Oscillatory behaviour of higher order neutral type nonlinear forced differential equation with oscillating coefficients

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Abstract
Oscillation criteria are given for higher order neutral type nonlinear forced differential equation with oscillating coefficients of the form

\[ y(t) + p(t)y(\tau(t))^{(n)} + \sum_{i=1}^{m} q_i(t) f_i(y(\sigma_i(t))) = s(t). \]

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1. Introduction
In this paper we are interested in oscillation of solutions of higher order neutral type nonlinear differential equations of the form

\[ y(t) + p(t)y(\tau(t))^{(n)} + \sum_{i=1}^{m} q_i(t) f_i(y(\sigma_i(t))) = s(t). \] (1.1)

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where \( n \geq 2 \) and the following conditions are always assumed to hold:

(i) \( p(t), q_i(t), \tau(t), s(t) \in C[t_0, +\infty) \) for \( i = 1, 2, \ldots, m \),

(ii) \( p(t) \) and \( s(t) \) are oscillating functions,

(iii) \( q_i(t) \geq 0 \) for \( i = 1, 2, \ldots, m \),

(iv) \( \sigma_i(t) \in C'[t_0, +\infty), \sigma_i'(t) > 0, \lim_{t \to \infty} \sigma_i(t) = \infty \) for \( i = 1, 2, \ldots, m \) and \( \lim_{t \to \infty} \tau(t) = \infty \),

(v) \( f_i(u) \in C(R) \) is nondecreasing function, \( uf_i(u) > 0 \) for \( u \neq 0 \) and \( i = 1, 2, \ldots, m \).

Recently, many studies have been made on the oscillatory and asymptotic behaviour of solutions of higher order neutral type functional differential equations. Most of the known results which were studied are the cases when \( p(t) = c \in R \) and \( p(t) > 0 \) (or \( < 0 \)), which are special cases of Eq. (1.1) and related equations; see, for example, [1–10] and references cited therein.

The purpose of this paper is to study oscillatory behaviour of solutions of Eq. (1.1). For the general theory of differential equations, one can refer to [1–5]. Many references to some applications of the differential equations can be found in [5].

As is customary, a solution of Eq. (1.1) is said to oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

For the sake of convenience, the function \( z(t) \) is defined by
\[
z(t) = y(t) + p(t)y(\tau(t)) - r(t),
\]
where \( r(t) \in C[t_0, +\infty) \) is \( n \) time differentiable and an oscillating function with the property of \( r^{(n)}(t) = s(t) \).

2. Some auxiliary lemmas

Lemma 2.1 [1]. Let \( y(t) \) be a function such that it and each of its derivative up to order \( (n - 1) \) inclusive is absolutely continuous and of constant sign in an interval \( [t_0, +\infty) \). If \( y^{(n)}(t) \) is of constant sign and not identically zero on any interval of the form \([t_1, +\infty)\) for some \( t_1 \geq t_0 \), then there exist a \( t_x \geq t_0 \) and an integer \( m \), \( 0 \leq m \leq n \), with \( n + m \) even for \( y^{(n)}(t) \geq 0 \), or \( n + m \) odd for \( y^{(n)}(t) \leq 0 \), and such that for every \( t \geq t_x \), \( m > 0 \) implies \( y^{(k)}(t) > 0 \), \( k = 0, 1, 2, \ldots, m - 1 \), and \( m \leq n - 1 \) implies \((-1)^{m+k} y^{(k)}(t) > 0 \), \( k = m, m + 1, \ldots, n - 1 \).

Lemma 2.2 [1]. If the function \( y(t) \) is as in Lemma 2.1 and \( y^{(n-1)}(t)y^{(n)}(t) \leq 0 \) for all \( t \geq t_x \),
then for every \( \lambda, 0 < \lambda < 1 \), there exists a constant \( M > 0 \) such that
\[
\left| y(\lambda t) \right| \geq M t^{n-1} \left| y^{(n-1)}(t) \right| \text{ for all large } t.
\]

3. Main results

Theorem 3.1. Assume that \( n \) is odd and the following conditions are satisfied:
(C1) \( \lim_{t \to -\infty} p(t) = 0 \) and \( \lim_{t \to -\infty} r(t) = 0 \);
(C2) \( \int_{t_0}^{+\infty} q_i(v) \, dv = +\infty \) for \( i = 1, 2, \ldots, m \).

Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as \( t \to +\infty \).

**Proof.** Assume that Eq. (1.1) has a bounded nonoscillatory solution \( y(t) \). Further, we assume that \( y(t) \) does not tend to zero as \( k \to \infty \). Without loss of generality, let \( y(t) \) be eventually positive (the proof is similar when \( y(t) > 0 \), \( z \)).

Assume that Eq. (1.1) has a bounded nonoscillatory solution \( y(t) \). Further, we assume that \( y(t) \) does not tend to zero as \( k \to \infty \). Without loss of generality, let \( y(t) \) be eventually positive (the proof is similar when \( y(t) \) is eventually negative). That is, let \( y(t) > 0 \), \( y(\tau(t)) > 0 \) and \( y(\sigma_i(t)) > 0 \) for \( t \geq t_1 \geq t_0 \) and \( i = 1, 2, \ldots, m \). By (1.1), (1.2), we have

\[
\left(3.1\right)
\]

That is, \( z^{(n)}(t) < 0 \). It follows that \( z^{(j)}(t) \) \( (j = 0, 1, 2, \ldots, n - 1) \) is strictly monotone and of constant sign eventually. Since \( p(t) \) and \( r(t) \) are oscillating functions, there exists a \( t_2 \geq t_1 \) such that, as \( t \geq t_2 \), \( z(t) \) is not bounded, \( z(t) \) is also bounded for \( t \geq t_3 \). Since \( z(t) \) is bounded, we may write \( \lim_{t \to -\infty} z(t) = L \) \( (c < L < +\infty) \). Assume that \( 0 < L < +\infty \). Let \( L > 0 \). Then there exist a constant \( c > 0 \) and a \( t_5 \geq t_4 \), such that \( z(t) < c > 0 \) for \( t \geq t_5 \). Since \( y(t) \) is bounded, by (C1), \( \lim_{t \to -\infty} p(t)y(\tau(t)) = 0 \). Therefore, there exist a constant \( \varepsilon > 0 \) and a \( t_6 \geq t_5 \), such that \( y(t) = z(t) - p(t)y(\tau(t)) + r(t) > \varepsilon > 0 \) for \( t \geq t_6 \). Thus, we may take a \( t_7 \) with the property of \( t_7 \geq t_6 \), such that \( y(\sigma_i(t)) > \varepsilon > 0 \) for \( t \geq t_7 \). From (3.1), we have

\[
\left(3.2\right)
\]

Multiplying (3.2) by \( t^{n-1} \) and integrating it from \( t_7 \) to \( t \), we obtain

\[
F(t) - F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} q_i(v) v^{n-1} \, dv,
\]

where

\[
F(t) = t^{n-1} z^{(n-1)}(t) - (n-1) t^{n-2} z^{(n-2)}(t) + (n-1)(n-2) t^{n-3} z^{(n-3)}(t) + \cdots + (n-1)(n-2)(n-3) \cdots 2 r \, z(t) + (n-1)(n-2)(n-3) \cdots 3 \cdot 2 z(t).
\]

Since \( (-1)^k z^{(k)}(t) > 0 \) for \( k = 0, 1, 2, \ldots, n - 1 \) and \( t \geq t_4 \), \( F(t) > 0 \) for \( t \geq t_4 \). From (3.3), we have

\[
-F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} q_i(v) v^{n-1} \, dv.
\]
By \((C_2)\), we obtain
\[
-F(t) \leq -f(c_1) \int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(v) v^{n-1} \, dv = -\infty
\]
as \(t \to \infty\). This is a contradiction. Hence, \(L > 0\) is impossible. Therefore, \(L = 0\) is the only possible case. That is, \(\lim_{t \to \infty} z(t) = 0\). Since \(y(t)\) is bounded, by \((C_1)\), we obtain
\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) - \lim_{t \to \infty} p(t)y(t) + \lim_{t \to \infty} r(t) = 0
\]
from (1.2).

Now let us consider the case of \(y(t) < 0\) for \(t \geq t_1\). By (1.1) and (1.2), we have
\[
z^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) f_i \left( y \left( \sigma_i(t) \right) \right) > 0 \quad (t \geq t_1).
\]
That is, \(z^{(n)}(t) > 0\). It follows that \(z^{(j)}(t) (j = 0, 1, 2, \ldots, n-1)\) is strictly monotone and of constant sign eventually. Since \(p(t)\) and \(r(t)\) are oscillating functions, there exists a \(t_2 \geq t_1\), such that as \(t \geq t_2\), \(z(t) < 0\) eventually. Since \(y(t)\) is bounded, by \((C_1)\), there is a \(t_3 \geq t_2\), such that \(z(t)\) is also bounded for \(t \geq t_3\). Assume that \(x(t) = -z(t)\). Then \(x^{(n)}(t) = -z^{(n)}(t)\). Therefore, \(x(t) > 0\) and \(x^{(n)}(t) < 0\) for \(t \geq t_3\). Hence, we observe that \(x(t)\) is bounded. Since \(n\) is odd, by Lemma 2.1, there exist a \(t_4 \geq t_3\) and \(l = 0\) (otherwise \(x(t)\) is not bounded), such that \((-1)^k x^{(k)}(t) > 0\) for \(k = 0, 1, 2, \ldots, n-1\). That is, \((-1)^k x^{(k)}(t) < 0\) for \(k = 0, 1, 2, \ldots, n-1\). In particular, as \(t \geq t_4\), \(z^{'}(t) > 0\). Therefore, \(z(t)\) is increasing. Thus, we can assume that \(\lim_{t \to \infty} z(t) = L (\infty < L \leq 0)\). As in the proof of \(y(t) > 0\), we may prove that \(L = 0\). As for the rest, it is similar to the case of \(y(t) > 0\). That is, \(\lim_{t \to \infty} y(t) = 0\). This contradicts our assumption. Hence, the proof is completed. \(\square\)

**Theorem 3.2.** Assume that \(n\) is even and \((C_1)\) holds. Moreover, if the following conditions are satisfied:

\((C_3)\) There is a function \(\varphi(t)\), such that \(\varphi(t) \in C'[t_0, +\infty)\),
\[
\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \varphi(v) \sum_{i=1}^{m} q_i(v) \, dv = +\infty
\]
and
\[
\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ \frac{[\varphi(v)]^2}{\varphi(v) \sigma_i'(v) \sigma_i^{n-2}(v)} \right] \, dv < +\infty
\]
for the function \(\varphi(t)\),

then every bounded solution of Eq. (1.1) is oscillatory.
Proof. Assume that Eq. (1.1) has a bounded nonoscillatory solution $y(t)$. Further, we suppose that $y(t)$ does not tend to zero as $k \to \infty$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, let $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma_i(t)) > 0$ for $t \geq t_1 \geq t_0$. By (1.1), (1.2), we have for $t \geq t_1$

$$z^{(n)}(t) = - \sum_{i=1}^{m} q_i(t) f_i\left(y(\sigma_i(t))\right) < 0.$$  

That is, $z^{(n)}(t) < 0$. It follows that, $z^{(j)}(t)$ $(j = 0, 1, 2, \ldots, n-1)$ is strictly monotone and of constant sign eventually. Since $p(t)$ and $r(t)$ are oscillating functions, there exists a $t_2 \geq t_1$ such that, as $t \geq t_2$, $z(t) > 0$ eventually. Since $y(t)$ is bounded, by (C1), from (1.2), there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for sufficiently $t \geq t_3$. Because $n$ is even, by Lemma 2.1, when $l = 1$ (otherwise $z(t)$ is not bounded) there exists $t_4 \geq t_3$, such that as $t \geq t_4$,

$$(-1)^{k+1}z^{(k)}(t) > 0 \quad (k = 0, 1, 2, \ldots, n-1).$$  

In particular, since $z'(t) > 0$ for $t \geq t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded, by (C1) $\lim_{t \to \infty} p(t)y(\tau(t)) = 0$. Then, by (1.2), there exist a $t_5 \geq t_4$ and an integer $\delta (\delta > 1)$, such that

$$y(t) = z(t) - p(t)y(\tau(t)) + r(t) > \frac{1}{\delta}z(t) > 0$$  

for $t \geq t_5$. We may get a $t_6 \geq t_5$, such that

$$y(\sigma_i(t)) > \frac{1}{\delta}z(\sigma_i(t)) > 0$$  

for $t \geq t_6$ and $i = 1, 2, \ldots, m$. From (3.4), (3.6) and the properties of $f$, we have

$$z^{(n)}(t) \leq - \sum_{i=1}^{m} q_i(t) f_i\left(\frac{1}{\delta}z(\sigma_i(t))\right) = - \sum_{i=1}^{m} q_i(t) f_i\left(\frac{1}{\delta}z(\sigma_i(t))\right) \frac{1}{z(\sigma_i(t))} z(\sigma_i(t)),$$  

for $t \geq t_6$. Since $z(t)$ is bounded and increasing, $\lim_{t \to \infty} z(t) = L$ $(0 < L < +\infty)$. By the continuity of $f$, we have

$$\lim_{t \to \infty} f_i\left(\frac{1}{\delta}z(\sigma_i(t))\right) \frac{1}{z(\sigma_i(t))} = f_i\left(\frac{L}{\delta}\right) \frac{1}{L} > 0.$$  

Then, there is a $t_7 \geq t_6$, such that, as $t \geq t_7$ and $i = 1, 2, \ldots, m$,

$$\lim_{t \to \infty} f_i\left(\frac{1}{\delta}z(\sigma_i(t))\right) \frac{1}{z(\sigma_i(t))} \geq f_i\left(\frac{L}{\delta}\right) \frac{1}{2L} = \alpha > 0.$$  

By (3.7) and (3.8),

$$z^{(n)}(t) \leq -\alpha \sum_{i=1}^{m} q_i(t) z(\sigma_i(t)) \quad \text{for } t \geq t_7.$$  

Let us get

$$w(t) = \frac{z^{(n-1)}(t)}{z(\sigma_i(t))} \quad (i = 1, 2, \ldots, m).$$
We know that from (3.5) there is a \( t_8 \geq t_7 \), such that \( w(t) > 0 \) for sufficiently large \( t \geq t_8 \). Since \( z(t) \) is increasing, there exists a \( t_9 \geq t_8 \), such that \( z( \sigma_i(t) ) > z( \frac{1}{\delta} \sigma_i(t) ) > 0 \) for sufficiently large \( t \geq t_9 \) and \( i = 1, 2, \ldots, m \). We may get a result together with (3.9), such that

\[
w'(t) = \frac{z\left( \frac{1}{\delta} \sigma_i(t) \right) z^{(n)}(t) - z'\left( \frac{1}{\delta} \sigma_i(t) \right) z^{(n-1)}(t) \sigma_i'(t)}{z^2\left( \frac{1}{\delta} \sigma_i(t) \right)}
\]

\[
= \frac{z^{(n)}(t)}{z\left( \frac{1}{\delta} \sigma_i(t) \right)} - \frac{1}{\delta} \frac{w(t) z'\left( \frac{1}{\delta} \sigma_i(t) \right)}{z\left( \frac{1}{\delta} \sigma_i(t) \right)} \sigma_i'(t).
\]

(3.10)

From (3.5) we have \( z'(t) > 0 \) and \( z^{(n-1)}(t) > 0 \) for \( t \geq t_9 \). Since \( \sigma_i(t) \leq t \) and \( \sigma_i'(t) > 0 \) for \( i = 1, 2, \ldots, m \), by Lemma 2.2, there exist a constant \( M > 0 \) and a \( t_{10} \geq t_9 \) for \( \lambda = 1/\delta \) and \( z'(t) \), such that

\[
z'\left( \frac{1}{\delta} \sigma_i(t) \right) \geq M \sigma_i'^{-2}(t) z^{(n-1)}(t) \geq M \sigma_i'^{-2}(t) \sigma_i'(t) z^{(n-1)}(t)
\]

for \( t \geq t_{10} \). Therefore, we may get the following result together with (3.10):

\[
w'(t) \leq -\alpha \sum_{i=1}^{m} q_i(t) - \frac{M}{\delta^{n-1}} w^2(t) \sigma_i'^{-2}(t) \sigma_i'(t).
\]

(3.11)

From (3.11), we have

\[
\sum_{i=1}^{m} q_i(t) \leq -w'(t) - \frac{M}{\delta} w^2(t) \sigma_i'^{-2}(t) \sigma_i'(t) \quad (t \geq t_{10}).
\]

(3.12)

Multiplying (3.12) by \( \psi(t) \) and integrating it from \( t_{10} \) to \( t \), we obtain

\[
\alpha \int_{t_{10}}^{t} \psi(v) \sum_{i=1}^{m} q_i(v) \, dv \leq -\int_{t_{10}}^{t} \psi(v) w'(v) \, dv - \frac{M}{\delta^{n-1}} \int_{t_{10}}^{t} \psi(v) w^2(v) \sigma_i'^{-2}(v) \sigma_i'(v) \, dv
\]

\[
= -\psi(t) w(t) + \psi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \psi'(v) w(v) \, dv
\]

\[
- \frac{M}{\delta^{n-1}} \int_{t_{10}}^{t} \psi(v) w^2(v) \sigma_i'^{-2}(v) \sigma_i'(v) \, dv
\]

\[
\leq \psi(t_{10}) w(t_{10}) - \frac{M}{\delta^{n-1}} \int_{t_{10}}^{t} \psi(v) \sigma_i'^{-2}(v) \sigma_i'(v) \, dv
\]

\[
\times \left[ w(v) - \frac{\delta^{n-1} \psi'(v)}{2M \psi(v) \sigma_i'^{-2}(v) \sigma_i'(v)} \right]^2 \, dv
\]
\[ + \int_{t_0}^{t} \frac{\delta^{n-1}[\psi'(v)]^2}{4M \psi(v) \sigma_i^{n-2}(v) \sigma'_i(v)} \, dv \leq \psi(t(t_0))w(t(t_0)) + \int_{t_0}^{t} \frac{\delta^{n-1}[\psi'(v)]^2}{4M \psi(v) \sigma_i^{n-2}(v) \sigma'_i(v)} \, dv. \]

Therefore, by \((C_3)\)

\[ +\infty = \alpha \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \psi(v) \sum_{i=1}^{m} q_i(v) \, dv \leq \psi(t(t_0))w(t(t_0)) + \frac{\delta^{n-1}}{4M} \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \frac{[\psi'(v)]^2}{\psi(v) \sigma_i^{n-2}(v) \sigma'_i(v)} \, dv < +\infty \]

for \(i = 1, 2, \ldots, m\). This is a contradiction.

Now let us consider the case of \(y(t) < 0\) for \(t \geq t_1\). By (1.1) and (1.2), we have

\[ z^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) f_i(y(\sigma_i(t))) > 0 \quad (t \geq t_1). \]

That is, \(z^{(n)}(t) > 0\). It follows that \(z^{(j)}(t) \ (j = 0, 1, 2, \ldots, n - 1)\) is strictly monotone and of constant sign eventually. Since \(p(t)\) and \(r(t)\) are oscillating functions, there exists a \(t_2 \geq t_1\), such that as \(t \geq t_2\), \(z(t) < 0\) eventually. Since \(y(t)\) is bounded, by \((C_1)\), from (1.2), there is a \(t_3 \geq t_2\), such that \(z(t)\) is also bounded for \(t \geq t_3\). Assume that \(x(t) = -z(t)\). Then \(x^{(n)}(t) = -z^{(n)}(t)\). Therefore, \(x(t) > 0\) and \(x^{(n)}(t) < 0\) for \(t \geq t_3\). Hence, we observe that \(x(t)\) is bounded. Since \(n\) is even, by Lemma 2.1, there exist a \(t_4 \geq t_3\) and \(l = 1\) (otherwise \(x(t)\) is not bounded), such that \((-1)^k x^{(k)}(t) > 0\) for \(k = 0, 1, 2, \ldots, n - 1\) and \(t \geq t_4\). That is, \((-1)^k z^{(k)}(t) > 0\) for \(k = 0, 1, 2, \ldots, n - 1\) and \(t \geq t_4\). In particular, as \(t \geq t_4\), \(z^{(l)}(t) > 0\). Therefore, \(z(t)\) is increasing. For the rest of proof, we can proceed the proof similar to the case of \(y(t) > 0\). Hence, the proof is completed. ∎

References


