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Hypersurfaces in Mori dream spaces

Michela Artebani^{*,1}, Antonio Laface²

Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción,
Avenida Esteban Iturra s/n, Barrio Universitario, Casilla 160 C, Concepción, Chile

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ABSTRACT

Let X be a hypersurface of a Mori dream space Z . We provide necessary and sufficient conditions for the Cox ring $\mathcal{R}(X)$ of X to be isomorphic to $\mathcal{R}(Z)/(f)$, where $\mathcal{R}(Z)$ is the Cox ring of Z and f is a defining section for X . We apply our results to Calabi–Yau hypersurfaces of toric Fano fourfolds. Our second application is to general degree d hypersurfaces in \mathbb{P}^n containing a linear subspace of dimension $n - 2$.

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Introduction

Let Z be a Mori dream space, that is a normal variety without global invertible non-constant regular functions, finitely generated class group and finitely generated Cox ring (which in case the class group is torsion-free is given by):

$$\mathcal{R}(Z) = \bigoplus_{[D] \in \text{Cl}(Z)} \Gamma(Z, \mathcal{O}_Z(D)).$$

Given an inclusion $i : X \rightarrow Z$ of a closed irreducible normal subvariety X of Z such that the restriction of classes of Weil divisors to X is well defined, there is a natural homomorphism $i_{\mathcal{R}} : \mathcal{R}(Z) \rightarrow \mathcal{R}(X)$. In order to relate the two Cox rings it is natural to ask whether such homomorphism is surjective

* Corresponding author. Fax: +56 (41) 2203321.

E-mail addresses: michela.artebani@gmail.com (M. Artebani), antonio.laface@gmail.com (A. Laface).

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and what is its kernel. In [Hau08] J. Hausen studied the case when Z is a smooth toric variety. In particular, he proved that $\mathcal{R}(X)$ is isomorphic to a quotient of $\mathcal{R}(Z)$ via the homomorphism $i_{\mathcal{R}}$ if the inclusion $i : X \rightarrow Z$ is *neat*, which essentially means that the pull-back $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ is well defined and is an isomorphism. This result and its proof can be extended to the case when Z is a factorial Mori dream space. In this paper we use this generalized version of J. Hausen’s theorem to study the case when X is a hypersurface in Z . More precisely, when X is a normal, irreducible and closed hypersurface of a Mori dream space Z , we find necessary and sufficient conditions for the homomorphism $i_{\mathcal{R}}$ to induce an isomorphism $\mathcal{R}(Z)/(f) \cong \mathcal{R}(X)$, where f is a defining section for X . In case Z is factorial, such conditions are the following ones: the pull-back $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ is an isomorphism and the intersection of the irrelevant locus with $V(f)$ has codimension ≥ 2 in $V(f)$, where $V(f) \subset \bar{Z} = \text{Spec}(\mathcal{R}(Z))$. This generalizes the main result of [Jow10] to the non-smooth case. Moreover, in case X is the generic element of an ample and spanned linear series on Z , we show that the latter condition on the codimension of the irrelevant locus is enough to guarantee the isomorphism.

The paper is organized as follows. In Section 1 we introduce good and neat embeddings. Section 2 contains our main Theorem 2.1 together with its corollary about general elements in ample linear series of Mori dream spaces. An application of Theorem 2.1 to smooth Mori dream Calabi–Yau hypersurfaces of smooth toric Fano varieties is given in Section 3. Finally in Section 4 we compute the Cox ring of a general degree d hypersurface in \mathbb{P}^n containing a linear subspace of dimension $n - 2$.

1. Embeddings in Mori dream spaces

In what follows Z will be a normal variety over an algebraically closed field \mathbb{K} of characteristic zero with finitely generated class group $\text{Cl}(Z)$ and without global invertible non-constant regular functions, that is $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$.

We briefly recall the definition of the Cox ring of Z as given in [ADHL, Chapter I, §4]. Let K be a subgroup of $\text{WDiv}(Z)$ such that the natural homomorphism $c : K \rightarrow \text{Cl}(Z)$, mapping D to its class $[D]$, is surjective. Let $K^0 = \ker(c)$ and fix a character $\chi : K^0 \rightarrow \mathbb{K}(Z)^*$ with $\text{div}(\chi(D)) = D$ for all $D \in K^0$. Consider the following sheaves graded by K and $\text{Cl}(Z)$ respectively:

$$S := \bigoplus_{D \in K} \mathcal{O}_Z(D), \quad \mathcal{R} = S/\mathcal{I},$$

where \mathcal{I} is the ideal sheaf locally generated by elements of the form $1 - \chi(D)$, with $D \in K^0$. The isomorphism class of $\mathcal{R}(Z)$ as a $\text{Cl}(Z)$ -graded ring does not depend on the choices of K and χ , so that it is called the *Cox ring* of Z . The variety Z is called a *Mori dream space* if $\mathcal{R}(Z)$ is finitely generated.

For any effective class $w \in \text{Cl}(Z)$ we define the \mathbb{Z} -graded algebra $\mathcal{R}(Z, w) := \bigoplus_{n \in \mathbb{Z}} \mathcal{R}(Z)_{nw}$ and let $\mathcal{R}(Z, w)_{>0}$ be the subalgebra consisting of all the positively graded elements. We recall that the *irrelevant ideal* $\mathcal{J}_{\text{irr}}(Z)$ of $\mathcal{R}(Z)$ is defined to be the radical of the ideal generated by the subalgebra $\mathcal{R}(Z, w)_{>0}$, for any choice of an ample class w (see [ADHL, Chapter I, §6]). Let

$$\bar{Z} := \text{Spec} \mathcal{R}(Z), \quad \hat{Z} = \bar{Z} - V(\mathcal{J}_{\text{irr}}(Z)).$$

The open subset \hat{Z} is known to be *big* in \bar{Z} , that is its complementary has codimension at least 2. Moreover there exists a good quotient $p_Z : \hat{Z} \rightarrow Z$ with respect to the action of the quasi-torus $H := \text{Spec}(\mathbb{K}[\text{Cl}(Z)])$ induced by the $\text{Cl}(Z)$ -grading of the Cox ring (see [ADHL, Chapter I, §6]).

In what follows we will denote by Z_F the locus of factorial points of Z , that is points $z \in Z$ such that the local ring $\mathcal{O}_{Z,z}$ is a unique factorization domain. Observe that since Z is normal, then Z_F is big in Z . Let X be a closed irreducible subvariety of Z and let $i : X \rightarrow Z$ be the inclusion map. We define $\hat{X} := p_Z^{-1}(X)$ and let \bar{X} be the closure of \hat{X} in \bar{Z} . We have the following commutative diagram:

$$\begin{array}{ccccc}
 \bar{X} & \longrightarrow & \bar{Z} & & \\
 \uparrow & & \uparrow & & \\
 \hat{X} & \longrightarrow & \hat{Z} & \longleftarrow & \hat{Z}_F \\
 \downarrow p_X & & \downarrow p_Z & & \downarrow p_{Z_F} \\
 X & \xrightarrow{i} & Z & \longleftarrow & Z_F
 \end{array}$$

where p_X and p_{Z_F} are restrictions of p_Z , while all the remaining maps are inclusions. In order to define a pull-back map on classes of Weil divisors of Z we will need the following assumption.

Definition 1.1. The embedding $i : X \rightarrow Z$ is *good* if $i^{-1}(Z_F)$ is big in X .

Assume that i is good and let $D = \sum_i a_i D_i$ be a Weil divisor of Z , where D_i are prime divisors and a_i are positive integers, not containing X in its support. Let $D \cap Z_F$ be the restriction of D to Z_F defined as $\sum_i a_i (D_i \cap Z_F)$. Observe that $D \cap Z_F$ is a Cartier divisor since Z_F is factorial. Thus we define the pull-back of D via i^* to be:

$$i^*(D) := \overline{i^*(D \cap Z_F)},$$

where $i^*(D \cap Z_F)$ denotes the usual pull-back of Cartier divisors and the overline denotes the closure of the corresponding Cartier divisor in X . There is a unique divisor in X extending $i^*(D \cap Z_F)$ due to the assumption that $i^{-1}(Z_F)$ is big in X .

The pull-back i^* defined between Weil divisors induces a pull-back map between the class groups of Z and X (that will be denoted with the same symbol). Such map can be obtained as follows. Observe that $\text{Cl}(Z) \cong \text{Cl}(Z_F) \cong \text{Pic}(Z_F)$, where the first isomorphism is due to the fact that Z_F is big in Z and the second to the fact that Z_F is factorial. The same holds by substituting Z with X and Z_F with $i^{-1}(Z_F) \cap X_F$, where X_F is the factorial locus of X . This allows to define a pull-back map $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ by means of the following commutative diagram:

$$\begin{array}{ccc}
 \text{Cl}(Z) & \xrightarrow{i^*} & \text{Cl}(X) \\
 \cong \downarrow & & \uparrow \cong \\
 \text{Pic}(Z_F) & \longrightarrow & \text{Pic}(i^{-1}(Z_F) \cap X_F),
 \end{array}$$

where the lower horizontal arrow is induced by the pull-back of Cartier divisors, which clearly respects linear equivalence (observe that any divisor is linearly equivalent to a divisor not containing X in its support). In what follows, we define $D_X := i^*(D)$ for any divisor D not containing X in its support.

Consider now the class w of a divisor $D \in K$. Given a non-zero element $f \in \mathcal{R}(Z)_w$ there exists a unique $\tilde{f} \in \mathcal{S}(Z)_D$ which is projected to f via the quotient map $\mathcal{S}(Z) \rightarrow \mathcal{R}(Z)$ (see [ADHL, Chapter I, §3]).

Definition 1.2. With the same notation as above, we say that an effective Weil divisor E is *defined* by $f \in \mathcal{R}(Z)_w$ if $E = \text{div}(\tilde{f}) + D$. Moreover, we will denote by \bar{E} the Cartier divisor of \bar{Z} which is defined as the zero locus of the same f thought as a regular function on \bar{Z} .

The following definition is just a reformulation of [Hau08, Definition 2.5].

Definition 1.3. Let Z be a Mori dream space and $X \subset Z$ be a normal, irreducible closed subvariety. The inclusion $i : X \rightarrow Z$ is a *neat embedding* if

- (i) i is good;
- (ii) the pull-back $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ is an isomorphism.

Remark 1.4. In the definition of neat embedding given in [Hau08, Definition 2.5] Z is a toric variety and point i is replaced by the requirement that $D_X^k := D^k \cap X$ are distinct and irreducible divisors in X , where the D^k are divisors defined by a minimal set of generators $\{f_1, \dots, f_k\}$ of the Cox ring of Z . It is not hard to show that this condition implies that i is good.

The proof of the following theorem is essentially the same as that of [Hau08, Theorem 2.6] in case the ambient variety Z is a Mori dream space. Since i is good, we can assume that $Z = Z_F$ and $X = i^{-1}(Z_F)$. In particular all divisors on Z are assumed to be Cartier. Let K be a subgroup of $\text{WDiv}(Z)$ generated by divisors which do not contain X in their support and whose classes generate $\text{Cl}(Z)$. Observe that i^*K is a well-defined subgroup of $\text{WDiv}(X)$. Moreover, the natural homomorphism $i^*K \rightarrow \text{Cl}(X)$ is surjective. The Cox sheaves of Z and X can be thus defined by means of the subgroups K and i^*K . With this choice, there is an isomorphism

$$i^* \mathcal{O}_Z(D) \cong \mathcal{O}_X(i^*D),$$

since all divisors are locally principal on Z . The rest of the proof follows as in [Hau08, Theorem 2.6].

Theorem 1.5. Let Z be a Mori dream space and $X \subset Z$ be a normal, irreducible closed subvariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$. If $X \subset Z$ is a neat embedding and Z is factorial, then there is an isomorphism of K -graded \mathcal{O}_X -algebras:

$$\mathcal{R}_X \cong (p_X)_* \mathcal{O}_{\hat{X}},$$

where \mathcal{R}_X is any Cox sheaf on X . Moreover, \hat{X} is normal and $p_X : \hat{X} \rightarrow X$ is a characteristic space for X .

2. Hypersurfaces in Mori dream spaces

We will now specialize the results of the previous section to the case when X is a normal, irreducible closed hypersurface in Z . In what follows we will assume the inclusion $i : X \rightarrow Z$ to be good, so that the pull-back of Weil divisors is well defined. Observe that the inclusion i induces a pull-back homomorphism

$$i_{\mathcal{R}} : \mathcal{R}(Z) \rightarrow \mathcal{R}(X).$$

We recall that Z_F is the factorial locus of Z and $\hat{Z}_F = p_Z^{-1}(Z_F)$. We will denote by $U_X := Z_F \cap X$ and by $\hat{U}_X := p_X^{-1}(U_X)$.

Theorem 2.1. Let Z be a Mori dream space and let X be a normal, irreducible closed hypersurface of Z defined by $f \in \mathcal{R}(Z)_w$ such that the inclusion $i : X \rightarrow Z$ is good. Then $i_{\mathcal{R}}$ induces an isomorphism $\mathcal{R}(Z)/(f) \cong \mathcal{R}(X)$ if and only if the following conditions hold:

- (i) \hat{X} is big in \bar{X} ,
- (ii) \hat{U}_X is big in \hat{X} ,
- (iii) $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ is an isomorphism.

Proof. Assume first that the three conditions (i), (ii) and (iii) hold. Consider the following commutative diagram:

$$\begin{array}{ccc}
 \hat{U}_X & \longrightarrow & \hat{Z}_F \\
 p_X \downarrow & & \downarrow p_Z \\
 U_X & \xrightarrow{i} & Z_F.
 \end{array}$$

Since Z_F is factorial, we can apply Theorem 1.5 to the inclusion map i . Observe that i is neat by (iii), Z_F is big in Z and U_X is big in X since i is good. Thus we obtain a sheaf isomorphism $\mathcal{R}_{U_X} \cong (p_X)_* \mathcal{O}_{\hat{U}_X}$. Since $\hat{U}_X \subset \hat{X}$ and $\hat{X} \subset \bar{X}$ are big inclusions by (i) and (ii), then:

$$\mathcal{R}(X) = \Gamma(X, \mathcal{R}) = \Gamma(U_X, \mathcal{R}) = \Gamma(\hat{U}_X, \mathcal{O}) = \Gamma(\hat{X}, \mathcal{O}) = \Gamma(\bar{X}, \mathcal{O}),$$

where the first equality is by definition, the second fourth and fifth are due to the fact that the corresponding inclusions of subsets are big, and finally the third equality is due to the sheaf isomorphism given above. The last ring is isomorphic to $\mathcal{R}(Z)/(f)$ and the isomorphism is induced by $i_{\mathcal{R}}$.

Conversely, if $i_{\mathcal{R}}$ induces an isomorphism $\mathcal{R}(Z)/(f) \cong \mathcal{R}(X)$, then $\mathcal{R}(X)$ is graded by $i^* \text{Cl}(Z)$, which is thus isomorphic to $\text{Cl}(X)$. Moreover $p_X : \hat{X} \rightarrow X$ is a characteristic space, so that \hat{X} is big in \bar{X} and p_X does not contract divisors by [ADHL, Chapter I, §6]. The latter property and the fact that U_X is big in X imply that \hat{U}_X is big in \hat{X} . \square

Remark 2.2. Let $\{f_1, \dots, f_k\}$ be a minimal set of generators of $\mathcal{R}(Z)$ and \bar{D}^j 's be the zero sets of the f_j 's in \bar{Z} . Conditions (i) and (ii) in Theorem 2.1 can be given in terms of $\bar{D}^j \cap \bar{X}$: if these are $\text{Cl}(X)$ -prime and distinct divisors, then conditions (i) and (ii) in hold. We recall that a Weil divisor on \bar{X} is called $\text{Cl}(X)$ -prime if it is a finite sum of distinct prime divisors which are transitively permuted by the action of the quasi-torus H_X (see [ADHL, Chapter I, §4]). In fact, it can be proved that the complement $V \subset \bar{Z}$ of all the intersections $\bar{D}^i \cap \bar{D}^j$, with i and j distinct, is contained in \hat{Z}_F and that $V \cap \hat{X}$ is big in \bar{X} because of the hypotheses on the divisors $i^*(\bar{D}^j)$'s. This implies that $V \cap \hat{X}$ is big in \hat{X} and that \hat{X} is big in \bar{X} , giving (ii) and (i) respectively.

Consider now an ample and spanned class $w \in \text{Cl}(Z)$ and let X be the effective divisor defined by a general $f \in \mathcal{R}(Z)_w$. Given such a w , we will denote by $\varphi_w : Z \rightarrow \mathbb{P}^n$ the morphism defined by the complete linear series $|w|$.

Corollary 2.3. Let Z be a projective Mori dream space of dimension ≥ 3 and let X be a closed hypersurface of Z defined by a general $f \in \mathcal{R}(Z)_w$, where w is an ample and spanned class of $\text{Cl}(Z)$. If $\dim(Z) = 3$ we also assume that f is very general and that $(\varphi_w)_* K_Z(1)$ is spanned. Then the following are equivalent:

- (i) \hat{X} is big in \bar{X} .
- (ii) $i_{\mathcal{R}}$ induces an isomorphism $\mathcal{R}(Z)/(f) \cong \mathcal{R}(X)$.

Proof. We have already seen that (ii) implies (i). So we now show that (i) implies (ii). Since w is ample and spanned, and f is general in its Riemann Roch space, then X is irreducible and normal by Bertini's first theorem and [BS95, Theorem 1.7.1]. The genericity assumption on f and the fact that w is spanned imply that i is good. Finally, the restriction map $i^* : \text{Cl}(Z) \rightarrow \text{Cl}(X)$ is an isomorphism by the generalized Lefschetz hyperplane theorems [RS06, Theorem 1] and [RS09, Theorem 1].

We now show that condition (ii) of Theorem 2.1 holds. Recall that the factorial locus Z_F is big in Z . Since p_Z does not contract divisors, then \hat{Z}_F is big in \hat{Z} . Consider an irreducible, not necessarily closed, subvariety B of $Z - Z_F$, which intersects X , and such that all the fibers of p_Z over B have

the same dimension d . Then $\dim(p_Z^{-1}(B)) = \dim(B) + d \leq \dim(\hat{Z}) - 2$. Since X is general and w is spanned, the intersection $B \cap X$ has codimension one in B . Thus

$$\dim(p_X^{-1}(B \cap X)) = \dim(B \cap X) + d \leq \dim(\hat{X}) - 2.$$

Whence $\hat{U}_X = p_X^{-1}(Z_F \cap X)$ is big in \hat{X} and the result follows from Theorem 2.1. \square

Remark 2.4. In [Jow10, Theorem 6] Shin-Yao Jow proved that if Z is a smooth complex projective Mori dream space of dimension ≥ 4 such that

$$V(\mathcal{J}_{\text{irr}}(Z)) = \bar{Z} - \hat{Z} \text{ has codimension } \geq 3 \text{ in } \bar{Z}, \tag{2.1}$$

then every smooth ample divisor $X \subset Z$ is a Mori dream space such that, via the restriction map $\text{Pic}(Z)_{\mathbb{R}} \cong \text{Pic}(X)_{\mathbb{R}}$, the nef cones of Z and that of X coincide and each Mori chamber of X is a union of Mori chambers of Z .

We observe that, under these hypotheses, condition (i) of Theorem 2.1 clearly holds, and condition (iii) is given by the classical Lefschetz hyperplane theorem. Thus Theorem 2.1 states that $i_{\mathcal{R}}$ induces an isomorphism $\mathcal{R}(Z)/(f) \cong \mathcal{R}(X)$ if and only if \hat{X} is big in \bar{X} . The latter condition is equivalent to (2.1). In fact, since the class of X in $\text{Cl}(Z)$ is ample, then the irrelevant locus $V(\mathcal{J}_{\text{irr}}(Z))$ is contained in \bar{X} and equals $\bar{X} - \hat{X}$, so that it has codimension ≥ 3 in \bar{Z} if and only if it has codimension ≥ 2 in \bar{X} .

Remark 2.5. The condition X ample in Z is not necessary to have $\mathcal{R}(X) \cong \mathcal{R}(Z)/(f)$. For example consider the smooth toric fourfold Z whose Cox ring $\mathcal{R}(Z) \cong \mathbb{C}[x_1, \dots, x_6]$ has grading matrix and irrelevant ideal:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{J}_{\text{irr}}(Z) = (x_1, x_2) \cap (x_3, x_4, x_5, x_6).$$

If we put $w_i := \deg(x_i)$, then the movable and the nef cone of Z are $\text{Mov}(Z) = \langle w_1, w_3 \rangle$ and $\text{Nef}(Z) = \langle w_1, w_4 \rangle$, see [ADHL, Propositions 3.2.3 and 3.2.6]. Let

$$f := x_1 x_2 x_5^2 + x_3 x_4 + x_6^2$$

be in $\Gamma(Z, \mathcal{O}_Z(D))$ and let $X := (f) + D$ be the prime divisor of Z defined by f . By the Samuel criterion [Sam64] the quotient ring $R := \mathcal{R}(Z)/(f)$ is factorial, by choosing the \mathbb{Z} -grading $(1, 2, 2, 3, 1)$ on the first five variables. If we denote by

$$\bar{X} := V(f), \quad \hat{X} := \bar{X} - V(\mathcal{J}_{\text{irr}}(Z)),$$

then the restriction $p_X : \hat{X} \rightarrow X$ of the characteristic map $p_Z : \hat{Z} \rightarrow Z$ is a geometric quotient with respect to the action of $H := (\mathbb{C}^*)^2$. Looking at f and $\mathcal{J}_{\text{irr}}(Z)$ we see that \hat{X} is big in \bar{X} . This implies that \bar{X} does not admit invertible global regular functions. Moreover H acts freely on \hat{Z} , and consequently on \hat{X} , since Z is smooth, so that the action of H is strongly stable in the sense of [ADHL, Definition 6.4.1]. Hence the quotient map $p : \hat{X} \rightarrow X$ is a characteristic space by [ADHL, Theorem 6.4.3], so that R is isomorphic to the Cox ring of X . We conclude by observing that the class w_3 of X in $\text{Cl}(Z)$ is not nef.

3. Calabi–Yau threefolds in smooth toric Fano varieties

We apply results of the previous section to the case $X \subset Z$, where Z is a smooth toric Fano variety and X is a smooth hyperplane section of Z in the anticanonical embedding. Thus X is a Calabi–Yau threefold. We will often make use of the Magma database of smooth toric Fano varieties. To check any Magma calculation contained in this paper follow these steps:

- open this page: <http://www2.udec.cl/~alaface/software/T-Fano.txt> and copy its content into the online Magma calculator located here: <http://magma.maths.usyd.edu.au/calc/>;
- paste in the same window, below the previous text, the function occurring in the calculation.

The result will be the output of the corresponding Magma calculation done in the paper. All the software sessions in this section are in Magma code [BCP97].

Theorem 3.1. *Let X be a smooth Calabi–Yau threefold which is hyperplane section of a smooth toric Fano variety Z . Then $\mathcal{R}(X) \cong \mathcal{R}(Z)/(f)$ if and only if Z is one of the following:*

| Z | Grading matrix | Irrelevant ideal |
|--|--|--|
| $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ | $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}$ | $(x_1, x_2, x_5) \cap (x_3, x_4, x_6)$ |
| $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$ | $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ | $(x_1, x_2, x_5) \cap (x_3, x_4, x_6)$ |
| $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ | $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$ | $(x_1, x_2, x_5) \cap (x_3, x_4, x_6)$ |
| $\mathbb{P}^2 \times \mathbb{P}^2$ | $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ | $(x_1, x_2, x_5) \cap (x_3, x_4, x_6)$ |
| \mathbb{P}^4 | $[1 \ 1 \ 1 \ 1 \ 1]$ | (x_1, \dots, x_5) |

Proof. First we show that, by looking into the Magma database of smooth toric Fano varieties, there are exactly five such varieties whose Cox ring admits an irrelevant ideal of codimension at least 3.

```
> [n : n in [24..147] |
  Length(FanoX(n))-Dimension(IrrelevantIdeal(FanoX(n))) ge 3];
[ 44, 70, 141, 146, 147 ]
```

Each of these varieties, let us say Z , satisfies the hypothesis of Theorem 2.1 by Remark 2.4, since $-K_Z$ is very ample and in particular it is ample and spanned. Hence any smooth element X of the linear series $|-K_Z|$ has Cox ring isomorphic to $\mathcal{R}(Z)/(f)$, where $f \in H^0(Z, -K_Z)$ is a defining section for X in Z .

To obtain the grading matrix for the Cox ring of Z and the irrelevant ideal we again ask Magma to do it. For example the variety n. 44 is:

```
> FanoX(44);
Toric variety of dimension 4
Variables: $.1, $.2, $.3, $.4, $.5, $.6
The components of the irrelevant ideal are:
  ($.6, $.4, $.3), ($.5, $.2, $.1)
The 2 gradings are:
  0, 0, 1, 1, 0, 1,
  1, 1, 2, 0, 1, 2
```

This gives the second and third column of the central table of our theorem. To provide the projective models for the five varieties we calculate the value of $-K_Z^4$, which is just the degree of Z in the anticanonical embedding:

```
> Degree(-CanonicalDivisor(FanoX(44)));
594
```

and determine all the linear relations within the vertices of the polytope defined by $-K_Z$:

```
> Kernel(Matrix(Vertices(FanoP(44))));
RSpace of degree 6, dimension 2 over Integer Ring
Echelonized basis:
( 1 1 0 -2 1 0)
( 0 0 1 1 0 1)
```

Looking for varieties with these invariants in [Bat99, Proposition 3.1.1 and p. 1046] we obtain the left hand side column of our table. \square

Remark 3.2. Observe that $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ is isomorphic to the blow-up, along the vertex, of the quadratic cone of dimension four with vertex a line, while $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ is isomorphic to the blow-up of \mathbb{P}^4 along a line.

Theorem 3.3. *Let Z be one of the five varieties of Theorem 3.1 and let P_Z be the polytope defined by $-K_Z$. Let Z^* be the toric variety constructed from the polar polytope P_Z^* . Then the general element of $| -K_{Z^*} |$ is a Mori dream Calabi–Yau variety whose Cox ring admits just one relation: $\mathcal{R}(X^*) \cong \mathcal{R}(Z^*)/(f^*)$.*

Proof. By [Ba2] Z^* is a toric Fano variety, so that $-K_{Z^*}$ is ample. To prove the theorem it is enough to show, by Corollary 2.3, that the irrelevant ideal of the Cox ring $\mathcal{R}(Z^*)$ has codimension at least 3. The following Magma command calculates the codimension of the irrelevant ideal for any such Z^* :

```
> [Length(FanoDualX(n))-Dimension(IrrelevantIdeal(FanoDualX(n))):
n in [44, 70, 141, 146, 147]];
[ 3, 3, 3, 3, 5 ]  $\square$ 
```

4. Hypersurfaces of \mathbb{P}^n containing a codimension two linear space

Denote by X_d a general degree d smooth hypersurface of \mathbb{P}^n containing a linear subspace L of codimension 2. Here we assume X_d to be at least three dimensional. An elementary calculation shows that X_d is singular at a finite set of points lying on L . However, after blowing up L in \mathbb{P}^n , the resulting strict transform X of X_d is smooth. Denote by $\pi : Z \rightarrow \mathbb{P}^n$ the blow-up map at L . The Cox ring of Z is a polynomial ring in $n + 2$ variables with grading matrix

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ -1 & -1 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

The last variable, x_{n+2} , corresponds to the exceptional divisor of the blow-up. Denote by $\bar{Z} := \text{Spec}(\mathcal{R}(Z))$ and by $\hat{Z} \subseteq \bar{Z}$ the characteristic space of Z together with the characteristic map

$p : \hat{Z} \rightarrow Z$. Observe that the irrelevant ideal of Z is

$$\mathcal{J}_{\text{irr}}(Z) = (x_1, x_2) \cap (x_3, \dots, x_{n+2}).$$

Let $\hat{X} := p^{-1}(X)$ and \bar{X} be its Zariski closure in \bar{Z} . The equation of \bar{X} in \bar{Z} is

$$x_1 f + x_2 g = 0,$$

where f and g are general polynomials of degree $(d - 1)e_1$. Observe that the quotient ring $\mathcal{R}(Z)/(x_1 f + x_2 g)$ is non-factorial, since \bar{x}_1 is irreducible in the quotient ring but it does not divide \bar{g} due to the generality assumption on X_d . We introduce a new variable x_0 in order to obtain factoriality. Consider the \mathbb{Z}^2 -graded ring

$$R := \mathbb{C}[x_0, \dots, x_{n+2}]/(x_0 x_2 - f, x_0 x_1 + g),$$

where the gradings of x_i , with $i = 1, \dots, n + 2$, are given before and $\deg(x_0) = (d - 2)e_1 + e_2 = \begin{bmatrix} d-2 \\ 1 \end{bmatrix}$.

Theorem 4.1. *The Cox ring $\mathcal{R}(X)$ is isomorphic to R .*

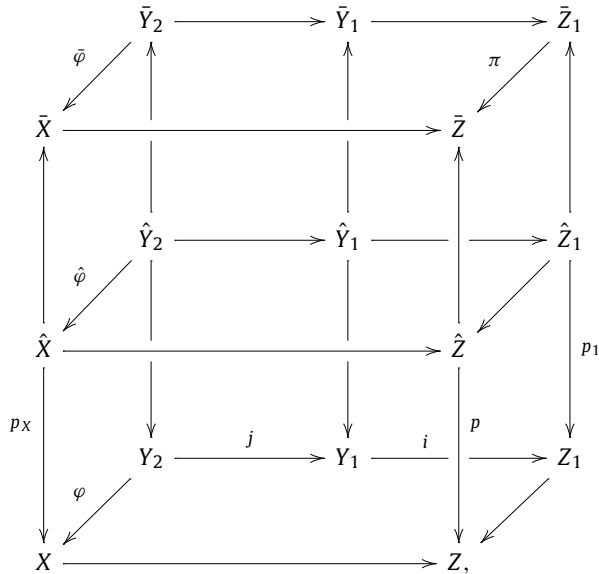
Proof. Let Z_1 be the toric variety whose Cox ring $\mathcal{R}(Z)$ is isomorphic to the \mathbb{Z}^2 -graded polynomial ring $\mathbb{C}[x_0, \dots, x_{n+2}]$ such that $w := e_1$ is an ample class. The Cox ring of Z_1 is a polynomial ring in $n + 3$ variables with grading matrix

$$\begin{bmatrix} d-2 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{4.1}$$

The first variable is the x_0 just introduced, while the remaining variables have the same grading of those of $\mathcal{R}(Z)$. The irrelevant ideal of $\mathcal{R}(Z_1)$ is

$$\begin{aligned} \mathcal{J}_{\text{irr}}(Z_1) &= (x_0, x_3, \dots, x_{n+2}) \\ &\cap (x_1, \dots, x_{n+1}), \end{aligned}$$

so that the corresponding irrelevant locus has codimension $n + 1$ in \bar{Z}_1 . Observe that a very general element of the linear series $|d - 1e_1|$ in Z_1 can be written in the form $\alpha(x_0 x_2 - f) + \beta(x_0 x_1 + g)$, with f and g very general. Let $Y_1 \in |d - 1e_1|$ be such a very general element. We observe that Y_1 is normal, by [BS95, Theorem 1.7.1]. Moreover the inclusion map $i : Y_1 \rightarrow Z_1$ induces an isomorphism $i^* : \text{Cl}(Z_1) \rightarrow \text{Cl}(Y_1)$, by [RS06, Theorem 1]. Let $Y_2 \subseteq Y_1$ be a very general element of the linear series $|d - 1e_1|$ cut out in Z_1 by the equations $x_0 x_2 - f = 0, x_0 x_1 + g = 0$. As before we see that Y_2 is normal and the inclusion map $j : Y_2 \rightarrow Z_2$ induces an isomorphism $j^* : \text{Cl}(Y_1) \rightarrow \text{Cl}(Y_2)$. Let $\hat{Y}_i := p^{-1}(Y_i)$ and let \bar{Y}_i be the Zariski closure of \hat{Y}_i in \bar{Z}_i . Observe that the intersection $\{x_k = 0\} \cap \bar{Y}_i$ is irreducible for any i and k . Moreover, as k varies, the intersections are distinct. Thus, since $\deg(x_0 x_2 - f) = \deg(x_0 x_1 + g) = (d - 1)e_1$ is ample and spanned in both Z_1 and Y_1 , then \hat{Y}_i is a characteristic space for Y_i , by Corollary 2.3. Consider now the commutative diagram:



where all the horizontal arrows are inclusion maps, π is the projection on the last $n + 2$ coordinates and $\bar{\varphi}$ is the restriction of π since $(x_0x_2 - f, x_0x_1 + g) \cap \mathbb{C}[x_1, \dots, x_{n+2}] = (x_1f + x_2g)$. If $x \in \hat{Y}_2 - \{x_1 = x_2 = 0\}$, then x_0 is uniquely determined by the equations of Y_2 , so that $\hat{\varphi}$ is one to one on this big open subset of \hat{Y}_2 . Observe that $\{x_1 = x_2 = 0\}$ is not contained in \hat{X} since it is contained in the irrelevant locus of \hat{Z} . Thus φ is an isomorphism in codimension 1 so that $\mathcal{R}(X) \cong \mathcal{R}(Y_2) \cong \mathcal{O}(\bar{Y}_2) = R$. \square

Remark 4.2. Unfortunately Corollary 2.3 cannot be applied to compute $\mathcal{R}(X)$ in the case $n = 3$. In fact, to show that the generalized Lefschetz Theorem still holds for $Y_2 \subset Y_1$ one needs the extra condition that $\varphi_*(K_{Y_1})(1)$ is globally generated [RS09, Theorem 1], where φ is the morphism on Y_1 associated to the complete linear series $|Y_2|$. According to the grading matrix (4.1) the class D of Y_1 has degree $(d - 1)e_1$ and the zero locus of all monomials in the Riemann–Roch space of D coincides with the irrelevant locus of \bar{Y}_2 . Thus D is very ample, so that the previous condition is equivalent to the base-point freeness of the divisor $K_{Y_2} + D$. However, since $K_{Y_2} + D = K_{Y_1} + 2D$ has degree $(d - 4)e_1$ then the common zero locus of its monomials is the union of the two components $V(x_1, x_2, x_3, x_4)$ and $V(x_3, x_4, x_5)$. The first component is contained into the irrelevant locus, while the second one intersects $\bar{Y}_2 = V(x_0x_2 - f)$ along a two dimensional orbit which, under the action of the torus, becomes a point p of Y_2 . Hence the base locus of $K_{Y_1} + 2D$ consist exactly of p , so that the required condition fails. In the following subsection we will prove the analogous of Theorem 4.1 in the three dimensional case using a different technique.

4.1. Calculating the Cox ring when $n = 3$

We now proceed to calculate a presentation for the Cox ring in the three dimensional case.

Lemma 4.3. *Let a, b be positive integers such that $a + (2 - d)b > 0$ and $a + b > d - 4$. Then $D = aH + bL$ is non-special, that is $h^1(D) = 0$.*

Proof. By the adjunction formula the canonical divisor K_X is linearly equivalent to $(d - 4)H$. Write $D - K_X = N + \varepsilon L$, where $N := (a - d + 4)H + (b - \varepsilon)L$ and $\varepsilon := (d - 4)/(d - 2)$. Intersecting with the generators of the extremal rays of the effective cone gives

$$N \cdot L = (a - d + 4) + (b - \varepsilon)(2 - d) > 0, \quad N \cdot (H - L) = a + b - (d - 4) - \varepsilon > 0,$$

where the second inequality follows since $0 < \varepsilon < 1$. Thus N lies in the interior of the nef cone, so that it is ample. Thus we conclude by the Kawamata–Viehweg vanishing theorem. \square

Theorem 4.4. *The Cox ring of X is the following \mathbb{Z}^2 -graded ring*

| $\mathcal{R}(X)$ | Grading matrix |
|---|--|
| $\frac{\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]}{(x_0x_1 - f, x_0x_2 - g)}$ | $\begin{bmatrix} d-2 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$ |

where f and g are very general polynomials in x_1x_5, x_2x_5, x_3, x_4 .

Proof. The grading matrix in the statement of the theorem is given with respect to the classes of H and L . We now choose s_0, \dots, s_5 non-zero sections of the Cox ring such that the degree of s_i is the $i + 1$ -th column of the grading matrix. Let $s_0 \in H^0((d - 2)H + L)$ be a section which does not vanish on L . There is such a section by Lemma 4.3 and Kawamata–Viehweg. Let s_1, s_2 be a basis of $H^0(H - L)$ and let s_3, s_4 be such that $H^0(H) = \langle s_1s_5, s_2s_5, s_3, s_4 \rangle$, where s_5 is a section defining L .

To prove that the s_i actually generate the Cox ring it is enough to show it for nef divisors, since if D is not nef, then $D \sim N + aL$ for some positive integer a and some nef divisor N . Thus $H^0(D) = s_5^a \cdot H^0(N)$. Consider now a nef divisor $D = aH + bL$. The exact sequence $0 \rightarrow \mathcal{O}_X(D - 2F) \rightarrow \mathcal{O}_X(D - F) \oplus \mathcal{O}_X(D - F) \rightarrow \mathcal{O}_X(D) \rightarrow 0$, where $F \sim H - L$, induces the following exact sequence in cohomology:

$$H^0(D - F) \oplus H^0(D - F) \xrightarrow{f} H^0(D) \longrightarrow H^1(D - 2F),$$

where $f(u, v) = us_1 + vs_2$. Observe that since D is nef then the following hold:

$$a + b \geq 0, \quad k := D \cdot L = a + (2 - d)b \geq 0.$$

We consider five cases.

- (i) If $b = 0$, then $D = aH$. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(n)) = 0$ for all $n \in \mathbb{Z}$ by [Har77, Theorem 5.1], then the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(a)) \rightarrow H^0(aH)$ is surjective for all $a \geq 0$. This implies that the elements of $H^0(aH)$ are polynomials in s_1, \dots, s_5 , for any non-negative a .
- (ii) If $a + b = 0$, then D is a multiple of F . Hence the complete linear series $|D|$ is composed with the pencil $|F|$ or equivalently any element of $H^0(D)$ is a polynomial in s_1, s_2 .
- (iii) If $k \geq 2d - 3$ and $a + b > d - 4$, then $h^1(D - 2F) = 0$, by Lemma 4.3. In this case, the map f is surjective, so that the elements of $H^0(D)$ are polynomials in s_1, s_2 and sections in $H^0(D - F)$.
- (iv) If $k \leq 2d - 4$ and $b \geq 1$, consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & & H^0(kH) & & & \\
 & & & \downarrow \cdot s_0^b & \searrow r' & & \\
 0 & \longrightarrow & H^0(D - L) & \xrightarrow{\cdot s_5} & H^0(D) & \xrightarrow{r} & H^0(\mathcal{O}_{\mathbb{P}^1}(k)) \longrightarrow H^1(D - L),
 \end{array}$$

where the bottom row is exact, the vertical map is multiplication by s_0^b , since $D = kH + b((d - 2)H + L)$, and r' is the restriction map to L . Since $a \geq (d - 2)b$ and $a + b \geq (d - 1)b \geq d - 1$, then $h^1(D - L) = 0$ by Lemma 4.3, so that r is surjective. Moreover r' is surjective for any $k \geq 0$

since the restrictions of s_3, s_4 to L span $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Thus any element $s \in H^0(D)$ is a sum $s = u s_5 + v s_0^b$, with $u \in H^0(D - L)$ and $v \in H^0(kH)$.

(v) If $a + b \leq d - 4$, then $b \leq 0$ since $a \geq (d - 2)b$. Let $C \in |H - L|$ be the curve defined by s_1 . Observe that C is a plane curve of degree $d := a + b$. Thus we have a commutative diagram:

$$\begin{array}{ccccc}
 & & H^0(\mathcal{O}_{\mathbb{P}^3}(d)) & \xrightarrow{r_1} & H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \\
 & & \downarrow \gamma & & \downarrow r_2 \\
 0 & \longrightarrow & H^0(D - F) & \xrightarrow{s_1} & H^0(D) & \xrightarrow{r} & H^0(D|_C),
 \end{array}$$

where r_1, r_2, r are restriction maps, γ is the restriction to X composed with the multiplication by s_2^{-b} and the bottom row is exact. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(h)) = 0$ for any $h \in \mathbb{Z}$ by [Har77, Theorem 5.1], then r_2 is surjective. Since both r_1 and r_2 are surjective, then r is surjective as well. Thus, any section of $s \in H^0(D)$ is a sum $s = u s_1 + v s_2^{-b}$, where $u \in H^0(D - F)$ and $v \in H^0(dH)$.

The previous arguments show that the Cox ring $\mathcal{R}(X)$ is generated by s_0, \dots, s_5 . Let I_X be the kernel of the ring homomorphism

$$\mathbb{C}[x_0, \dots, x_5] \rightarrow \mathcal{R}(X), \quad x_i \mapsto s_i.$$

Since the vector space $H^0((d - 1)H)$ is generated by degree $d - 1$ polynomials in $s_1 s_5, s_2 s_5, s_3, s_4$ and $s_0 s_1, s_0 s_2$ belong to this space, then I_X contains two polynomials of type $x_0 x_1 - f, x_0 x_2 - g$ as in the statement. The quotient ring $R = \mathbb{C}[x_0, \dots, x_5]/(x_0 x_1 - f, x_0 x_2 - g)$ is an integral domain, by the generality assumption on f and g . The surjectivity of $R \rightarrow \mathcal{R}(X)$ implies that the corresponding morphism $\text{Spec}(\mathcal{R}(X)) \rightarrow \text{Spec}(R)$ is injective. Since both $\text{Spec}(\mathcal{R}(X))$ and $\text{Spec}(R)$ are four dimensional integral affine varieties, then we deduce $R = \mathcal{R}(X)$. \square

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