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LETTER TO THE EDITOR

Nonexistence of Uniform Exponential Dichotomies for Delay Equations

Gyula Farkas[†]

Department of Mathematics, István Széchenyi University of Applied Sciences, H-9026 Győr, Hédervári u. 3., Hungary
E-mail: gyfarkas@math.ist.utl.pt; gyfarkas@math.bme.hu

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In this note, we report an error in the article by Zhang and Wu [2]. Before we recall the invalid result in [2, Theorem 2.1], we introduce some notation.

Consider a linear functional differential equation of the form

$$\dot{x}(t) = Lx_t,$$

where $L: C([-r,0],\mathbf{R}^n) \to \mathbf{R}^n$ is a bounded linear map, $x_t \in C([-r,0],\mathbf{R}^n)$ defined as $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$ and $C([-r,0],\mathbf{R}^n)$ is the Banach space of continuous functions $\phi : [-r,0] \to \mathbf{R}^n$ with norm $\|\phi\| = \sup_{\theta\in[-r,0]} |\phi(\theta)|$. Denote the C_0-semigroup generated by the linear functional differential equation by $\{T(t)\}_{t\geqslant 0}$.

In [2] the authors claimed that for any $\mu \in \mathbf{R}$ there exist constants $\gamma_2 > \gamma_1 > 0$ and K, K can be chosen independent of μ, such that

$$\|T(t)P_-\| \leqslant Ke^{(\mu-\gamma_2)t}, \qquad t \geqslant 0,$$

$$\|T(-t)P_+\| \leqslant Ke^{-(\mu-\gamma_1)t}, \qquad t \geqslant 0,$$

where P_+ is the eigenprojection associated with the spectral set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant \mu\}$, $P_- = I - P_+$ and A is the generator of $\{T(t)\}_{t\geqslant 0}$, see Theorem 2.1 in [2]. We call this family of exponential dichotomies as uniform exponential dichotomies since constant K is independent of μ. In what follows, we show that this claim is false, i.e. there are no uniform exponential dichotomies for delay equations.

Set $C_- = P_- C([-r,0],\mathbf{R}^n)$. We claim that there exists a function $\phi_* \in C_-$ such that $\|\phi_*\| = 1$ and $|\phi_*(-a^*)| = 1$ for some $a^* \in [0, r/2]$. In fact, consider

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a sequence $\{\phi_k\} \subset C([-r,0],\mathbf{R}^n)$ with the properties that $\|\phi_k\| = 1$, supp (ϕ_k) $\subset -r/2, 0]$ for $k = 1,2,\ldots$, and $\text{supp}(\phi_k) \cap \text{supp}(\phi_\ell) = \emptyset$ whenever $k \neq \ell$. Since C_- is of finite codimension, there is no loss of generality in assuming that $P_+\phi_k \to g$ for some $g \in C_+ = P_+C([-r,0],\mathbf{R}^n)$. Hence $\phi_k - \phi_\ell = c_{k\ell} + g_{k\ell}$, where $c_{k\ell} \in C_-$, $g_{k\ell} \in C_+$ and $\|g_{k\ell}\| \to 0$ as $k,\ell \to \infty$. It is readily seen that ϕ_* can be chosen for $c_{k\ell}/\|c_{k\ell}\|$ $(k \neq \ell)$, k,ℓ sufficiently large.

From the inequalities

$$1 = |\phi_*(-a^*)| = |[T(r-a^*)\phi_*](-r)| \leqslant \|T(r-a^*)\phi_*\| = \|T(r-a^*)P_-\phi_*\|$$
$$\leqslant \|T(r-a^*)P_-\| \leqslant Ke^{(\mu-\gamma_2)(r-a^*)} \leqslant Ke^{\mu(r-a^*)},$$

it follows that $K \geqslant e^{-\mu r - 2}$, for $\mu \leqslant 0$; i.e., K cannot be chosen independent of μ as $\mu \to -\infty$.

The proof of Theorem 2.1 in [2] contains two mistakes. First, the norms of P_+ and P_- are estimated by 1, see (2.8) on p. 417, and the continuity of $K_2(s)$, see (2.12) on p. 418, is used at $s = 0$ where it is not continuous. The discontinuity of $K_2(s)$ can be easily seen from the above construction. Let us study the estimates of the norms of the spectral projections in some detail.

Consider the following scalar delay differential equation:

$$\dot{x}(t) = ax(t) + bx(t-1),$$

where $b \neq 0$. Since $b \neq 0$, we know that there are infinitely many characteristic roots such that $\text{Re}\,\lambda_1 \geqslant \text{Re}\,\lambda_2 \geqslant \cdots$, $\text{Re}\,\lambda_j \to -\infty$ as $j \to \infty$. Let us switch to the complexification of $C([-1,0],\mathbf{R})$ and the semigroup $\{T(t)\}_{t \geqslant 0}$. Denote the eigenprojection associated with the spectral set $\{\lambda_j\}$ by P_j. It is known, see [1], that projections P_j have explicit matrix representations. Note that at most one characteristic root is not simple. Thus if j is large then $P_j\phi = a_j\phi_j$, where $\phi_j(\theta) = e^{\lambda_j\theta}$, $\theta \in [-1,0]$ and $a_j = \psi_j(0)\phi(0) + b\int_{-1}^{0} \psi_j(\tau+1)\phi(\tau)\,d\tau$. Here $\psi_j(\theta) = c_j e^{-\lambda_j\theta}$, $\theta \in [0,1]$ and (via an elementary calculation) $c_j = \frac{1}{1+be^{-\lambda_j}}$. From the representation of a_j we see that for all ε small enough we can choose a function ϕ_ε, $\|\phi_\varepsilon\| = 1$, $\phi_\varepsilon(0) = 1$, such that $|a_j| = |a_j(\phi_\varepsilon)| \geqslant \frac{1}{1+|b|e^{-\text{Re}\,\lambda_j}} - \varepsilon$. Thus $\|P_j\| \geqslant \|P_j\phi_\varepsilon\| \geqslant |a_j(\phi_\varepsilon)|e^{-\text{Re}\,\lambda_j} \geqslant (\frac{1}{1+|b|e^{-\text{Re}\,\lambda_j}} - \varepsilon)e^{-\text{Re}\,\lambda_j}$. Since ε was arbitrary we get that $\|P_j\| \geqslant \frac{e^{-\text{Re}\,\lambda_j}}{1+|b|e^{-\text{Re}\,\lambda_j}} \to \frac{1}{|b|}$ as $j \to \infty$.

Concerning the upper bounds, the following question arises: Is there a constant M such that $\|P_+\| \leqslant M$ for all $\mu \in \mathbf{R}$? We admit that we do not know the answer in general but for equations with complete system of (generalized) eigenfunctions of the generator (for definition see [1, Section 7.8]) the existence of the uniform bound for the norms of the eigenprojections implies convergent series expansion for all $\phi \in C$. Indeed, it is known that $\sum_{j=1}^{N} P_j \phi \to \phi$ in C for all $\phi \in D(A^\infty)$. Since $D(A^\infty)$ is dense in C and the norms of the projections $(P_+ =)\sum_{j=1}^{N} P_j$ have a uniform bound it follows that $\sum_{j=1}^{N} P_j\phi \to \phi$ in C for all $\phi \in C$. This motivates our expectation that the answer is negative.

The aim of paper [2] was to show that the homoclinic solution $x(t) = (\mathrm{sh}\,1)(\mathrm{ch}\,t)^{-1}$ of $\dot{x}(t) = (\mathrm{ch}\,1)(\mathrm{sh}\,1)^{-1}x(t) - (\mathrm{sh}\,1)^{-1}(1 + x^2(t))x(t-1)$ lies on a finite-dimensional invariant manifold. Since Theorem 2.1 is false the whole proof of the main result (Theorem 5.1) breaks down as well where the uniformity was used to control the cut-off regions. Thus, it is still not known whether the homoclinic orbit lies on a finite-dimensional invariant manifold.

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