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On whether or not $\mathscr{L}(E,F)=\mathscr{L}^r(E,F)$ for some classical Banach lattices E and F

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ABSTRACT

For Banach lattices E and F, $\mathscr{L}(E,F)$ is the space of all continuous linear operators $E\to F$, $\mathscr{L}^r(E,F)$ is the vector space of all regular continuous linear operators $E\to F$ which is endowed with the r-norm. This paper concerns the problems: (1) is every continuous linear operator $E\to F$ regular? (2) if the answer to (1) is "yes", there is a further problem: is its operator norm in $\mathscr{L}(E,F)$ equal to its r-norm in $\mathscr{L}^r(E,F)$? A series of conclusions is obtained for cases in which each of E and F is one of Banach lattices l_p $(1\le p<\infty)$, l_∞, c_0, c, $C[0,1]$ and $C(X)$.

INTRODUCTION

For Banach lattices E and F, $\mathscr{L}(E,F)$ is the space of all continuous linear operators $E\to F$, $\mathscr{L}^+(E,F)$ is the set of all positive elements of $\mathscr{L}(E,F)$. In general, $\mathscr{L}(E,F)$ possibly is not a vector lattice. Let $\mathscr{L}^r(E,F)$ denote the vector space of all regular operators $E\to F$ (i.e., every element T of $\mathscr{L}^r(E,F)$ possesses a decomposition $T=T_1-T_2$ where T_1 and T_2 are positive and continuous) in which the r-norm $\|\cdot\|_r$ is defined (cf. IV. § 1 of [S]) by

$$\|T\|_r=\inf\{\|T_1+T_2\|: T=T_1-T_2, T_i\in\mathscr{L}^+(E,F)\ (i=1,2)\}.$$

Equivalently, $\|T\|_r=\inf\{\|2T_1-T\|: T_1\in\mathscr{L}^+(E,F)$ with $T_1\ge T\}$. It is easy to see: for every $T\in\mathscr{L}^r(E,F)$, (1) $\|T\|\le\|T\|_r$; (2) if $T_0\ge T$ and $T_0\ge -T$ then $\|T_0\|\ge\|T\|_r$. (Indeed, since $T=\frac{1}{2}[(T_0+T)-(T_0-T)]$ and T_0+T, $T_0-T\in\mathscr{L}^+(E,F)$, we have $\|T\|_r\le\|\frac{1}{2}[(T_0+T)+(T_0-T)]\|=\|T_0\|$.) The r-norm makes $\mathscr{L}^r(E,F)$ into an ordered Banach space (cf. IV. Exerc. 3 of [S]). If F is order

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complete, then $\|T\|_r = \||T|\|$ and in this case $\mathscr{L}^r(E,F)$ is an order complete Banach lattice (cf. IV. 1.4 of [S]).

When studying the relationship between $\mathscr{L}(E,F)$ and $\mathscr{L}^r(E,F)$, two basic problems come to us: for Banach lattices E and F, is every continuous linear operator $E \to F$ regular? and if the answer to this problem is "yes", there is a further problem: is its operator norm in $\mathscr{L}(E,F)$ equal to its r-norm in $\mathscr{L}^r(E,F)$? (or, whether or not $\mathscr{L}(E,F) = \mathscr{L}^r(E,F)$ and whether or not $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$ where the meanings of "$=$" and "$\equiv$" will be explained below.) So far, the two problems are still far from their solutions. An essential theorem concerning the problems is Theorem 1.5 of Chap. IV of [S] which will be quoted in Section 1. In this paper we consider the problem, taking for E and F certain classical Banach lattices, that is, each of E and F is one of Banach lattices l_p $(1 \le p < \infty)$, l_∞, c_0, c, $C[0,1]$ and $C(X)$. Most of these cases we can settle in Section 2.

While considering the classical Banach lattices, we find that some special observations can be extended to general situations. As necessary preparations for Section 2 we put them into Section 1.

The basic terminology and elementary facts can be found, for instance, in [S]. By $\mathscr{L}(E,F) = \mathscr{L}^r(E,F)$ we mean that every element of $\mathscr{L}(E,F)$ is regular. By $\mathscr{L}(E,F) \neq \mathscr{L}^r(E,F)$ we mean that there exists an element of $\mathscr{L}(E,F)$ which is not regular. By $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$ we mean that $\mathscr{L}(E,F) = \mathscr{L}^r(E,F)$ and $\|T\| = \|T\|_r$ for every $T \in \mathscr{L}(E,F)$.

In this paper, X and Y are always compact Hausdorff spaces and they are always infinite. (If a compact Hausdorff space X is finite, it is clear that, for every Banach lattice F, $\mathscr{L}(C(X),F) = \mathscr{L}^r(C(X),F)$ and $\mathscr{L}(F,C(X)) \equiv \mathscr{L}^r(F,C(X))$.) We define

$$X_l := \{x \in X\colon \text{there exist } x_1, x_2, \ldots \text{ in } X \text{ with } x_i \neq x_j\ (i \neq j) \text{ and } \lim_{n\to\infty} x_n = x\}.$$

The cardinal of X_l is denoted by Card X_l. If X_l is infinite, we denote simply Card $X_l = \infty$. When we say "a nontrivial sequence $\{x_n\}_{n\in\mathbb{N}}$" we always assume that $x_i \neq x_j$ $(i \neq j)$. If a nontrivial sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞, we assume that $x_n \neq x_\infty$ $(n \in \mathbb{N})$.

The characteristic function of a subset A of X is denoted by 1_A.

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1. NECESSARY THEOREMS AND LEMMAS

Before starting our discussion, we mention an essential theorem (cf. Theorem 1.5 of Chap. IV of [S]) as follows.

THEOREM 1.0. *Let E,F be Banach lattices. Then $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$ whenever at least one of the following conditions is satisfied:*

(1) *F is an order complete AM-space with unit.*

(2) *E is an AL-space, and there exists a positive contractive projection $P\colon F^{**} \to F$.*

For $T\in \mathscr{L}(E,F)$, $T^*:F^*\to E^*$ is the adjoint operator. We know that if T is regular then T^* is also regular, but the map $T\to T^*$ need not preserve the r-norm. If F is reflexive, the situation can be improved much.

THEOREM 1.1. *Let E be a Banach lattice and F a reflexive Banach lattice.*
(1) *For any $\Omega\in\mathscr{L}(F^*,E^*)$, there exists a $T\in\mathscr{L}(E,F)$ such that $T^*=\Omega$, i.e. $\mathscr{L}(F^*,E^*)=\{T^*: T\in\mathscr{L}(E,F)\}$.*
(2) *$\mathscr{L}(E,F)=\mathscr{L}^r(E,F)$ if and only if $\mathscr{L}(F^*,E^*)=\mathscr{L}^r(F^*,E^*)$.*
(3) *$\mathscr{L}(E,F)\equiv\mathscr{L}^r(E,F)$ if and only if $\mathscr{L}(F^*,E^*)\equiv\mathscr{L}^r(F^*,E^*)$.*

PROOF. (1). For $\Omega\in\mathscr{L}(F^*,E^*)$, set $T:=\Omega^*|_E$. Then $T\in\mathscr{L}(E,F)$ and $T^*=\Omega$. (2) and (3) are not difficult to prove; we leave the proofs to reader. □

THEOREM 1.2. *Let E_1,E_2 and F be Banach lattices. Suppose there exist $\Phi_1\in\mathscr{L}(E_1,E_2)$ and $\Phi_2\in\mathscr{L}(E_2,E_1)$ such that $\Phi_2\Phi_1$ is the identity map of E_1. Then*
(1) *if $\Phi_1\geq 0$ and $\mathscr{L}(E_2,F)=\mathscr{L}^r(E_2,F)$, then $\mathscr{L}(E_1,F)=\mathscr{L}^r(E_1,F)$;*
(2) *if $\Phi_1\geq 0$, $\|\Phi_1\|\cdot\|\Phi_2\|=1$ and $\mathscr{L}(E_2,F)\equiv\mathscr{L}^r(E_2,F)$ then $\mathscr{L}(E_1,F)\equiv\mathscr{L}^r(E_1,F)$;*
(3) *if $\Phi_2\geq 0$ and $\mathscr{L}(F,E_2)=\mathscr{L}^r(F,E_2)$, then $\mathscr{L}(F,E_1)=\mathscr{L}^r(F,E_1)$;*
(4) *if $\Phi_2\geq 0$, $\|\Phi_1\|\cdot\|\Phi_2\|=1$ and $\mathscr{L}(F,E_2)\equiv\mathscr{L}^r(F,E_2)$ then $\mathscr{L}(F,E_1)\equiv\mathscr{L}^r(F,E_1)$.*

The proof is direct without difficulty. We omit it.

COROLLARY 1.3. *Let F be a Banach lattice and X a compact Hausdorff space. Suppose there exists a nontrivial convergent sequence in X. Then*
(1) *if $\mathscr{L}(C(X),F)=\mathscr{L}^r(C(X),F)$, then $\mathscr{L}(c,F)=\mathscr{L}^r(c,F)$;*
(2) *if $\mathscr{L}(C(X),F)\equiv\mathscr{L}^r(C(X),F)$, then $\mathscr{L}(c,F)\equiv\mathscr{L}^r(c,F)$;*
(3) *if $\mathscr{L}(F,C(X))=\mathscr{L}^r(F,C(X))$, then $\mathscr{L}(F,c)=\mathscr{L}^r(F,c)$;*
(4) *if $\mathscr{L}(F,C(X))\equiv\mathscr{L}^r(F,C(X))$, then $\mathscr{L}(F,c)\equiv\mathscr{L}^r(F,c)$.*

PROOF. Let $\{t_n\}_{n\in\mathbb{N}}$ be a nontrivial convergent sequence in X. We construct pairwise disjoint open sets U_n ($n\in\mathbb{N}$) such that $t_n\in U_n$, and choose $h_n\in C(X)$ ($n\in\mathbb{N}$) such that $0\leq h_n\leq 1$, $h_n(t_n)=1$ and $h_n=0$ outside U_n. For all $\alpha=(\alpha_1,\alpha_2,\dots)\in c$ ($\alpha_\infty:=\lim_{n\to\infty}\alpha_n$), since the series $\sum_{n=1}^{\infty}(\alpha_n-\alpha_\infty)h_n$ is uniformly convergent, we can define $\Phi_1\in\mathscr{L}(c,C(X))$ by

$$\Phi_1\alpha=\alpha_\infty 1_X+\sum_{n=1}^{\infty}(\alpha_n-\alpha_\infty)h_n.$$

Obviously $\Phi_1\geq 0$. Now we define $\Phi_2\in\mathscr{L}^+(C(X),c)$ by

$$\Phi_2 f=(f(t_1),f(t_2),\dots)\text{ for all } f\in C(X).$$

Then $\Phi_2\Phi_1$ is the identity map of c and $\|\Phi_1\|=\|\Phi_2\|=1$. By Theorem 1.2, the conclusion follows immediately. □

COROLLARY 1.4. *Let F be a Banach lattice.*
(1) *If $\mathscr{L}(C[0,1],F)=\mathscr{L}^r(C[0,1],F)$, then $\mathscr{L}(c,F)=\mathscr{L}^r(c,F)$.*

(2) *If* $\mathscr{L}(C[0,1],F)\equiv\mathscr{L}^r(C[0,1],F)$, *then* $\mathscr{L}(c,F)\equiv\mathscr{L}^r(c,F)$.
(3) *If* $\mathscr{L}(F,C[0,1])=\mathscr{L}^r(F,C[0,1])$, *then* $\mathscr{L}(F,c)=\mathscr{L}^r(F,c)$.
(4) *If* $\mathscr{L}(F,C[0,1])\equiv\mathscr{L}^r(F,C[0,1])$, *then* $\mathscr{L}(F,c)\equiv\mathscr{L}^r(F,c)$.

The converses of (1) and (2) of Corollary 1.4 are not true. For instance, we shall see below that $\mathscr{L}(c,c)\equiv\mathscr{L}^r(c,c)$ (Theorem 2.13), but $\mathscr{L}(C[0,1],c)\neq\neq\mathscr{L}^r(C[0,1],c)$ (Theorem 2.12). We do not know whether the converses of (3) and (4) of Corollary 1.4 are true or false.

Applying Theorem 1.2, we take c_0 as E_1 and c as E_2 while the map Φ_1 is the identity map from c_0 into c and the map $\Phi_2:c\to c_0$ is defined by

$$\Phi_2x=(x(1)-x(\infty),x(2)-x(\infty),\dots)\ (x\in c,x(\infty):=\lim_{n\to\infty}x(n)).$$

Then we have a corollary as follows.

COROLLARY 1.5. *Let F be a Banach lattice. If* $\mathscr{L}(c,F)=\mathscr{L}^r(c,F)$ *then* $\mathscr{L}(c_0,F)=\mathscr{L}^r(c_0,F)$.

The converse is not true. For instance, we shall see below that $\mathscr{L}(c_0,c_0)\equiv\equiv\mathscr{L}^r(c_0,c_0)$ (Theorem 2.2), but $\mathscr{L}(c,c_0)\neq\mathscr{L}^r(c,c_0)$ (Theorem 2.8).

As preparations for the next section, we make some observations about a compact Hausdorff space X.

THEOREM 1.6. *Let X be a compact Hausdorff space. If there exists a sequence in X which has no convergent subsequence, then there exist a regular Borel measure* μ *and Borel measurable functions* $g_1,g_2,\dots$ *on X such that (a)* $\mu(X)=1$, *(b)* $g_1=1_X$, *(c)* $|g_n(x)|=1$ μ*-almost everywhere, (d)* $\int g_ng_m d\mu=0$ $(n\neq m)$.

PROOF. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in X which has no convergent subsequence. We may assume that $\{y_n\}_{n\in\mathbb{N}}$ is nontrivial. Set $X_2:=X$. By compactness we can find two distinct accumulation points a and b of the sequence in X_2. It follows that there exist compact neighborhoods X_3 and X_4 of a and b, respectively, with $X_3\cap X_4=\varnothing$. Obviously, each of X_3 and X_4 contains infinitely many points of $\{y_n\}_{n\in\mathbb{N}}$. Applying the same argument to X_3 and X_4, respectively, we obtain compact subsets X_5, X_6 of X_3 and X_7, X_8 of X_4 such that $X_5\cap X_6=\varnothing$, $X_7\cap X_8=\varnothing$ and each of X_5, X_6, X_7 and X_8 contains infinitely many points of $\{y_n\}_{n\in\mathbb{N}}$. Continuing the method we can obtain compact sets $X_2,X_3,X_4,X_5,\dots$ in which each X_i contains infinitely many points of $\{y_n\}_{n\in\mathbb{N}}$ and contains two disjoint sets, X_{2i-1} and X_{2i}.

We now construct a regular Borel measure μ on X such that

$$(*)\qquad \mu(X_i)=\frac{1}{2^{n-1}}\text{ if }i\in\{2^{n-1}+1,\dots,2^n\},\ n\in\mathbb{N}.$$

To this end, we take $x_i\in X_i$ $(i=2,3,\dots)$ and for $f\in C(X)$ define

$$\phi_n(f)=\frac{1}{2^{n-1}}\,(f(x_{2^{n-1}+1})+\dots+f(x_{2^n})),\ n\in\mathbb{N}.$$

Let $D := \{f \in C(X) : \lim_{n\to\infty} \phi_n(f) \text{ exists}\}$. Obviously, D is a linear subspace of $C(X)$ and $1_X \in D$. Define $\phi : D \to \mathbb{R}$ by

$$\phi(f) = \lim_{n\to\infty} \phi_n(f) \text{ for all } f \in D.$$

It is easy to see that $\phi \in D^*$ with $\phi(1_X) = 1$ and $\|\phi\| = 1$. Use the Hahn-Banach Theorem to extend ϕ to $\Phi \in C(X)^*$, so $\Phi(1_X) = 1$ and $\|\Phi\| = 1$. Thus, Φ corresponds to a regular Borel measure μ. To show that the measure μ satisfies (*), for $n \in \mathbb{N}$ and $i \in \{2^{n-1}+1, \ldots, 2^n\}$ we take $g \in C(X)$ such that $g = 1$ on X_i and $g = 0$ on $X_{2^{n-1}+1} \cup \ldots \cup X_{i-1} \cup X_{i+1} \cup \ldots \cup X_{2^n}$. Thus, $g = 1_{X_i}$ on the support of μ. It is easy to see that

$$\frac{1}{2^{n-1}} = \phi_n(g) = \phi_{n+1}(g) = \phi_{n+2}(g) = \ldots.$$

Consequently, $g \in D$ and

$$\frac{1}{2^{n-1}} = \lim_{j\to\infty} \phi_j(g) = \phi(g) = \Phi(g) = \int g\,d\mu = \int 1_{X_i}\,d\mu = \mu(X_i).$$

Make Borel measurable functions $g_1, g_2, \ldots$ on X such that

$$g_n = (-1)^i \text{ on } X_i \text{ if } i \in \{2^{n-1}+1, \ldots, 2^n\},\ n \in \mathbb{N}.$$

It is easy to check that $g_1, g_2, \ldots$ are as desired. □

In the above proof, let $Y_i := X_{2^{i+1}-1}$ $(i \in \mathbb{N})$. We can prove easily the following corollary.

COROLLARY 1.7. *Let X be a compact Hausdorff space. If there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X which has no convergent subsequence, then there exist compact subsets $Y_1, Y_2, \ldots$ of X and mutually disjoint open subsets $U_1, U_2, \ldots$ of X such that each of Y_i contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$ and $Y_i \subset U_i$ $(i \in \mathbb{N})$.*

2. ABOUT SOME CLASSICAL BANACH LATTICES

In this section, we shall answer the problems: whether or not $\mathscr{L}(E,F) = \mathscr{L}^r(E,F)$ and whether or not $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$ for some classical Banach lattices, that is, each of E and F is one of the Banach lattices l_p $(1 \le p \le \infty)$, l_∞, c_0, c, $C[0,1]$ and $C(X)$.

In l_p, l_∞, c_0 and c, we denote $e_1 := (1,0,0,0,\ldots)$, $e_2 := (0,1,0,0,\ldots)$, … and $e := (1,1,1,\ldots)$.

THEOREM 2.1. *$\mathscr{L}(l_1,F) \equiv \mathscr{L}^r(l_1,F)$ for every Banach lattice F.*

PROOF. Suppose $S \in \mathscr{L}(l_1,F)$. For every $x = (x(1), x(2), \ldots) \in l_1$, in the sense of norm convergence, $x = \sum_{i=1}^\infty x(i)e_i$, and hence $Sx = \sum_{i=1}^\infty x(i)Se_i$. As

$$\sum_{i=1}^\infty \| |x(i)| |Se_i| \| \le \sum_{i=1}^\infty |x(i)|\,\|Se_i\| \le \Big(\sum_{i=1}^\infty |x(i)|\Big)\|S\| < \infty,$$

we can define $T: l_1 \to F$ by

$$Tx = \sum_{i=1}^{\infty} x(i)|Se_i| \text{ for all } x \in l_1.$$

Obviously, $T \in \mathscr{L}(l_1, F)$ with $T \geq 0$, $T \geq S$ and $\|T\| \leq \|S\|$. Hence, $S \in \mathscr{L}^r(l_1, F)$.

Since $T \geq S$ and $T \geq -S$, we claim $\|S\|_r \leq \|T\| \leq \|S\|$. Consequently, $\|S\| = \|S\|_r$. □

A positive element of a Riesz space E is said to be *discrete* if every $g \in E$ satisfying $0 \leq g \leq f$ is a scalar multiple of f, i.e., there is a $\lambda \in \mathbb{R}$ such that $g = \lambda f$. If $f_1, \ldots, f_n$ are discrete and linearly independent in E, it is clear that $f_i \wedge f_j = 0$ $(i \neq j)$. If F is an AM-space (e.g. $C(Y)$, $C[0,1]$, c, c_0) and E is one of l_p $(1 \leq p < \infty)$ and c_0, so $e_1, e_2, \ldots$ are discrete in E and the linear hull of $e_1, e_2, \ldots$ is norm dense in E, we can conclude $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$. In fact, there is a more general theorem as follows.

THEOREM 2.2. *Let E be a Banach lattice and F an AM-space. If the linear hull E_0 of all discrete elements of E is norm dense in E, then $\mathscr{L}(E,F) \equiv \mathscr{L}^r(E,F)$.*

PROOF. Suppose $S \in \mathscr{L}(E,F)$. For every $f \in E_0$, $f = \sum_{i=1}^{n} \lambda_i f_i$ where $f_1, \ldots, f_n$ are discrete and linearly independent in E and $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, we define $T_0: E_0 \to F$ by

$$T_0 f = \sum_{i=1}^{n} \lambda_i |Sf_i|.$$

By the orthogonality of $f_1, \ldots, f_n$ it is easy to see that

$$\left|\sum_{i=1}^{n} \varepsilon_i \lambda_i f_i\right| = \sum_{i=1}^{n} |\lambda_i| f_i = \left|\sum_{i=1}^{n} \lambda_i f_i\right| \text{ for } \varepsilon_i \in \{-1, 1\} \ (i = 1, \ldots, n).$$

Hence, since F is an AM-space,

$$\|T_0 f\|_F \leq \left\|\sum_{i=1}^{n} |\lambda_i S f_i|\right\|_F = \left\|\sup_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \sum_{i=1}^{n} \varepsilon_i \lambda_i S f_i\right\|_F$$

$$\leq \|S\| \sup_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \left\|\sum_{i=1}^{n} \varepsilon_i \lambda_i f_i\right\|_E = \|S\| \cdot \|f\|_E.$$

For every $g \in E$, since E_0 is norm dense in E there are $g_1, g_2, \ldots \in E_0$ such that $\lim_{n\to\infty} g_n = g$ (in this proof, every convergence means norm convergence) and we define $T: E \to F$ by

$$Tg = \lim_{n\to\infty} T_0 g_n.$$

Then T is an extension of T_0 and $\|T_0\| = \|T\| \leq \|S\|$. Since $T_0 \geq 0$ on E_0 and $f \in E_0$ implies $|f| \in E_0$, for $g \geq 0$ if $g = \lim_{n\to\infty} g_n$ $(g_n \in E_0)$ we have

$$Tg = T|g| = T \lim_{n\to\infty} |g_n| = \lim_{n\to\infty} T_0 |g_n| \geq 0.$$

That is, $T \geq 0$. For the same reason, it follows from $T_0 \geq \pm S$ on E_0 that $T \geq \pm S$. Therefore, $S \in \mathscr{L}^r(E,F)$ and $\|S\|_r \leq \|T\| \leq \|S\|$, so $\|S\|_r = \|S\|$. The proof is complete. □

THEOREM 2.3. *If* $1<p<\infty$, $1 \leq q < \infty$ *and* $\frac{1}{2}+1/q-1/p>0$, *then* $\mathscr{L}(l_p, l_q) \neq \mathscr{L}^r(l_p, l_q)$.

PROOF. Define $b_1, b_2, b_3, \ldots \in l_2$ by

$$b_1 = 2^{-\frac{1}{2}}(1,1,0,0,\ldots),$$

$$b_2 = 2^{-\frac{1}{2}}(1,-1,0,0,\ldots),$$

$$b_3 = 4^{-\frac{1}{2}}(0,0,1,1,1,1,0,0,\ldots),$$

$$b_4 = 4^{-\frac{1}{2}}(0,0,1,-1,1,-1,0,0,\ldots),$$

$$b_5 = 4^{-\frac{1}{2}}(0,0,1,1,-1,-1,0,0,\ldots),$$

$$b_6 = 4^{-\frac{1}{2}}(0,0,1,-1,-1,1,0,0,\ldots),$$

$$b_7 = 8^{-\frac{1}{2}}(0,0,0,0,0,0,1,1,1,1,1,1,1,1,0,0,\ldots),$$

. . . . . .

Then $\{b_n\}_{n \in \mathbb{N}}$ is an orthonormal basis in l_2. In this proof the symbol $(\cdot,\cdot)$ represents the inner product in l_2.

(1) Let p $(1<p\leq 2)$ and q $(1 \leq q < 2)$ be given.

Let q_1 be determined by the formula $1/q_1+\frac{1}{2}=1/q$. Then $q<q_1$. Since $\frac{1}{2}+1/q-1/p-1/q_1=\frac{1}{2}+1/q-1/p-(1/q-\frac{1}{2})=1-1/p>0$, or $\frac{1}{2}+1/q-1/p>1/q_1$, we can choose a number α $(q<\alpha<q_1)$ so close to q_1 that $\frac{1}{2}+1/q-1/p>1/\alpha>1/q_1$. Let $\lambda(n):=n^{-1/\alpha}$ $(n \in \mathbb{N})$. Then $\lambda=(\lambda(1),\lambda(2),\ldots) \in l_{q_1}$. Define $S \in \mathscr{L}(l_2, l_q)$ by

$$Sx = (\lambda(1)(x,b_1), \lambda(2)(x,b_2), \ldots) \text{ for all } x \in l_2.$$

In particular, we have

$$|Se_n| = (2^k)^{-\frac{1}{2}}(0,\ldots,0,\lambda(2^k-1),\ldots,\lambda(2^{k+1}-2),0,0,\ldots)$$

$$\text{if } n \in \{2^k-1,\ldots,2^{k+1}-2\},\ k \in \mathbb{N}.$$

Let I be the identity map of l_p into l_2 and $\Phi:=SI$. Then $\Phi \in \mathscr{L}(l_p,l_q)$.

Now, we prove that Φ is not regular. Suppose Φ is regular. Then there is a $T \in \mathscr{L}^+(l_p,l_q)$ with $T \geq \Phi$. Set $T_1:=T-\Phi$. Then $(T+T_1)e_n \geq |\Phi e_n| = |Se_n|$. (Indeed, $(T+T_1)e_n \geq Te_n \geq \Phi e_n$ and $(T+T_1)e_n \geq T_1 e_n = (T-\Phi)e_n \geq -\Phi e_n$.) Hence

$$\sum_{n=2^k-1}^{2^{k+1}-2} (T+T_1)e_n \geq \sum_{n=2^k-1}^{2^{k+1}-2} |Se_n| =$$

$$=(2^k)^{\frac{1}{2}}(0,\ldots,0,\lambda(2^k-1),\ldots,\lambda(2^{k+1}-2),0,0,\ldots).$$

Thus, for $k \in \mathbb{N}$, since $\lambda(1) \geq \lambda(2) \geq \ldots$ we have

$$\|T+T_1\|(2^k)^{1/p} \geq \|\sum_{n=2^k-1}^{2^{k+1}-2} (T+T_1)e_n\|_{l_q}$$

$$\geq (2^k)^{\frac{1}{2}}(2^k\lambda(2^{k+1})^q)^{1/q} = (2^k)^{\frac{1}{2}+1/q}(2^{k+1})^{-1/\alpha}.$$

We notice that $\frac{1}{2}+1/q-1/p-1/\alpha>0$, so

$$\|T+T_1\| \geq 2^{-1/\alpha}(2^k)^{\frac{1}{2}+1/q-1/\alpha-1/p} \to \infty \text{ (as } k \to \infty).$$

This contradicts the fact $T+T_1 \in \mathscr{L}(l_p, l_q)$.

(2) Let p $(1<p\leq 2)$ and q $(2 \leq q < \infty)$ be given such that p and q satisfy $\frac{1}{2}+1/q-1/p>0$.

In the proof of Part (1), we consider $q_1 \to \infty$ and $\alpha \to \infty$, so that $\lambda = (1,1,1,\ldots) \in l_\infty$. Define $S \in \mathscr{L}(l_2, l_2)$ by

$$Sx = ((x, b_1), (x, b_2), \ldots) \text{ for all } x \in l_2.$$

Let I be the identity map of l_p into l_2 and $\Phi := SI$. Then $\Phi \in \mathscr{L}(l_p, l_q)$ (since $l_2 \subset l_q$). Suppose there exists a $T \in \mathscr{L}^+(l_p, l_q)$ with $T \geq \Phi$. Set $T_1 := T - \Phi$. Continuing as in the proof of Part (1), we can obtain a contradiction:

$$\|T+T_1\| \geq (2^k)^{\frac{1}{2}+1/q-1/p} \to \infty \text{ (as } k \to \infty).$$

Therefore, Φ is not regular in $\mathscr{L}(l_p, l_q)$.

(3) Let p $(2 \leq p < \infty)$ and q $(2 \leq q < \infty)$ be given.

We know from Part (1) and (2) that $\mathscr{L}(l_p, l_q) \neq \mathscr{L}^r(l_p, l_q)$ $(1<p\leq 2,\ 1<q\leq 2)$. By Theorem 1.1, $\mathscr{L}(l_p, l_q) \neq \mathscr{L}^r(l_p, l_q)$ $(2 \leq p < \infty,\ 2 \leq q < \infty)$ follows immediately.

(4) Let p $(2<p<\infty)$ and q $(1 \leq q < 2)$ be given.

Let the number q_1 be determined by the formula $1/q_1 + \frac{1}{2} = 1/q$. Then $q_1 > q$. We can choose a number α such that $q < \alpha < q_1$. Let $\lambda(n) := n^{-1/\alpha}$ $(n \in \mathbb{N})$. Then $\lambda = (\lambda(1), \lambda(2), \ldots) \in l_{q_1}$.

Let the number p_1 be determined by the formula $1/p_1 + 1/p = \frac{1}{2}$. It is clear that $p_1 > 2$. For the chosen α, since $1/q - 1/\alpha > 0$, we can choose a number β $(2 < \beta < p_1)$ so close to p_1 that $1/q - 1/\alpha > 1/\beta + 1/p - \frac{1}{2} > 0$. Hence, we obtain $\frac{1}{2} + 1/q - 1/\alpha - 1/\beta - 1/p > 0$.

Let $\theta(n) := n^{-1/\beta}$ $(n \in \mathbb{N})$. Then $\theta = (\theta(1), \theta(2), \ldots) \in l_{p_1}$.

For $x \in l_p$, the symbol $\langle \theta, x \rangle$ is defined by $\langle \theta, x \rangle = (\theta(1)x(1), \theta(2)x(2), \ldots)$. It is clear that the operator $x \to \langle \theta, x \rangle$ is a linear continuous operator from l_p into l_2. Now, we define $S \in \mathscr{L}(l_p, l_q)$ by

$$Sx = (\lambda(1)(\langle \theta, x \rangle, b_1), \lambda(2)(\langle \theta, x \rangle, b_2), \ldots) \text{ for all } x \in l_p.$$

In particular, we have

$$|Se_n| = (2^k)^{-\frac{1}{2}}\theta(n)(0, \ldots, 0, \lambda(2^k-1), \ldots, \lambda(2^{k+1}-2), 0, 0, \ldots)$$

$$\text{if } n \in \{2^k-1, \ldots, 2^{k+1}-2\},\ k \in \mathbb{N}.$$

We proceed to prove that S is not regular. Suppose S is regular. Then there is a $T \in \mathscr{L}^+(l_p, l_q)$ with $T \geq S$. Set $T_1 := T - S$. Then $(T+T_1)e_n \geq |Se_n|$ $(n \in \mathbb{N})$.

Also since $\theta(1)\geq\theta(2)\geq\dots$, we have

$$\sum_{n=2^k-1}^{2^{k+1}-2}(T+T_1)e_n\geq\sum_{n=2^k-1}^{2^{k+1}-2}|Se_n|$$
$$\geq(2^k)^{\frac{1}{2}}\theta(2^{k+1})(0,\dots,0,\lambda(2^k-1),\dots,\lambda(2^{k+1}-2),0,0,\dots).$$

Thus, for $k\in\mathbb{N}$, since $\lambda(1)\geq\lambda(2)\geq\dots$ we have

$$\|T+T_1\|(2^k)^{1/p}\geq\Big\|\sum_{n=2^k-1}^{2^{k+1}-2}(T+T_1)e_n\Big\|_{l_q}$$
$$\geq(2^k)^{\frac{1}{2}}\theta(2^{k+1})(2^k\lambda(2^{k+1})^q)^{1/q}=(2^k)^{\frac{1}{2}+1/q}(2^{k+1})^{-1/\beta-1/\alpha}.$$

We notice that $\frac{1}{2}+1/q-1/\alpha-1/\beta-1/p>0$, so

$$\|T+T_1\|\geq2^{-1/\beta-1/\alpha}(2^k)^{\frac{1}{2}+1/q-1/\alpha-1/\beta-1/p}\to\infty\ (\text{as }k\to\infty).$$

This contradicts the fact $T+T_1\in\mathscr{L}(l_p,l_q)$. □

REMARK. We still do not know what happens to the relationship between $\mathscr{L}(l_p,l_q)$ and $\mathscr{L}^r(l_p,l_q)$ if p and q satisfy the condition: $1<p\leq2$, $2<q<\infty$ and $\frac{1}{2}+1/q-1/p\leq0$. We put it here as an open question.

LEMMA 2.4. $\mathscr{L}(l_\infty,l_q)\neq\mathscr{L}^r(l_\infty,l_q)$ $(1\leq q<2)$.

PROOF. Let a number α be chosen such that $\frac{1}{2}<1/\alpha<1/q$. Then

$$\sum_{n=1}^{\infty}(n^{-1/\alpha})^q=\infty\ \text{and}\ \sum_{n=1}^{\infty}(n^{-1/\alpha})^2<\infty.$$

By Dvoretzky-Rogers' theorem (cf. Theorem 1.c.2 of [LT]), there is an unconditionally convergent series $\sum_{n=1}^\infty x_n$ in l_q such that $\|x_n\|_{l_q}=n^{-1/\alpha}$ for every n. Define $S:l_\infty\to l_q$ by

$$Sy=\sum_{n=1}^{\infty}y(n)x_n\ \text{for all}\ y=(y(1),y(2),\dots)\in l_\infty.$$

Then $Se_n=x_n$ $(n\in\mathbb{N})$ and $S\in\mathscr{L}(l_\infty,l_q)$ (cf. p. 16 of [LT]).

Suppose S is regular. Then there exists a $T\in\mathscr{L}^+(l_\infty,l_q)$ with $T\geq S$. Set $T_1:=T-S$. Then $(T+T_1)e_n\geq|Se_n|$ and

$$\|T+T_1\|^q\geq\Big\|(T+T_1)\Big(\sum_{n=1}^{N}e_n\Big)\Big\|^q=\sum_{i=1}^{\infty}\Big[\sum_{n=1}^{N}((T+T_1)e_n)(i)\Big]^q$$
$$\geq\sum_{i=1}^{\infty}\sum_{n=1}^{N}[((T+T_1)e_n)(i)]^q=\sum_{n=1}^{N}\|(T+T_1)e_n\|^q$$
$$\geq\sum_{n=1}^{N}\|Se_n\|^q=\sum_{n=1}^{N}(n^{-1/\alpha})^q\to\infty\ (\text{as }N\to\infty).$$

This contradicts the fact $T+T_1\in\mathscr{L}(l_\infty,l_q)$. Therefore, S is not regular. □

COROLLARY 2.5. *$\mathscr{L}(c, l_q) \neq \mathscr{L}^r(c, l_q)$, $\mathscr{L}(c_0, l_q) \neq \mathscr{L}^r(c_0, l_q)$ and for every (infinite) compact Hausdorff space X, $\mathscr{L}(C(X), l_q) \neq \mathscr{L}^r(C(X), l_q)$ $(1 \leq q < 2)$.*

PROOF. In the proof of Lemma 2.4, replacing l_∞ by c and c_0, respectively, we can prove $\mathscr{L}(c, l_q) \neq \mathscr{L}^r(c, l_q)$ and $\mathscr{L}(c_0, l_q) \neq \mathscr{L}^r(c_0, l_q)$ $(1 \leq q < 2)$.

It remains to prove $\mathscr{L}(C(X), l_q) \neq \mathscr{L}^r(C(X), l_q)$ $(1 \leq q < 2)$. Since the compact Hausdorff space X is infinite we can construct a nontrivial sequence $\{t_n\}_{n \in \mathbb{N}}$ and mutually disjoint open sets U_n $(n \in \mathbb{N})$ such that $t_n \in U_n$ $(n \in \mathbb{N})$. Thus, for each $n \in \mathbb{N}$ we choose $h_n \in C(X)$ such that $0 \leq h_n \leq 1$, $h_n(t_n) = 1$ and $h_n = 0$ outside U_n. In the proof of Lemma 2.4, we only need to change the definition of S into

$$Sf = \sum_{n=1}^{\infty} f(t_n) x_n \text{ for all } f \in C(X).$$

Then $Sh_n = x_n$ $(n \in \mathbb{N})$ and $S \in \mathscr{L}(C(X), l_q)$. Now, we can complete the proof in the same way as the proof of Lemma 2.4. □

We know from Theorem 2.3 that $\mathscr{L}(l_p, l_1) \neq \mathscr{L}^r(l_p, l_1)$ $(1 < p < \infty)$. By Theorem 1.1 it follows that $\mathscr{L}(c_0, l_q) \neq \mathscr{L}^r(c_0, l_q)$ $(1 < q < \infty)$. Combining this result with Corollary 2.5 we obtain

COROLLARY 2.6. *$\mathscr{L}(c_0, l_q) \neq \mathscr{L}^r(c_0, l_q)$ $(1 \leq q < \infty)$.*

COROLLARY 2.7. *$\mathscr{L}(c, l_q) \neq \mathscr{L}^r(c, l_q)$ $(1 \leq q < \infty)$.*

PROOF. Corollary 1.5 and Corollary 2.6. □

THEOREM 2.8. *For every (infinite) compact Hausdorff space X, $\mathscr{L}(C(X), c_0) \neq$ $\neq \mathscr{L}^r(C(X), c_0)$.*

PROOF. Case I. There exists a nontrivial convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ in X. Suppose $\{x_n\}_{n \in \mathbb{N}}$ converges to x_0. Define $S \in \mathscr{L}(C(X), c_0)$ by

$$Sf = (f(x_1) - f(x_0), f(x_2) - f(x_0), \ldots) \text{ for all } f \in C(X).$$

We proceed to prove that S is not regular. Suppose S is regular. Then there is a $T \in \mathscr{L}^+(C(X), c_0)$ with $T \geq S$. It is clear that for each couple of x_n and x_0 there exists a function $f_n \in C(X)$ such that $0 \leq f_n \leq 1$, $f_n(x_n) = 1$ and $f_n(x_0) = 0$. Now, for every $n \in \mathbb{N}$, we have

$$(T1_X)(n) \geq (Tf_n)(n) \geq (Sf_n)(n) = f_n(x_n) - f_n(x_0) = 1.$$

But $T1_X \in c_0$. Contradiction.

Case II. There exists a sequence in X which has no convergent subsequence.

By Theorem 1.6, there exist a regular Borel measure μ and Borel measurable functions $g_1, g_2, \ldots$ on X such that (a) $\mu(X) = 1$, (b) $g_1 = 1_X$, (c) $|g_n(x)| = 1$ μ-almost everywhere, (d) $\int g_n g_m d\mu = 0$ $(n \neq m)$. Thus, $g_1, g_2, \ldots$ is an orthonormal sequence in $L^2(\mu)$. Hence, for every $f \in L^2(\mu)$, so for every $f \in C(X)$, $\lim_{n\to\infty} \int fg_n d\mu = 0$. We define $S \in \mathscr{L}(C(X), c_0)$ by

$$Sf = (\int fg_1 d\mu, \int fg_2 d\mu, \dots) \text{ for all } f \in C(X).$$

(Actually, it is obvious that $Sf \in l_2$ and $S \in \mathscr{L}(C(X), l_2)$.) It is easy to check that $\|S\| = 1$. We proceed to prove that S is not regular. Since $C(X)$ is dense in $L^2(\mu)$, for $n \in \mathbb{N}$ there exists an $h_n \in C(X)$ with $\|h_n - g_n\|_{L^2} \leq \frac{1}{2}$. Set

$$\bar{h}_n(x) := \begin{cases} h_n(x) \text{ if } & -1 \leq h_n(x) \leq 1 \\ 1 \quad \text{if} & h_n(x) > 1 \\ -1 \quad \text{if} & h_n(x) < -1. \end{cases}$$

Then $\bar{h}_n \in C(X)$, $|\bar{h}_n| \leq 1$ and $\|\bar{h}_n - g_n\|_{L^2} \leq \frac{1}{2}$. By (a) and (c),

$$1 - \int \bar{h}_n g_n d\mu = \int (g_n - \bar{h}_n) g_n d\mu \leq \|g_n - \bar{h}_n\|_{L^2} \|g_n\|_{L^2} \leq \tfrac{1}{2}.$$

Setting $f_n := 1_X + \bar{h}_n$, by (b) and (d) we obtain $\int f_n g_n d\mu = \int \bar{h}_n g_n d\mu \geq \frac{1}{2}$. Now, if S is regular, then there exists a $T \in \mathscr{L}^+(C(X), c_0)$ with $T \geq S$. For every $n \in \mathbb{N}$, we have

$$2(T1_X)(n) \geq (Tf_n)(n) \geq (Sf_n)(n) = \int f_n g_n d\mu \geq \tfrac{1}{2}.$$

But $T1_X \in c_0$. Contradiction. □

THEOREM 2.9. *For every (infinite) compact Hausdorff space X, $\mathscr{L}(C(X), l_q) \neq \mathscr{L}^r(C(X), l_q)$ $(1 \leq q < \infty)$.*

PROOF. (1) If there is a nontrivial convergent sequence in X, by Corollary 1.3 and Corollary 2.7 the conclusion holds.

(2) If there is a sequence in X which has no convergent subsequence and $2 \leq q < \infty$, we have shown that S, which is defined in Case II of the proof of Theorem 2.8 and is not regular in $\mathscr{L}(C(X), c_0)$, is actually an element of $\mathscr{L}(C(X), l_2)$, so $S \in \mathscr{L}(C(X), l_q)$ $(2 \leq q < \infty)$. Now, it is easy to see that this S is not regular in $\mathscr{L}(C(X), l_q)$.

Combining (1), (2) and Corollary 2.5, we complete the proof. □

Now, we turn to observe the relationship between $\mathscr{L}(C(X), c)$ and $\mathscr{L}^r(C(X), c)$, and the relationship between $\mathscr{L}(C(X), C[0,1])$ and $\mathscr{L}^r(C(X), C[0,1])$. We need some lemmas first.

LEMMA 2.10. *Let X be a compact Hausdorff space. If there exists a sequence in X which has no convergent subsequence, then $\mathscr{L}(C(X), c) \neq \mathscr{L}^r(C(X), c)$.*

PROOF. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X which has no convergent subsequence. By Corollary 1.7 there exist compact subsets $Y_1, Y_2, \dots$ and mutually disjoint open subsets $U_1, U_2, \dots$ of X such that each Y_i contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$ and $Y_i \subset U_i$ $(i\in\mathbb{N})$. Hence, for each $i\in\mathbb{N}$, there exists an $h_i\in C(X)$ such that $0\le h_i\le 1$, $h_i=1$ on Y_i and $h_i=0$ on $X\setminus U_i$. We can see from the proof of Theorem 2.8 that for each $i\in\mathbb{N}$ there exists an $S_i\in\mathscr{L}(C(Y_i),c_0)$ with the properties:

(a) $\|S_i\|=1$

(b) there exist $f_{i1}, f_{i2},\ldots$ in $C(Y_i)$ such that $0\le f_{in}\le 1$ and $(S_if_{in})(n)\ge\frac{1}{4}$ $(n\in\mathbb{N})$.

Since every compact set Y_i is C^*-embedded in X (cf. 6.9(b) of [GJ]), we assume that each f_{in} extends to $f_{in}^0\in C(X)$ with $0\le f_{in}^0\le 1$ $(i,n\in\mathbb{N})$.

Define $S: C(X)\to c_0$ by

$$(Sf)(2^{i-1}(2j-1)) = 1/i(S_i(f|_{Y_i}))(j) \quad (i,j\in\mathbb{N}) \text{ for all } f\in C(X)$$

i.e., $Sf=((S_1(f|_{Y_1}))(1), \frac{1}{2}(S_2(f|_{Y_2}))(1), (S_1(f|_{Y_1}))(2), \frac{1}{3}(S_3(f|_{Y_3}))(1), (S_1(f|_{Y_1}))(3), \frac{1}{2}(S_2(f|_{Y_2}))(2), (S_1(f|_{Y_1}))(4), \frac{1}{4}(S_4(f|_{Y_4}))(1),\ldots)$.

Observe that $\|S\|\le 1$. Hence $S\in\mathscr{L}(C(X),c_0)$, so $S\in\mathscr{L}(C(X),c)$. We proceed to prove that S is not regular in $\mathscr{L}(C(X),c)$. Suppose S is regular. Then there is a $T\in\mathscr{L}^+(C(X),c)$ with $T\ge S$. Take $i\in\mathbb{N}$. We have $Th_i\ge T(f_{ij}^0h_i)\ge S(f_{ij}^0h_i)$, $(j\in\mathbb{N})$. Since $f_{ij}^0h_i|_{Y_i}=f_{ij}$ and $f_{ij}^0h_i|_{Y_k}=0$ $(k\ne i;\ j,k\in\mathbb{N})$ and by (b),

$$Th_i(2^{i-1}(2j-1))\ge 1/i(S_if_{ij})(j)\ge 1/i\cdot\tfrac{1}{4},$$

so $Th_i(\infty) := \lim_{n\to\infty}(Th_i)(n)\ge 1/i\cdot\frac{1}{4}$. Hence,

$$\begin{aligned}(T1_X)(\infty) := \lim_{n\to\infty}(T1_X)(n) &\ge (T(h_1+\ldots+h_i))(\infty)\\ &= (Th_1)(\infty)+\ldots+(Th_i)(\infty)\ge\tfrac{1}{4}(1+\ldots+1/i),\end{aligned}$$

so $(T1_X)(\infty)=\infty$. Contradiction. □

LEMMA 2.11. *Let X be a compact Hausdorff space. Then*
(1) *if $\mathscr{L}(C(X),c)=\mathscr{L}^r(C(X),c)$, then X_l is finite* (i.e., Card $X_l<\infty$),
(2) *if $\mathscr{L}(C(X),c)\equiv\mathscr{L}^r(C(X),c)$, then X_l has only one element* (i.e., Card $X_l=1$).

PROOF. (1) Since $\mathscr{L}(C(X),c)=\mathscr{L}^r(C(X),c)$ and we know $\|\cdot\|\le\|\cdot\|_r$, the identity map of $\mathscr{L}^r(C(X),c)$ into $\mathscr{L}(C(X),c)$ is closed, hence there exists an $m>0$ such that, for every $\Phi\in\mathscr{L}(C(X),c)$, $\|\Phi\|_r\le m\|\Phi\|$ (by the closed-graph theorem).

Suppose X_l is infinite. For an arbitrary $N\in\mathbb{N}$, take $x_{1\infty},\ldots,x_{N\infty}\in X_l$ and $X_i=\{x_{i\infty},x_{i1},x_{i2},\ldots\}$ such that $\{x_{in}\}_{n\in\mathbb{N}}$ is a nontrivial sequence which converges to $x_{i\infty}$ and $X_i\cap X_j=\emptyset$ if $i\ne j$ $(i,j=1,\ldots,N)$. Define $\Phi\in\mathscr{L}(C(X),c)$ by, for all $f\in C(X)$,

$$\begin{aligned}\Phi f=(&f(x_{11})-f(x_{1\infty}),\ldots,f(x_{N1})-f(x_{N\infty}),\\ &f(x_{12})-f(x_{1\infty}),\ldots,f(x_{N2})-f(x_{N\infty}),\ldots),\end{aligned}$$

i.e. $\Phi f((j-1)N+i)=f(x_{ij})-f(x_{i\infty})$ $(i\in\{1,\ldots,N\},\ j\in\mathbb{N})$. Obviously, $\|\Phi\|=2$ and $(\Phi f)(\infty) := \lim_{n\to\infty}(\Phi f)(n)=0$.

Take an arbitrary $\Psi \in \mathscr{L}^+(C(X), c)$ with $\Psi \geq \Phi$ (the condition $\mathscr{L}(C(X), c) = \mathscr{L}^r(C(X), c)$ guarantees the existence of Ψ). Define $\psi_\infty, \psi_1, \psi_2, \ldots \in C(X)^{*+}$ by, for all $f \in C(X)$,

$$\psi_n(f) = (\Psi f)(n) \ (n \in \mathbb{N}) \text{ and } \psi_\infty(f) = (\Psi f)(\infty) \ (:= \lim_{n \to \infty} (\Psi f)(n)).$$

Thus, $\Psi f = (\psi_1(f), \psi_2(f), \ldots)$ and $\psi_\infty(f) = \lim_{n\to\infty} \psi_n(f)$. For $x \in X$, define $\delta_x \in C(X)^{*+}$ by

$$\delta_x(f) = f(x) \text{ for all } f \in C(X).$$

Obviously, $\delta_x \perp \delta_y$ if $x \neq y$. Since, for each i, $\psi_{(j-1)N+i} \geq 0$ and $\psi_{(j-1)N+i} \geq \delta_{x_{ij}} - \delta_{x_{i\infty}}$, we have $\psi_{(j-1)N+i} \geq (\delta_{x_{ij}} - \delta_{x_{i\infty}})^+ = \delta_{x_{ij}}$. Now, for every $f \in C(X)^+$,

$$\psi_\infty(f) = \lim_{j\to\infty} \psi_{(j-1)N+i}(f) \geq \lim_{j\to\infty} \delta_{x_{ij}}(f) = \lim_{j\to\infty} f(x_{ij}) = f(x_{i\infty}) = \delta_{x_{i\infty}}(f).$$

Hence, $\psi_\infty \geq \delta_{x_{i\infty}}$ for each i. Furthermore, $\psi_\infty \geq \delta_{x_{1\infty}} + \ldots + \delta_{x_{N\infty}}$. It follows that

$$[(2\Psi - \Phi)f](\infty) = 2\psi_\infty(f) - 0 \geq 2(f(x_{1\infty}) + \ldots + f(x_{N\infty})),$$

so $[(2\Psi - \Phi)1_X](\infty) \geq 2N$. This implies $\|2\Psi - \Phi\| \geq 2N$. Therefore, $\|\Phi\|_r \geq 2N = N\|\Phi\|$. By the arbitrariness of N, the conclusion $\|\Phi\|_r \geq N\|\Phi\|$ and the conclusion $\|\Phi\|_r \leq m\|\Phi\|$ are contradictory.

(2) If X_l is infinite, by (1), $\mathscr{L}(C(X), c) \neq \mathscr{L}^r(C(X), c)$. If X_l is finite and Card $X_l = N \geq 2$, we know from the proof of (1) that there exists a $\Phi \in \mathscr{L}(C(X), c)$ such that $\|\Phi\|_r \geq N\|\Phi\| > \|\Phi\|$. This is a contradiction to $\mathscr{L}(C(X), c) \equiv \mathscr{L}^r(C(X), c)$. □

THEOREM 2.12. *Let X be a compact Hausdorff space. Then the following assertions are equivalent.*

(1) *Every sequence in X has a convergent subsequence and* Card $X_l = n$ *where n is a positive integer.*
(2) *X is a compactification of a discrete space A such that A is a dense subspace of X and* Card $(X \setminus A) = n$ *for some $n \in \mathbb{N}$.*
(3) *There are open and compact subspaces $X_1, \ldots, X_n$ of X such that $X = \bigcup_{i=1}^n X_i$ and each X_i is the one-point compactification of a discrete space.*
(4) *For every compact Hausdorff space Z, $\mathscr{L}(C(X), C(Z)) = \mathscr{L}^r(C(X), C(Z))$.*
(5) $\mathscr{L}(C(X), C[0,1]) = \mathscr{L}^r(C(X), C[0,1])$.
(6) $\mathscr{L}(C(X), c) = \mathscr{L}^r(C(X), c)$.

THEOREM 2.13. *Let X be a compact Hausdorff space. Then the following assertions are equivalent.*

(i) *Every sequence has a convergent subsequence and* Card $X_l = 1$.
(ii) *X is the one-point compactification of a discrete space.*
(iii) *For every compact Hausdorff space Z, $\mathscr{L}(C(X), C(Z)) \equiv \mathscr{L}^r(C(X), C(Z))$.*
(iv) $\mathscr{L}(C(X), C[0,1]) \equiv \mathscr{L}^r(C(X), C[0,1])$.
(v) $\mathscr{L}(C(X), c) \equiv \mathscr{L}^r(C(X), c)$.

PROOF OF THEOREM 2.12 AND 2.13. (1)$\Rightarrow$(2). It is enough to prove that $\{x\}$ is an open set for every $x\in X\setminus X_l$. As X_l is finite, $X\setminus X_l$ is an open set. Hence, for $x\in X\setminus X_l$ there exists an open set V with $x\in V\subset\bar{V}\subset X\setminus X_l$. By (1), $\bar{V}$ is finite. Then $\bar{V}\setminus\{x\}$ is finite, hence closed, and $\{x\}=V\setminus(\bar{V}\setminus\{x\})$ is open.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(1) are easy to prove.

(i)$\Leftrightarrow$(ii). It is a special case of (1)$\Leftrightarrow$(3).

(i)$\Rightarrow$(iii). Assume $X_l=\{x_0\}$. For $f\in C(X)$, it is clear that the closed set $\{x\in X:|f(x)-f(x_0)|\geq 1/n\}$ is finite, so $A:=\{x\in X:f(x)\neq f(x_0)\}$ is countable. Make a countably infinite set $A_f=\{x_0,x_{1f},x_{2f},\dots\}$ which contains A. Obviously, $\lim_{i\to\infty}x_{if}=x_0$ and A_f is compact. Suppose $S\in\mathscr{L}(C(X),C(Y))$. For $f\in C(X)$, in the sense of norm convergence

$$f=\sum_{x\in A_f}(f(x)-f(x_0))1_{\{x\}}+f(x_0)1_X.$$

For each point $y\in Y$ and each $N\in\mathbb{N}$, take $g\in C(X)$ such that $|g|\leq 1$ and

$$g(x_{if})=1 \text{ if } i\in\{1,\dots,N\} \text{ and } S1_{\{x_{if}\}}(y)\geq 0$$

$$g(x_{if})=-1 \text{ if } i\in\{1,\dots,N\} \text{ and } S1_{\{x_{if}\}}(y)<0$$

$$g(x)=1 \text{ if } x\in A_f\setminus\{x_{1f},\dots,x_{Nf}\} \text{ and } S1_X(y)-\sum_{i=1}^{N}S1_{\{x_{if}\}}(y)\geq 0$$

$$g(x)=-1 \text{ if } x\in A_f\setminus\{x_{1f},\dots,x_{Nf}\} \text{ and } S1_X(y)-\sum_{i=1}^{N}S1_{\{x_{if}\}}(y)<0.$$

Then

$$\|S\|\geq\|Sg\|\geq Sg(y)=\sum_{i=1}^{N}(g(x_{if})-g(x_0))S1_{\{x_{if}\}}(y)+g(x_0)S1_X(y)$$

$$=\sum_{i=1}^{N}|S1_{\{x_{if}\}}(y)|+|S1_X(y)-\sum_{i=1}^{N}S1_{\{x_{if}\}}(y)|.$$

As $N\to\infty$, in the sense of pointwise convergence, we obtain

(a) $$\|S\|1_Y\geq\sum_{x\in A_f}|S1_{\{x\}}|+|S1_X-\sum_{x\in A_f}S1_{\{x\}}|.$$

Furthermore, in the sense of pointwise convergence,

(b) $$\tfrac{1}{2}(S1_X+\|S\|1_Y)\geq S1_X+\sum_{x\in A_f}(S1_{\{x\}})^-$$

(c) $$\tfrac{1}{2}(S1_X+\|S\|1_Y)\geq\sum_{x\in A_f}(S1_{\{x\}})^+.$$

Thus, by (a) we can define $T:C(X)\to C(Y)$ by

$$Tf=\sum_{x\in A_f}(f(x)-f(x_0))(S1_{\{x\}})^+ +\tfrac{1}{2}f(x_0)(S1_X+\|S\|1_Y),\ (f\in C(X)).$$

By (b) and (c), it is easy to verify that $T\geq S$ and $T\geq 0$, so $T\in\mathscr{L}(C(X),C(Y))$. Hence, $S\in\mathscr{L}^r(C(X),C(Y))$. It remains to prove $\|S\|=\|S\|_r$. For any $f\in C(X)$

with $\|f\| \leq 1$, setting $h := \frac{1}{2}(S1_X + \|S\| 1_Y)$, we have

$$\|(2T-S)f\| =$$

$$= \sup_{y\in Y} \Big| \sum_{x\in A_f} f(x)|S1_{\{x\}}(y)| + f(x_0)(2h(y) - S1_X(y) - \sum_{x\in A_f} |S1_{\{x\}}(y)|)\Big|$$

$$\leq \|f\| \sup_{y\in Y} \Big| \sum_{x\in A_f} |S1_{\{x\}}(y)| + 2h(y) - S1_X(y) - \sum_{x\in A_f} |S1_{\{x\}}(y)| \Big|$$

$$\leq \sup_{y\in Y} |2h(y) - S1_X(y)| = \|S\|.$$

Hence, $\|S\|_r \leq \|2T-S\| \leq \|S\|$. Consequently, $\|S\| = \|S\|_r$.

(1)⇒(4). Assume $X_l = \{x_1, \dots, x_n\}$. Since (3) holds, there are open and compact subspaces $X_1, \dots, X_n$ such that $X = \bigcup_{i=1}^n X_i$ and each X_i is the one-point compactification of a discrete space. For every $g \in C(X_i)$, we define $g^* \in C(X)$ by

$$g^*(x) = g(x) \text{ if } x \in X_i$$

$$g^*(x) = 0 \quad \text{ if } x \in X \setminus X_i.$$

Thus, every $f \in C(X)$ can be expressed by $f = (f|_{X_1})^* + \dots + (f|_{X_n})^*$.

For $S \in \mathscr{L}(C(X), C(Y))$, define $S_i \in \mathscr{L}(C(X_i), C(Y))$ by

$$S_i g = S g^* \text{ for all } g \in C(X_i) \ (i = 1, \dots, n).$$

By (i)⇒(iii), for each i there is a $T_i \in \mathscr{L}^+(C(X_i), C(Y))$ with $T_i \geq S_i$. Now, we define $T \in \mathscr{L}^+(C(X), C(Y))$ by

$$Tf = T_1(f|_{X_1}) + \dots + T_n(f|_{X_n}) \text{ for all } f \in C(X).$$

For every $f \in C(X)^+$,

$$Tf = T_1(f|_{X_1}) + \dots + T_n(f|_{X_n}) \geq S_1(f|_{X_1}) + \dots + S_n(f|_{X_n})$$

$$= S(f|_{X_1})^* + \dots + S(f|_{X_n})^* = Sf,$$

we obtain $T \geq S$. Consequently, $S \in \mathscr{L}^r(C(X), C(Y))$.

(4)⇒(5) and (iii)⇒(iv) are trivial.

(5)⇒(6) and (iv)⇒(v) by Corollary 1.4.

(6)⇒(1) and (v)⇒(i) by Lemma 2.10 and Lemma 2.11. □

By Theorem 2.12, we have $\mathscr{L}(C[0,1], c) \neq \mathscr{L}^r(C[0,1], c)$ and $\mathscr{L}(l_\infty, c) \neq \mathscr{L}^r(l_\infty, c)$. Furthermore, using Corollary 1.4, respectively, we obtain

COROLLARY 2.14. *$\mathscr{L}(C[0,1], C[0,1]) \neq \mathscr{L}^r(C[0,1], C[0,1])$ and $\mathscr{L}(l_\infty, C[0,1]) \neq \mathscr{L}^r(l_\infty, C[0,1])$.*

About the relationship between $\mathscr{L}(C(X), C(Y))$ and $\mathscr{L}^r(C(X), C(Y))$ we only have a partial result as follows.

THEOREM 2.15. *Let X and Y be compact Hausdorff spaces. Suppose there exists a nontrivial convergent sequence in Y. Then the following assertion* (7) *is equivalent with the assertions* (1) *to* (6) *of Theorem* 2.12 *and the following assertion* (vi) *is equivalent with the assertions* (i) *to* (v) *of Theorem* 2.13.

(7) $\mathscr{L}(C(X), C(Y)) = \mathscr{L}^r(C(X), C(Y))$.

(vi) $\mathscr{L}(C(X), C(Y)) \equiv \mathscr{L}^r(C(X), C(Y))$.

PROOF. (4)$\Rightarrow$(7) is trivial. (7)$\Rightarrow$(6) by Corollary 1.3.
(iii)$\Rightarrow$(vi) is trivial. (vi)$\Rightarrow$(v) by Corollary 1.3. □

It is interesting that, in case that there exists a nontrivial convergent sequence in Y, the relationship between $\mathscr{L}(C(X), C(Y))$ and $\mathscr{L}^r(C(X), C(Y))$ depends only on X and not on any further properties of Y. Thus, we have

COROLLARY 2.16. *Let Y be a compact Hausdorff space in which there exists a nontrivial convergent sequence. Then* $\mathscr{L}(l_\infty, C(Y)) \neq \mathscr{L}^r(l_\infty C(Y))$ *and* $\mathscr{L}(C[0,1], C(Y)) \neq \mathscr{L}^r(C[0,1], C(Y))$.

But, in case that there is no nontrivial convergent sequence in Y, and X does not satisfy (1) of Theorem 2.12, we still do not know what happens between $\mathscr{L}(C(X), C(Y))$ and $\mathscr{L}^r(C(X), C(Y))$ except for the following special case. If Y is extremally disconnected (so, $C(Y)$ is an order complete AM-space with unit), by Theorem 1.5 of Chap. iv of [S], $\mathscr{L}(C(X), C(Y)) \equiv \mathscr{L}^r(C(X), C(Y))$.

As the end of this paper, we put a table below about what we have discussed.

	$l_q(1 \leq q < \infty)$	l_∞	c_0	c	$C[0,1]$	$C(Y)$
l_1	$\equiv$, Th. 2.1	$\equiv$, Th. 1.0	$\equiv$, Th.2.1	$\equiv$, Th. 2.1	$\equiv$, Th. 2.1	$\equiv$, Th. 2.1
$l_p(1<p<\infty)$	Th. 2.3	$\equiv$, Th. 1.0	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2
c_0	$\neq$, Cor. 2.6	$\equiv$, Th. 1.0	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2	$\equiv$, Th. 2.2
c	$\neq$, Cor. 2.7	$\equiv$, Th. 1.0	$\neq$, Th. 2.8	$\equiv$, Th. 2.13	$\equiv$, Th. 2.13	$\equiv$, Th. 2.13
l_∞	$\neq$, Th. 2.9	$\equiv$, Th. 1.0	$\neq$, Th. 2.8	$\neq$, Th. 2.12	$\neq$, Th. 2.12	Cor. 2.16
$C[0,1]$	$\neq$, Th. 2.9	$\equiv$, Th. 1.0	$\neq$, Th. 2.8	$\neq$, Th. 2.12	$\neq$, Th. 2.12	Cor. 2.16
$C(X)$	$\neq$, Th. 2.9	$\equiv$, Th. 1.0	$\neq$, Th. 2.8	Th. 2.12, 2.13	Th. 2.12, 2.13	Th. 2.15

REFERENCES

[GJ] Gillman, L. and M. Jerison – Rings of continuous functions, Van Nostrand, Princeton-Toronto-New York-London, 1960.

[LT] Lindenstrauss, J. and L. Tzafriri – Classical Banach spaces I, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[S] Schaefer, H.H. – Banach lattices and positive operators, Springer-Verlag, New York-Heidelberg-Berlin, 1974.