf{gr}$, $G: S\text{-}\mathbf{gr} \to R\text{-}\mathbf{gr}$ such that $F$ and $G$ commute for every $\sigma \in G$ with the $\sigma$-suspension functors. This definition was set by Gordon and Green in [4] in the particular case when $G = \mathbb{Z}$.

Anyway, it is easily seen that their main result on graded equivalences [4, Theorem 5.4] still holds in this general setting. In particular any graded equivalence between $R$-**gr** and $S$-**gr** gives rise to a Morita equivalence between $R$-**mod** and $S$-**mod**

In Section 2 we deal with question (2) above. We call a graded ring $R$ left F.G.G. iff there is a ring $A$ and a category equivalence between $R$-**gr** and $A$-**mod**. Theorem 2.2 is the central result on left F.G.G. rings. Among other facts, it is proved that a graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$ of type $G$ is left F.G.G. iff $R$-**gr** has a *finitely generated generator* iff $R$ is left graded equivalent to a strongly graded ring $S$ iff there is a finite subset $F$ of $G$ such that for every $\tau \in G, R_e = \sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}$ (in this case ${}_R U = \bigoplus_{\sigma \in F} R_\sigma$ is a generator of $R$-**gr**).

In particular this last characterization is very useful for applications. The first important one is that any left F.G.G. ring is right F.G.G. so that we can just speak about F.G.G. rings without regarding of the side. Clearly any strongly graded ring and any graded ring over a finite group $G$ are F.G.G., but not every graded ring is F.G.G.

Remark 2.4 illustrates why the polynomial ring in one variable over a division ring with usual $\mathbb{Z}$-gradation is not F.G.G. Other applications are the following ones.

If $g: R \to S$ is a morphism of graded rings of the same type and $R$ is F.G.G., then $S$ is F.G.G. (Corollary 2.7).

The graded direct product of two graded rings $R$ and $S$ of the same type is F.G.G. iff both $R$ and $S$ are (Corollary 2.7).

If $R$ is a graded ring of type $G$, $H$ is a subgroup of $G$ of finite index and $R^{(H)}$ is F.G.G., then $R$ is F.G.G.

If $R$ is a graded ring of type $G$ and $H$ is a finite normal subgroup of $G$ and $R_{\langle G/H \rangle}$ is F.G.G., then $R$ is F.G.G. (Corollary 2.8).

Conditions on $M \in R$-**gr** are given for END($M$) to be F.G.G. (Proposition 2.12).

At this point we focus our attention on those rings $A$ such that $R$-**gr** is equivalent to $A$-**mod** for a fixed F.G.G. graded ring $R$. Proposition 3.2 shows that $R$-**gr** is equivalent to $A$-**mod** iff **gr**-$R$ is equivalent to **mod**-$A$ and thus we call such a ring $A$ an admissible ring for the F.G.G. graded ring $R$.

It is proved that admissible rings for $R$ are just the rings in the Morita equivalence class of rings of graded endomorphisms of the modules ${}_RU = \bigoplus_{\sigma\in F} R(\sigma)$, ${}_RU$ being a generator of $R$-**gr**.

An F.G.G. graded ring $R$ is called S.F.G.G. if $R_e$ is an admissible ring for $R$, i.e. S.F.G.G. rings are precisely the rings in question (1) above.

Not every F.G.G. ring is S.F.G.G. (Example 3.15), and, as we stated before, not every S.F.G.G. ring is strongly graded (Example 3.5). Nevertheless in some particular cases (S.F.G.G.) F.G.G. rings are strongly graded. For example, it is proved that if an F.G.G. graded ring $R$ has an admissible ring $A$ with $A$ modulo its Jacobson radical simple artinian, then $R$ is strongly graded (Corollary 3.13) and also, if $R$ is S.F.G.G. and $R_e$ modulo its Jacobson radical is semisimple artinian, then $R$ is strongly graded.

## 0. Notations and preliminaries

All rings considered in this paper are associative with identity $1 \neq 0$ and all modules are unital.

Let $R$ be a ring. $R$-**mod** will denote the category of left $R$-modules. The notation ${}_RM$ will be used to emphasize that $M$ is a left $R$-module.

Moreover if $R$ and $S$ are two rings, we will write ${}_RM_S$ to mean that $M$ is an $R$-$S$ bimodule (left $R$-module and right $S$-module). $J(R)$ will denote the Jacobson radical of $R$.

Let $G$ be a multiplicative group with identity element $e$. Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a graded ring of type $G$. We denote by $R$-**gr** (**gr**-$R$) the category of graded left (right) $R$-modules. If $M = \bigoplus_{\sigma\in G} M_\sigma$, $N = \bigoplus_{\sigma\in G} N_\sigma$ are two graded left modules, $\mathrm{Hom}_{R\text{-gr}}(M, N)$ is the set of morphisms in the category $R$-**gr** from $M$ to $N$, i.e.

$$\mathrm{Hom}_{R\text{-gr}}(M, N) = \{f : M \to N \mid f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma, \forall \sigma \in G\}.$$

If $M = \bigoplus_{\lambda\in G} M_\lambda$ is a graded left $R$-module and $\sigma \in G$, then $M(\sigma)$ is the graded left module obtained from $M$ by setting $M(\sigma)_\lambda = M_{\lambda\sigma}$; the graded left module $M(\sigma)$ is called the *$\sigma$-suspension* of $M$ [6].

It is well known [6] that the mapping $M \mapsto M(\sigma)$ defines a functor $T_\sigma^R : R\text{-}\mathbf{gr} \to R\text{-}\mathbf{gr}$ which is an equivalence of categories.

The *forgetful functor* $R$-**gr** $\to$ $R$-**mod** will be denoted by $\Phi_R$. If $H$ is a subgroup of $G$, then the ring $R^{(H)}$, with gradation $(R^{(H)})_\chi = R_\chi$ for all $\chi \in H$, is a graded ring of type $H$.

If $H$ is a normal subgroup of $G$, we will denote by $R_{\langle G/H\rangle}$ the ring $R$ endowed with the grading of type $G/H$ defined by $R_{\hat\sigma} = \bigoplus_{h\in H} R_{h\sigma}$, $\hat\sigma = H\sigma \in G/H$.

Recall that the graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$ is called *strongly graded* if $R_\sigma R_\tau = R_{\sigma\tau}$ for any $\sigma, \tau \in G$ or, equivalently, if $R_\sigma R_{\sigma^{-1}} = R_e$ for any $\sigma \in G$ (see [3, 6]).

A graded left module $M = \bigoplus_{\sigma \in G} M_\sigma$ over a graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$ is called *strongly graded* if $R_\sigma M_\tau = M_{\sigma\tau}$ for any $\sigma, \tau \in G$.

Since we will use it several times we recall the following result of Dade:

**Theorem 0.1.** *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type $G$. Then the following statements are equivalent:*

(a) *$R$ is strongly graded;*

(b) *The functor $R \otimes_{R_e} - : R_e$-**mod** $\to R$-**gr** given by $M \mapsto R \otimes_{R_e} M$, where $M \in R_e$-**mod** and $R \otimes_{R_e} M$ is a graded left $R$-module by the grading $(R \otimes_{R_e} M)_\sigma = R_\sigma \otimes_{R_e} M$, is an equivalence;*

(c) *Every graded left $R$-module is strongly graded;*

(*d*) *$R$ is a generator of $R$-**gr**.*

**Proof.** See [3, Theorems 2.8 and 4.6] or [6, Theorems I.3.4 and I.5.1]. □

**Remark 0.2.** It is easy to see that condition (c) can be weakened to the following form:

(c′) *Every* gr-*simple left $R$-module is strongly graded.*

In fact, in this case, if $S$ is a gr-simple module, then $S_e \neq 0$ so that $R$ generates $S$ in $R$-**gr.** Thus, if (c′) is fulfilled, $R$ generates every gr-simple left $R$-module. As any finitely generated module $M$ in $R$-**gr** contains a gr-maximal left submodule and $R$ is projective in $R$-**gr**, it follows that $R$ generates every finitely generated module in $R$-**gr** and hence $R$ is a generator of $R$-**gr**.

Let $M$ and $N$ be graded left modules over the graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$. For every $\tau \in G$ we set

$$\mathrm{HOM}_R(M, N)_\tau = \{ f : M \to N f \text{ is } R\text{-linear and } f(M_\sigma) \subset N_{\sigma\tau}, \forall \sigma \in G \} .$$

$\mathrm{HOM}_R(M, N)_\tau$ is an additive subgroup of the group $\mathrm{Hom}_R(M, N)$ of all $R$-linear maps from $M$ to $N$.

$$\mathrm{HOM}_R(M, N) = \bigoplus_{\tau \in G} \mathrm{HOM}_R(M, N)_\tau$$

is a graded abelian group of type $G$. If either $G$ is finite or $M$ is finitely generated, then $\mathrm{HOM}_R(M, N) = \mathrm{Hom}_R(M, N)$ (see [6, Corollary I.2.11]). If $M = N$, then $\mathrm{END}_R(M) = \mathrm{Hom}_R(M, M)$ with multiplication $fg = g \circ f$, $f, g \in \mathrm{END}_R(M)$, is a graded ring of type $G$ and $M = \bigoplus_{\sigma \in G} M_\sigma$ is a graded right $\mathrm{END}_R(M)$-module. In fact if $\tau, \sigma \in G$ and $f \in \mathrm{END}_R(M)_\tau$, then $f(M_\sigma) \subset M_{\sigma\tau}$.

If $M, N \in R$-**gr**, then we say that $N$ *weakly divides $M$ in $R$-**gr*** if it is isomorphic to a direct summand of a direct sum of a finite number of copies of $M$. $M$ and $N$ are *weakly isomorphic in $R$-**gr*** if each of them weakly divides the other in $R$-**gr.**

$M \in R$-**gr** is said to be *weakly G-invariant* if $M$ and $M(\sigma)$ are weakly isomorphic for all $\sigma \in G$ (see [3] and [6]).

A graded ring is called *left* gr-*noetherian* if it satisfies the ascending chain condition for left graded ideals. Let $R$ and $S$ be two graded rings of the same type $G$. A ring homomorphism $f: R \to S$ is called a *graded ring homomorphism* if $f(R_\sigma) \subset S_\sigma$ for all $\sigma \in G$.

Let $A$ be a ring, $G$ a group. We denote by $A[G]$ the group ring of $A$ over $G$ endowed with the $G$-grading: $(A[G])_\sigma = A_\sigma$ for all $\sigma \in G$.

Let $A$ be a ring, $G$ a subgroup of Aut($A$). We denote by $A * G$ the graded ring whose underlying abelian group is that one of the free $A$-module $A^{(G)}$ with multiplication defined by

$$(ag) * (bh) = (ag(b))(gh) \quad g, h \in G, a, b \in A$$

and $G$-grading $(A * G)_g = Ag$ for all $g \in G$.

$\mathbb{N}$ will denote the set of non-negative integers, $\mathbb{Z}$ the ring of integers. If $A$ is a ring and $n \in \mathbb{N}$, $n \neq 0$, then $M_n(A)$ will denote the ring of $n \times n$ matrices with entries in $A$. We will adopt the convention $M_0(A) = \{0\}$.

## 1. Graded equivalence

**Definitions 1.1.** Let $R$ and $S$ be graded rings of type $G$. A functor $F: R$-**gr** $\to S$-**gr** is called a *graded functor* if for every $\sigma \in G$, $F$ commutes with the $\sigma$-suspension functor, i.e. if $F \circ T^R_\sigma = T^S_\sigma \circ F$.

A graded functor $F: R$-**gr** $\to S$-**gr** is a *graded equivalence* if there is a graded functor $G: S$-**gr** $\to R$-**gr** such that $F \circ G = 1_{S\text{-gr}}$ and $G \circ F = 1_{R\text{-gr}}$.

We say that $R$ and $S$ are *left graded equivalent* if there is a graded equivalence $R$-**gr** $\to S$-**gr**.

**Remark 1.2.** Let $A$ be a ring and $G$ a subgroup of Aut($A$). Consider the graded rings of type $G$, $R = A[G]$ and $S = A * G$. As $R$ and $S$ are strongly graded (see [6]), by Theorem 0.1 $R$-**gr** and $S$-**gr** are both equivalent to $A$-**mod**. In fact $R_e = A = S_e$. Thus $R$-**gr** is equivalent to $S$-**gr**. Anyway $R$ and $S$ are not left graded equivalent.

Graded equivalences were introduced by Gordon and Green in [4]. Even if they considered only graded rings of type $\mathbb{Z}$ it is easily checked that all their results concerning graded equivalences still hold in the general case of graded rings of any type.

We quote from [4] some results we will use later.

**Theorem 1.3.** *Let R and S be graded rings of type G. Then the following statements are equivalent*:

(a) *R and S are left graded equivalent*;

(b) *There is a Morita equivalence* $L: R$-**mod** $\to S$-**mod** *and a graded functor* $F: R$-**gr** $\to S$-**gr** *such that* $\Phi_S \circ F = L \circ \Phi_R$;

(c) *There exists an object* $P \in R$-**gr** *such that* $\Phi_R(P)$ *is a finitely generated projective generator in* $R$-**mod** *and the graded ring* $\mathrm{END}_R(P)$ *is isomorphic to S as graded rings.*

**Proof.** See [4, Theorem 5.4]. □

**Remark 1.4.** Two graded rings can be Morita equivalent without being left graded equivalent. For example, let $k$ be any field and consider $R = k[X, X^{-1}]$ where $X$ is a variable commuting with $k$ and with $\mathbb{Z}$-gradation given by $R_n = \{aX^n, a \in k\}$ for all $n \in \mathbb{Z}$. Let $S = k[X, X^{-1}]$ with the trivial $\mathbb{Z}$-gradation: $S_0 = S$ and $S_n = 0$ for $n \neq 0$, $n \in \mathbb{Z}$. Then $R$-**mod** = $S$-**mod** but $R$-**gr** is not equivalent to $S$-**gr** in any sense. In fact, $R$ being strongly graded, by Theorem 0.1 $R$-**gr** is equivalent to $k$-**mod** while it is easily checked that $S$-**gr** is equivalent to $(A\text{-}\mathbf{mod})^{\mathbb{Z}}$, where $A$ is the ring $k[X, X^{-1}]$ without any grading. More examples can be found in [4].

## 2. F.G.G. rings

**Definition 2.1.** Let $R$ be a graded ring. We will say that $R$ is *left F.G.G.* if there is a ring $A$ and a category equivalence between $R$-**gr** and $A$-**mod**.

**Theorem 2.2.** *Let R be a graded ring of type G. The following statements are equivalent*:

(a) *R is left F.G.G.*;

(b) $R$-**gr** *has a finitely generated generator*;

(c) *There is a finite subset F of G such that* $U = \bigoplus_{\sigma \in F} R(\sigma)$ *is a generator of* $R$-**gr**;

(d) *R is left graded equivalent to a strongly graded ring S*;

(e) *There is a strongly graded ring S of type G such that* $R$-**gr** *is equivalent to* $S$-**gr**;

(f) *There is a finite subset F of G such that, for every* $\tau \in G$, $R_e = \sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}$. *Moreover, if* (f) *is fulfilled,* $U = \bigoplus_{\sigma \in F} R(\sigma)$ *is a generator of* $R$-**gr**.

**Proof.** (a)$\Rightarrow$(b). Let $A$ be a ring and let $F: R$-**gr** $\to A$-**mod**, $G: A$-**mod** $\to R$-**gr** be a category equivalence. Then $G(A)$ is a finitely generated (projective) generator of $R$-**gr**.

(b)$\Rightarrow$(c). Let $V$ be a finitely generated generator of $R$-**gr**. Then, as $\bigoplus_{\sigma \in G} R(\sigma)$ generates $R$-**gr** (see [6]), there is a finite subset $F$ of $G$ such that $U = \bigoplus_{\sigma \in F} R(\sigma)$ generates $V$.

(c)$\Rightarrow$(d). Let $F$ be a finite subset of $G$ such that ${}_RU = \bigoplus_{\sigma\in F} R(\sigma)$ is a generator of $R$-**gr**. Then ${}_RU$ is weakly $G$-invariant. In fact, for every $\tau \in G$, the $\tau$-suspension $T_\tau^R: R\text{-}\mathbf{gr} \to R\text{-}\mathbf{gr}$ being an equivalence of categories, $T_\tau^R(U) = U(\tau)$ is a finitely generated projective generator of $R$-**gr**. Thus, as $U$ generates—in $R$-**gr**—$U(\tau)$ which is finitely generated and projective, $U(\tau)$ weakly divides $U$ in $R$-**gr** and conversely, as $U(\tau)$ generates—in $R$-**gr**—$U$, which is finitely generated and projective, $U$ weakly divides $U(\tau)$ in $R$-**gr**. This holds for every $\tau \in G$. Thus $U$ is weakly $G$-invariant and hence $\mathrm{END}({}_RU) = S$ is a strongly graded ring of type $G$ (cf. [3, Theorem 4.6] or [6, Theorem I.5.1, p. 43]). Apply now Theorem 1.3.

(d)$\Rightarrow$(e). Trivial.

(e)$\Rightarrow$(a). Let $S$ be a strongly graded ring such that $R$-**gr** is equivalent to $S$-**gr** and set $A = S_e$. By Theorem 0.1 $S$-**gr** is equivalent to $A$-**mod**. Thus $R$-**gr** is equivalent to $A$-**mod**.

(c)$\Leftrightarrow$(f). ${}_RU = \bigoplus_{\sigma\in F} R(\sigma)$ is a generator of $R$-**gr** iff for every $\tau \in G$ there is an $n_\tau \in \mathbb{N}$ and a surjective morphism ${}_RU^{n_\tau} \to R(\tau)$ in $R$-**gr**. Now let $n \in \mathbb{N}$ and let $f: {}_RU^n \to R(\tau)$ be a morphism. Identify ${}_RU^n$ with $\bigoplus_{\sigma\in F}(R(\sigma))^n$. Then, for each $\sigma$, there exists $r_\sigma = (r_{\sigma 1}, \ldots, r_{\sigma n}) \in (R_{\sigma^{-1}\tau})^n$ such that, for every $x = (x_\sigma)_{\sigma\in F} \in U^n$, where $x_\sigma = (x_{\sigma 1}, \ldots, x_{\sigma n}) \in R(\sigma)^n$

$$f(x) = \sum_{\sigma\in F}\sum_{i=1}^{n} x_{\sigma i} r_{\sigma i}\,. \tag{3}$$

Clearly $f$ is surjective iff there is an $x \in U^n$ such that $f(x) = 1$ and then $x \in (U^n)_{\tau^{-1}}$, $f$ being a graded morphism. Thus $f$ is surjective iff, for each $\sigma \in G$, there exists $x_\sigma = (x_{\sigma 1}, \ldots, x_{\sigma n}) \in (R(\sigma)^n)_{\tau^{-1}} = (R_{\tau^{-1}\sigma})^n$ such that $\sum_{\sigma\in F}\sum_{i=1}^n x_{\sigma i} r_{\sigma i} = 1$.

Then

$$\sum_{\sigma\in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau} = R_e\,.$$

Conversely if $\sum_{\sigma\in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau} = R_e$, then, for each $\sigma \in F$, there is an $y_\sigma \in R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}$ such that $\sum_{\sigma\in F} y_\sigma = 1$. For each $\sigma \in F$ we can write

$$y_\sigma = \sum_{i=1}^{n_\sigma} x_{\sigma i} r_{\sigma i}$$

where, for each $i = 1, \ldots, n$, $x_{\sigma i}$ and $r_{\sigma i}$ are suitable elements of $R_{\tau^{-1}\sigma}$ and $R_{\sigma^{-1}\tau}$ respectively and $n_\sigma \in \mathbb{N}$. Now, $F$ being finite, we can assume that $n_\sigma$ is constant equal to a suitable $n$ for every $\sigma \in F$. Then defining $f: U^n \to R(\tau)$ via (3) we get the required surjective morphism. □

**Corollary 2.3.** *Let $R$ be a graded ring of type $G$. If $R$ is strongly graded or $G$ is finite, then $R$ is left F.G.G.* □

**Remark 2.4.** Not every graded ring is left F.G.G. For example let $\mathcal{K}$ be a division ring and consider the polynomial ring $R = \mathcal{K}[x]$ with the usual gradation of type $\mathbb{Z}$: $R_n = 0$ for $n < 0$ and $R_n = \{ax^n \mid a \in \mathcal{K}\}$ for $n > 0$. Clearly $R$ does not satisfy condition f of Theorem 2.2.

**Notation and definition 2.5.** Let $R$ be a graded ring of type $G$. For every subset $X$ of $G$ we set $R(X) = \bigoplus_{\sigma \in X} R(\sigma)$.

If $F$ is a finite subset of $G$ and $R(F)$ satisfies condition f of Theorem 2.2, then we will say that $F$ satisfies *left*-(f) for $R$. An analoguos definition holds for *right*-(f) for $R$.

**Corollary 2.6.** *Let $R$ be a graded left F.G.G. ring of type $G$ and assume that $F \subseteq G$ satisfies left-(f) for $R$. Then $R$ is right F.G.G. and $F' = \{\sigma^{-1} \mid \sigma \in F\}$ satisfies right-(f) for $R$.*

**Proof.** For each $\tau \in G$ we have

$$R_e = \sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau} .$$

Then, for each $\theta \in G$, we have

$$R_e = \sum_{\sigma \in F} R_{(\sigma\theta^{-1}\sigma^{-1})^{-1}\sigma} R_{\sigma^{-1}(\sigma\theta^{-1}\sigma^{-1})} ,$$

i.e.

$$R_e = \sum_{\sigma \in F} R_{\sigma\theta} R_{\theta^{-1}\sigma^{-1}}$$

which is easily seen to be the 'right' version of condition f of Theorem 2.2 with respect to the finite set $F'$. $\square$

In view of Corollary 2.6 from now on we will simply say that a graded ring $R$ is F.G.G. without mentioning left or right, but we will still distinguish between left-(f) or right-(f).

Let $R$ and $S$ be two graded rings of the same type $G$. We will denote by $R \oplus^{\mathrm{gr}} S$ the ring $R \oplus S$ endowed with the grading over $G$ defined by

$$(R \overset{\mathrm{gr}}{\oplus} S)_\sigma = R_\sigma \oplus S_\sigma , \quad \sigma \in G .$$

Let $G$ be an abelian group, $R$ and $S$ two graded algebras of type $G$ over a ring $\mathcal{K}$. We will denote with $R \otimes_{\mathcal{K}}^{\mathrm{gr}} S$ the $\mathcal{K}$-algebra $R \otimes_{\mathcal{K}} S$ endowed with the grading of type $G$ defined by

$$(R \overset{\mathrm{gr}}{\underset{\mathcal{K}}{\otimes}} S)_\sigma = \bigoplus_{\theta, \tau \in G, \theta\tau = \sigma} (R_\theta \underset{\mathcal{K}}{\otimes} S_\tau) .$$

**Corollary 2.7.** *Let $R$ and $S$ be two graded rings of the same type $G$. Then*

(1) *If $g: R \to S$ is a morphism of graded rings ($g(1) = 1$), $R$ is F.G.G. and $F$ satisfies left-(f) for $R$, then $S$ is F.G.G. and $F$ satisfies left-(f) for $S$;*

(2) *$R \oplus^{gr} S$ is F.G.G. iff both $R$ and $S$ are. Moreover, $F \subseteq G$ satisfies left-(f) for $R \oplus^{gr} S$ iff it satisfies left-(f) for both $R$ and $S$;*

(3) *If $G$ is abelian, $R$ and $S$ are graded $\mathcal{K}$-algebras and either $R$ or $S$ is F.G.G., then $R \otimes^{gr}_{\mathcal{K}} S$ is F.G.G.*

**Proof.** (1) If $F$ satisfies left-(f) for $R$, then, for every $\tau \in G$, $R_e = \sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}$ and hence

$$1 = g(1) \in \sum_{\sigma \in F} g(R_{\tau^{-1}\sigma}) g(R_{\sigma^{-1}\tau}) .$$

As $g(R_{\tau^{-1}\sigma}) \subseteq S_{\tau^{-1}\sigma}$ and $g(R_{\sigma^{-1}\tau}) \subseteq S_{\sigma^{-1}\tau}$ the conclusion follows.

(2) If a subset of $G$ satisfies left-(f) for a certain ring, then it is clear that any finite subset of $G$ which contains this one satisfies left-(f) for the same ring. Thus if both $R$ and $S$ are F.G.G., then we can assume that there is a finite subset $F$ of $G$ which satisfies left-(f) for both $R$ and $S$. Then, for every $\tau \in G$, we have

$$R_e = \sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}$$

and

$$S_e = \sum_{\sigma \in F} S_{\tau^{-1}\sigma} S_{\sigma^{-1}\tau} .$$

Hence

$$\begin{aligned}(R \overset{gr}{\oplus} S)_e = R_e \oplus S_e &= \Big(\sum_{\sigma \in F} R_{\tau^{-1}\sigma} R_{\sigma^{-1}\tau}\Big) \oplus \Big(\sum_{\sigma \in F} S_{\tau^{-1}\sigma} S_{\sigma^{-1}\tau}\Big) \\ &= \sum_{\sigma \in F} (R_{\tau^{-1}\sigma} \oplus S_{\tau^{-1}\sigma})(R_{\sigma^{-1}\tau} \oplus S_{\sigma^{-1}\tau}) \\ &= \sum_{\sigma \in F} (R \overset{gr}{\oplus} S)_{\tau^{-1}\sigma} (R \overset{gr}{\oplus} S)_{\sigma^{-1}\tau} .\end{aligned}$$

Thus $F$ satisfies left-(f) for $R \oplus^{gr} S$, hence, by Theorem 2.2, $R \oplus^{gr} S$ is F.G.G. Conversely, if $R \oplus^{gr} S$ is F.G.G. and $F$ satisfies left-(f) for $R \oplus^{gr} S$, then, as the projections over $R$ and $S$ are graded ring-homomorphisms, both $R$ and $S$ are F.G.G. and $F$ satisfies left-(f) both for $R$ and for $S$, in view of (1).

(3) Assume that $R$ is F.G.G. The map

$$g: R \to R \overset{gr}{\underset{\mathcal{K}}{\otimes}} S$$

defined by $g(r) = r \otimes 1$, $r \in R$, is a morphism of graded rings. Apply now (1). □

**Corollary 2.8.** *Let $R$ be a graded ring of type $G$ and $H$ a subgroup of $G$. Then*

(1) *If $H$ has finite index in $G$ and $R^{(H)}$ is F.G.G., then $R$ is F.G.G.;*
(2) *If $H$ is finite and normal in $G$ and if $R_{\langle G/H \rangle}$ is F.G.G., then $R$ is F.G.G.*

**Proof.** (1) Let $E$ be a finite set of representatives of the left cosets of $G$ modulo $H$: $G = \bigcup_{\varepsilon \in E} \varepsilon H$. Let $F \subseteq H$ satisfy left-(f) for $R^{(H)}$. Then for every $\chi \in H$

$$R_e = (R^{(H)})_e = \sum_{\sigma \in F} (R^{(H)})_{\chi^{-1}\sigma} (R^{(H)})_{\sigma^{-1}\chi} = \sum_{\sigma \in F} R_{\chi^{-1}\sigma} R_{\sigma^{-1}\chi} .$$

Let $L = \{\varepsilon\sigma \mid \varepsilon \in E, \sigma \in F\}$. Then, for every $\tau \in G$

$$R_e = \sum_{\lambda \in L} R_{\tau^{-1}\lambda} R_{\lambda^{-1}\tau} .$$

In fact, if $\tau \in G$, then $\tau = \varepsilon\chi$ for a suitable $\varepsilon \in E$ and $\chi \in H$. Then

$$\sum_{\lambda \in L} R_{\tau^{-1}\lambda} R_{\lambda^{-1}\tau} = \sum_{\sigma \in F} R_{\tau^{-1}\varepsilon\sigma} R_{\sigma^{-1}\varepsilon^{-1}\tau} = \sum_{\sigma \in F} R_{\chi^{-1}\sigma} R_{\sigma^{-1}\chi} = R_e .$$

(2) Let $\pi : G \to G/H$ be the canonical projection. Since $R_{\langle G/H \rangle}$ is F.G.G., by Theorem 2.2 there is a finite subset $F$ of $G$ such that for every $\tau \in G$

$$(R_{\langle G/H \rangle})_{\pi(e)} = \sum_{\sigma \in F} (R_{\langle G/H \rangle})_{\pi(\tau^{-1}\sigma)} (R_{\langle G/H \rangle})_{\pi(\sigma^{-1}\tau)} .$$

This means

$$\sum_{h \in H} R_h = \sum_{\sigma \in F} \Big( \sum_{h \in H} R_{\tau^{-1}\sigma h} \Big) \Big( \sum_{k \in H} R_{k^{-1}\sigma^{-1}\tau} \Big) .$$

As $h \neq k$ implies $(R_{\tau^{-1}\sigma h} R_{k^{-1}\sigma^{-1}\tau}) \cap R_e = 0$, it is easy to see that

$$R_e = \sum_{\sigma \in F} \Big( \sum_{h \in H} R_{\tau^{-1}\sigma h} R_{h^{-1}\sigma^{-1}\tau} \Big) ,$$

i.e.

$$R_e = \sum_{\lambda \in L} R_{\tau^{-1}\lambda} R_{\lambda^{-1}\tau}$$

where $L = \{\sigma h \mid \sigma \in F, h \in H\}$. □

**Proposition 2.9.** *Let $R$ be a left* gr-*noetherian ring of type $G$, $H$ a subgroup of $G$. If $R$ is F.G.G., then $R^{(H)}$ is F.G.G.*

**Proof.** Let $V$ be a finitely generated generator of $R$-**gr**. Then, by [6, Corollary II.3.13], $V^{(H)}$ is finitely generated. Let us prove that $V^{(H)}$ is a generator of $R^{(H)}$-**gr**.

Let $\chi \in H$ and note that $(R^{(H)})(\chi) = (R(\chi))^{(H)}$. In fact, for every $h \in H$, it is

$$((R^{(H)})(\chi))_h = (R^{(H)})_{h\chi} = R_{h\chi} = (R(\chi))_h = (R(\chi))_h = ((R(\chi))^{(H)})_h .$$

Moreover, if $V^n \to R(\chi) \to 0$ is an epimorphism in $R$-**gr**, then $(V^n)^{(H)} \to R(\chi)^{(H)} \to 0$ is an epimorphism in $R^{(H)}$-**gr**. Then it easily follows that $V^{(H)}$ is a generator of $R^{(H)}$. Therefore, by Theorem 2.2, $R^{(H)}$ is F.G.G. □

**Definition 2.10.** Let $R$ be a graded ring of type $G$, $M, N \in R$-**gr**. We shall say that *$M$ w.-h. (weakly-homogeneously) divides $N$ in $R$-**gr*** if for every $\sigma, \tau \in G$, $M(\sigma)$ weakly divides $N(\sigma\tau)$ in $R$-**gr**, i.e. iff $M(\sigma)$ is isomorphic in $R$-**gr** to a direct summand of a direct sum of a finite number of copies of $N(\sigma\tau)$.

**Notation 2.11.** Let $R$ be a graded ring of type $G$, $M \in R$-**gr**, $F$ a subset of $G$. We shall denote with $M(F)$ the graded left $R$-module $\bigoplus_{\sigma \in F} M(\sigma)$.

**Proposition 2.12.** *Let $R$ be a graded ring of type $G$, $M \in R$-**gr**. Then the graded ring* END($M$) *is F.G.G. iff there is a finite subset $F$ of $G$ such that $M$ w.-h. divides $M(F)$ in $R$-**gr**. In this case $F$ satisfies left-*(f) *for* END($M$).

**Proof**. END($M$) is F.G.G. iff there is a finite subset $F$ of $G$ such that for every $\tau \in G$:

$$(\mathrm{END}(M))_e = \sum_{\sigma \in F} (\mathrm{END}(M))_{\tau^{-1}\sigma}(\mathrm{END}(M))_{\sigma^{-1}\tau}$$

or equivalently

$$(\mathrm{END}(M))_e = \sum_{\sigma \in F} (\mathrm{END}(M))_{\tau\sigma}(\mathrm{END}(M))_{\sigma^{-1}\tau^{-1}} \quad \text{for every } \tau \in G ,$$

i.e. such that for every $\tau \in G$ there is an $n_\tau \in \mathbb{N}$ and elements $f_{i\sigma} \in \mathrm{END}(M)_{\tau\sigma}$, $g_{i\sigma} \in \mathrm{END}(M)_{\sigma^{-1}\tau^{-1}}$, $i = 1, \ldots, n_\tau$, $\sigma \in F$, such that

$$1_M = \sum_{\sigma \in F} f_{i\sigma} g_{i\sigma} = \sum_{\sigma \in F} g_{i\sigma} \circ f_{i\sigma} .$$

Now, for every $\theta \in G$,

$$\mathrm{END}(M)_{\tau\sigma} = \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M(\theta), M(\theta\tau\sigma)) ,$$

$$\mathrm{END}(M)_{\sigma^{-1}\tau^{-1}} = \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M(\theta\tau\sigma), M(\theta))$$

and $M(\theta\tau\sigma) = (M(\sigma))(\theta\tau)$. Thus we get that END($M$) is F.G.G. iff there exists a finite subset $F$ of $G$ such that $M$ w.-h.divides $M(F)$. □

**Proposition 2.13.** *Let $R$ be a graded ring, $A$ a ring and $F: R$-**gr**$\to A$-**mod**, $G: A$-**mod**$\to R$-**gr** a category equivalence. Then there is a positive integer $n$ and an idempotent matrix $\alpha \in M_n(A)$ such that*

$$R_e \cong \alpha M_n(A)\alpha .$$

**Proof.** Let ${}_AQ = F({}_RR)$. Then ${}_AQ$ is projective, finitely generated and $R_e = \mathrm{End}_{R\text{-gr}}({}_RR, {}_RR) \cong \mathrm{End}_A({}_AQ)$. ${}_AQ$, being projective and finitely generated, is a direct summand of $A^{(n)}$ for a suitable $n \in \mathbb{N}$. □

## 3. Admissible rings and S.F.G.G. graded rings

**Lemma 3.1.** *Let $R$ be a graded ring of type $G$, $F' = \{\sigma_1, \ldots, \sigma_n\}$ a finite subset of $G$, $U = R(F)$. Then*

$$(1)\qquad \mathrm{End}_{R\text{-gr}}({}_RU) \cong \begin{bmatrix} R_e & R_{\sigma_1\sigma_2^{-1}} & \cdots & R_{\sigma_1\sigma_n^{-1}} \\ R_{\sigma_2\sigma_1^{-1}} & R_e & \cdots & R_{\sigma_2\sigma_n^{-1}} \\ \vdots & & & \\ R_{\sigma_n\sigma_1^{-1}} & R_{\sigma_n\sigma_2^{-1}} & \cdots & R_e \end{bmatrix}$$

*and*

$$(2)\qquad \mathrm{End}_{\mathrm{gr}\text{-}R}(U_R) \cong \begin{bmatrix} R_e & R_{\sigma_1^{-1}\sigma_2} & \cdots & R_{\sigma_1^{-1}\sigma_n} \\ R_{\sigma_2^{-1}\sigma_1} & R_e & \cdots & R_{\sigma_2^{-1}\sigma_n} \\ \vdots & & & \\ R_{\sigma_n^{-1}\sigma_1} & R_{\sigma_n^{-1}\sigma_2} & \cdots & R_e \end{bmatrix} .$$

**Proof.** (1) See [6, Lemma I.5.4].

(2) Easily proved in a way analogous to (1). One should only note that if $e_1, \ldots, e_n$ is a homogeneous basis of $U_R$ with $\deg e_i = \sigma_i$ and if $f \in \mathrm{End}_{\mathrm{gr}\text{-}R}(U_R)$, then for every $i = 1, \ldots, n$

$$f(e_i) = \sum_{j=1}^{n} e_j r_{ji} \quad \text{with } \deg r_{ji} = \sigma_j^{-1}\sigma_i . \qquad \square$$

**Proposition 3.2.** *Let $R$ be an F.G.G. graded ring of type $G$, $A$ any ring. Then the following statements are equivalent:*

(a) $R$-**gr** $\approx A$-**mod**;

(b) *There exists a finite subset $F$ of $G$ such that $U = R(F)$ is a generator of $R$-**gr** and $\mathrm{End}_{R\text{-gr}}(U)$ is Morita equivalent to $A$;*

(c) *For every finite subset $F$ of $G$, if $U = R(F)$ is a generator of $R$-**gr**, then* $\mathrm{End}_{R\text{-}\mathbf{gr}}(U)$ *is Morita equivalent to $A$;*

(d) $\mathbf{gr}\text{-}R \approx \mathbf{mod}\text{-}A$.

*If these conditions are fulfilled we shall say that $A$ is an admissible ring for $R$.*

**Proof.** (a)$\Rightarrow$(c). Let $U = R(F)$ be a generator of $R$-**gr**. Then $S = \mathrm{END}({}_R U)$ is strongly graded and $S_e = \mathrm{End}_{R\text{-}\mathbf{gr}}(U)$. Thus $A\text{-}\mathbf{mod} \approx R\text{-}\mathbf{gr} \approx S\text{-}\mathbf{gr} \approx S_e\text{-}\mathbf{mod}$.

(c)$\Rightarrow$(b). Trivial in view of Theorem 2.2.

(b)$\Rightarrow$(a). See (e)$\Rightarrow$(a) of Theorem 2.2.

(b)$\Rightarrow$(d). Let $U = R(F)$ be a generator of $R$-**gr**. Then $F$ satisfies left-(f) for $R$, and Corollary 2.6 shows that $\bar{U} = R(\bar{F})$ where $\bar{F} = \{\sigma^{-1} \mid \sigma \in F\}$.

$\bar{U}$ is a generator of **gr**-$R$. If $F = \{\sigma_1, \ldots, \sigma_n\}$, then $\bar{F} = \{\sigma_1^{-1}, \ldots, \sigma_n^{-1}\}$, and by Lemma 3.1

$$\mathrm{End}_{\mathbf{gr}\text{-}R}(\bar{U}) \cong \begin{bmatrix} R_e & R_{\sigma_1\sigma_2^{-1}} & \cdots & R_{\sigma_1\sigma_n^{-1}} \\ R_{\sigma_2\sigma_1^{-1}} & R_e & \cdots & R_{\sigma_2\sigma_n^{-1}} \\ \vdots & & & \\ R_{\sigma_n\sigma_1^{-1}} & R_{\sigma_n\sigma_2^{-1}} & \cdots & R_e \end{bmatrix}$$

$$\cong \mathrm{End}_{R\text{-}\mathbf{gr}}(U)\,. \qquad \square$$

**Definition 3.3.** Let $R$ be a graded ring of type $G$. We shall say that $R$ is *S.F.G.G.* if $R$ is F.G.G. and $R_e$ is an admissible ring for $R$.

**Remark 3.4.** Clearly, by the well-known result of Dade (see Theorem 0.1) every strongly graded ring is S.F.G.G.

The following example shows that the converse, even if the group $G$ is finite, does not generally hold:

**Example 3.5.** Let $A$ be a commutative ring such that $A \times A \cong A$ (e.g. $A = \prod_{i\in I} B_i$, $B_i = B$ any commutative ring, $I$ an infinite set) and consider the graded ring $R = A \times \{0\}$ of type $G = \{0, 1\}$. Then for $U = R(G)$ we have

$$\mathrm{End}_{R\text{-}\mathbf{gr}}(U) \cong \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \cong A \times A \cong A\,.$$

Thus $R$ is S.F.G.G. Clearly $R$ is not strongly graded.

Anyway, if $R$ is an F.G.G. (resp. S.F.G.G.) ring having ‘special type’ of admissible rings, then $R$ is strongly graded. To be more precise we need the following:

**Definitions 3.6.** We shall say that a ring $A$ has

(1) *property* $(*)$ if any non-zero finitely generated projective left $A$-module is a generator of $A$-**mod**;

(2) *property* $(**)$ if any finitely generated projective left $A$-module with edomorphism ring isomorphic to $A$ is a generator of $A$-**mod.**

Before relating these definitions with our study of F.G.G. rings we want to investigate a little bit the class of rings with property $(*)$ (resp. $(**)$).

Clearly any ring, for which non-zero finitely generated projective modules are free, has property $(*)$. Moreover, we have

**Proposition 3.7.** *Let $A$ be a ring, $J = J(A)$ its Jacobson radical. If $A/J$ has property* $(*)$ (*resp.* $(**)$), *then also $A$ has this property.*

**Proof.** Let ${}_AP$ be a finitely generated projective left $A$-module. Then (see [1, Proposition 17.9]) ${}_AP$ is a generator of $A$-**mod** iff ${}_AP$ generates every simple left $A$-module. Clearly this holds iff ${}_AP/J({}_AP)$ generates every simple left $A$-module, i.e. every simple left $A/J$-module.

By [1, Proposition 17.10], $J({}_AP) = JP$. Thus ${}_AP$ generates $A$-**mod** iff ${}_AP/JP$ generates every simple left $A/J$-module. Clearly ${}_AP/JP \cong A/J \otimes_A P$ is a finitely generated projective left $A/J$-module. Moreover,

$$\mathrm{End}_{A/J}(P/JP) \cong \mathrm{End}({}_AP)/J(\mathrm{End}_A P)$$

(cf. [1, Corollary 17.12]). Thus if ${}_AP$ is non-zero, then $P/JP$ is non-zero and if $\mathrm{End}({}_AP) \cong A$, then $\mathrm{End}_{A/J}(P/JP) \cong A/J$.

Therefore if $A/J$ has property $(*)$ (resp. $(**)$), then $A$ has this property too. □

**Lemma 3.8.** *Any simple artinian ring $A$ has property* $(*)$.

**Proof.** Let ${}_AP$ be a non-zero finitely generated projective left $A$-module. Clearly ${}_AP$ has a simple quotient. As all simple left $A$-modules are isomorphic, ${}_AP$ generates every simple left $A$-module and hence ${}_AP$ is a generator (cf. [1, Proposition 17.9]). □

**Corollary 3.9.** *Let $A$ be a ring, $J = J(A)$ its Jacobson radical. If $A/J$ is simple artinian, then $A$ has property* $(*)$. □

**Lemma 3.10.** *Any semisimple artinian ring $A$ has property* $(**)$.

**Proof.** Obvious. □

**Corollary 3.11.** *Let $A$ be a ring, $J = J(A)$ its Jacobson radical. If $A/J$ is semisimple artinian, then $A$ has property* $(**)$. □

**Proposition 3.12.** *Let $R$ be a graded ring. Then*

(1) *If $R$ is F.G.G. and has an admissible ring $A$ with property* $(*)$, *then $R$ is strongly graded*;

(2) *If $R$ is S.F.G.G. and $R_e$ has property* $(**)$, *then $R$ is strongly graded.*

**Proof.** Let $F: R\text{-}\mathbf{gr} \to A\text{-}\mathbf{mod}$, $G: A\text{-}\mathbf{mod} \to R\text{-}\mathbf{gr}$ be a category equivalence. Then ${}_AP = F(R)$ is a finitely generated projective left $A$-module (and $\mathrm{End}({}_AP) = \mathrm{End}_{R\text{-}\mathbf{gr}}({}_RR) = R_e$). Thus if $A$ has property $(*)$ (resp. $A = R_e$ has property $(**)$), then ${}_AP$ is a generator of $A$-**mod**.

Thus ${}_RR \cong^{\mathrm{gr}} G({}_AP)$ is a generator of $R$-**gr**. By Theorem 0.1, this means that $R$ is strongly graded. □

**Corollary 3.13.** *If $R$ is an F.G.G. graded ring having an admissible ring $A$ with $A/J(A)$ simple artinian, then $R$ is strongly graded.* □

**Corollary 3.14.** *If $R$ is a S.F.G.G. graded ring and $R_e/J(R_e)$ is semisimple artinian, then $R$ is strongly graded.* □

**Example 3.15.** Let $D$ be a division ring and $R = D \times M$, the trivial extension of $D$ by the bimodule ${}_DM_D$. Then $R$ is a graded ring of type $G = \{0, 1\}$ with $R_0 = D \times \{0\}$ and $R_1 = \{0\} \times M$. Clearly $R$ *is F.G.G.* but not strongly graded, as $R_1R_1 = 0$. $R$ *is not S.F.G.G.* In fact, as $R_0 = D \times \{0\}$ if $R$ were S.F.G.G., by Corollary 3.14 $R$ would be strongly graded.

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