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A Class of Parabolic Quasi-Variational Inequalities*

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INTRODUCTION

Free boundary problems in the theory of elliptic and parabolic equations often can be reformulated as variational inequalities or, more generally, as quasi-variational inequalities. For examples of the later we refer to Baiocchi [1, 2]. In this paper we consider the following free boundary value problem. Find functions $u(x, t)$ and $s(t)$ such that

$$\begin{aligned} -u_{xx} + u_t &= 0 && \text{for } 0 < x < s(t), \quad 0 < t < T, \\ u &= \lambda && \text{for } x = s(t), \quad 0 < t < T, \\ u_x &= 0 && \text{for } x = s(t), \quad 0 < t < T, \\ u(x, 0) &= u_0(x) && \text{for } 0 < x < s_0, \\ u(0, t) &= h(t) && \text{for } 0 < t < T, \end{aligned} \tag{1}$$

where $\lambda(x, t)$, $u_0(x)$, and $h(t)$ are given functions and s_0, T are given positive numbers. This problem will be reformulated as a new type of quasi-variational inequality, namely: Find functions $w(x, t)$ and $s(t)$ such that $w(x, t) \geqslant 0$ and

$$\{(x, t): w(x, t) > 0\} = \{(x, t): 0 < x < s(t), 0 < t < T\}, \tag{2}$$

and

$$\begin{aligned} -w_{xx} + w_t &\geqslant f(x, t; s(t)) && \text{for } 0 < x < R, 0 < t < T, \\ &= f(x, t; s(t)) && \text{for } 0 < x < s(t), 0 < t < T, \\ w(x, 0) &= g(x) && \text{for } 0 < x < R, \\ w_x(0, t) &= \psi(t) && \text{for } 0 < t < T, \\ w(R, t) &= 0 && \text{for } 0 < t < T. \end{aligned} \tag{2'}$$

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The function f depends on the boundary of the set where $w(x, t) > 0$ in a certain specified manner, and g and ψ depend on the given u_0 and h. Herein lies the novelty and the quasi-variational nature of our problem. First of all, the function f depends on the solution w to (2′) by (2). Since this dependence is not smooth, the conditions (2), (2′) do not determine a variational inequality in the usual sense. However its solution may be found by approximation with solutions of variational inequalities (cf. Sect. 3), and for this reason we call our problem a quasi-variational inequality. The usual quasi-variational inequality entails a convex set of competing functions which may depend on the possible solution. The problem at hand will differ from this because the convex set will be fixed, but instead the inhomogenous term f will depend on the possible solution.

We prove that there exist minimal and maximal solutions for (2). A uniqueness theorem is also proved. As a by-product of the existence proof we establish that the free boundary, for any solution of (2′), is continuous and monotone increasing.

Problem (1) can also be reduced to a "Stefan type" problem for the function $v = -u_x$. One can easily check that

$$\begin{aligned} -v_{xx} + v_t &= 0 && \text{for } 0 < x < s(t), \quad 0 < t < T, \\ v &= 0 && \text{for } x = s(t), \quad 0 < t < T, \\ v_x &= -\lambda_x \dot{s} - \lambda_t && \text{for } x = s(t), \quad 0 < t < T, \\ v(x, 0) &= -u_0'(x) && \text{for } 0 < x < s_0, \\ v_x(0, t) &= -h'(t) && \text{for } 0 < t < T. \end{aligned} \tag{3}$$

When $\lambda_x > 0$, this problem has a unique local classical solution; it can be constructed by transforming the problem into a nonlinear integral equation for s (see, for instance, [5, 7, 8]). We shall assume later on that $\lambda_x > 0$, $\lambda_t \leqslant 0$. If also

$$u_0'(x) \leqslant 0, \qquad h'(t) \geqslant 0, \tag{4}$$

then there exists a unique classical global solution. Indeed, since in this case $v > 0$ if $0 < x < s(t)$, it follows that $v_x \leqslant 0$ on $x = s(t)$, and the a priori estimates (as in [5]) for $v_x(x, t)$ can be established. We shall not assume however the conditions in (4). Consequently the usual methods (such as in [5, 7, 8]) for establishing the existence of a global solution cannot be applied.

The methods of this paper apply also to problems other than (1). For example (see Sect. 6), they apply to the problem obtained from (1) upon replacing the condition $u = \lambda$ by

$$u(s(t), t) = 1 + \int_0^t g(s(\tau))\, d\tau \qquad (g > 0, g_x \leqslant 0). \tag{5}$$

Notice that if we set $v = -u_x$, then v satisfies the condition in (3) with the exception of $v_x = -\lambda_x \dot{s} - \lambda_t$, which is replaced by

$$v_x(s(t), t) = -g(s(t));$$

this is a "Stefan type" problem with zero latent heat.

Kružkov [6] has studied a free boundary problem similar to (1), but with $u = \lambda$ replaced by

$$u = 1 - \alpha \int_0^t \frac{d\tau}{1 - s(\tau)} \qquad \text{on} \quad x = s(t) \quad (\alpha > 0). \tag{6}$$

He proved a uniqueness theorem for smooth solutions. This particular problem arises when a viscoplastic rod hits a rigid obstacle [3].

1. Formulation in Variational Terms

Let $\lambda(x, t)$, $u_0(x)$, and $h(t)$ be given smooth functions for $0 \leqslant x \leqslant R$ and $0 \leqslant t \leqslant T$, and let $s_0 \in (0, R)$. With $D = \{(x, t): 0 < x < R, 0 < t < T\}$, consider the free boundary value problem classically stated.

PROBLEM 1. To find a curve Γ: $x = s(t)$, $0 < t < T$, and a function $u(x, t)$ which satisfy

$$\begin{aligned}
-u_{xx} + u_t &= 0 && \text{for} \quad 0 < x < s(t), \qquad 0 < t < T,\\
u &= \lambda && \text{for} \quad (x, t) \in \Gamma,\\
u_x &= 0 && \text{for} \quad (x, t) \in \Gamma,\\
u(x, 0) &= u_0(x) && \text{for} \quad 0 < x < s_0,\\
u(0, t) &= h(t) && \text{for} \quad 0 < t < T.
\end{aligned}$$

The above may be reformulated as a variational problem for the function

$$w(x, t) = \int_x^R (u(\xi, t) - \lambda(\xi, t))\, d\xi, \qquad (x, t) \in D, \tag{1.1}$$

where it is understood here that $s(t) < R$ if $0 < t < T$ and that $u(x, t)$ is extended to be $\lambda(x, t)$ in the set $s(t) \leqslant x \leqslant R$, $0 \leqslant t \leqslant T$. So $w(x, t) = 0$ if $s(t) \leqslant x \leqslant R, 0 < t < T$. We define $\Omega = \{(x, t): 0 < x < s(t), 0 < t < T\}$. We may now compute that, for $(x, t) \in \Omega$,

$$\begin{aligned}
w_t(x, t) &= \int_x^R (u_t(\xi, t) - \lambda_t(\xi, t))\, d\xi\\
&= \int_x^{s(t)} u_{\xi\xi}(\xi, t)\, d\xi - \int_x^{s(t)} \lambda_t(\xi, t)\, d\xi\\
&= u_x(s(t), t) - u_x(x, t) - \int_x^{s(t)} \lambda_t(\xi, t)\, d\xi\\
&= -u_x(x, t) - \int_x^{s(t)} \lambda_t(\xi, t)\, d\xi.
\end{aligned}$$

Also, for $(x, t) \in \Omega$,

$$w_x(x, t) = -(u(x, t) - \lambda(x, t)),$$
$$w_{xx}(x, t) = -u_x(x, t) + \lambda_x(x, t).$$

Therefore

$$-w_{xx}(x, t) + w_t(x, t) = f(x, t) \qquad \text{for} \quad (x, t) \in \Omega, \tag{1.2}$$

where

$$\begin{aligned} f(x, t) &= -\lambda_x(x, t) - \int_x^{s(t)} \lambda_t(\xi, t)\, d\xi \qquad && \text{if} \quad (x, t) \in \Omega \\ &= -\lambda_x(x, t) && \text{if} \quad (x, t) \in D\backslash\Omega. \end{aligned} \tag{1.3}$$

The definition of $f(x, t)$ for (x, t) in $D\backslash\Omega$ is made for technical convenience. We turn now to the boundary conditions asked of w. Assume that

$$u_0(x) > \lambda(x, 0) \qquad \text{for} \quad 0 \leqslant x < s_0 \qquad \text{and} \qquad u(s_0, 0) = \lambda(s_0, 0). \tag{1.4}$$

Now define

$$\begin{aligned} g(x) &= \int_x^{s_0} (u_0(\xi) - \lambda(\xi, 0))\, d\xi \qquad && \text{if} \quad 0 \leqslant x \leqslant s_0 \\ &= 0 && \text{if} \quad s_0 \leqslant x \leqslant R, \end{aligned} \tag{1.5}$$

so that $w(x, 0) = g(x)$. Observe that $g'(x) \leqslant 0$ by (1.4). Insisting that $u_0(x) > \lambda(x, 0)$ in $(0, s_0)$ will ensure that the free boundary Γ "starts" at s_0. Set

$$\psi(t) = -h(t) + \lambda(0, t), \qquad 0 \leqslant t \leqslant T. \tag{1.6}$$

Then

$$w_x(0, t) = \psi(t), \qquad 0 < t < T. \tag{1.7}$$

We assume that

$$\psi(t) \leqslant 0 \qquad \text{for} \quad 0 < t < T.$$

An important feature of this problem is the equation satisfied by w_x in Ω. Differentiating with respect to x in (1.2) shows that

$$-w_{xxx} + w_{xt} = f_x = -\lambda_{xx} + \lambda_t \qquad \text{in } \Omega.$$

Imposing the condition

$$-\lambda_{xx} + \lambda_t \leqslant 0 \qquad \text{in } D \tag{1.8}$$

and recalling that $w_x(x, t) = \lambda(x, t) - u(x, t) = 0$ for $(x, t) \in \Gamma$, we obtain

$$\begin{aligned} -w_{xxx} + w_{xt} &\leqslant 0 \qquad && \text{in } \Omega, \\ w_x &\leqslant 0 && \text{on } \partial'\Omega, \end{aligned}$$

where $\partial'\Omega$ denotes the parabolic boundary of Ω. Therefore, assuming w_x to be continuous in $\bar{\Omega}$, the maximum principle implies $w_x < 0$ in Ω. After an integration with respect to x, $w(x, t) > w(s(t), t) = 0$ in Ω. We state the information derived above, albeit partially in a formal manner, about w in terms of variational inequalities. For this, set

$$K = \{v \in H^1(D);\ v \geqslant 0 \text{ in } D\}.$$

Then

$$\begin{aligned} (-w_{xx} + w_t)(v - w) &\geqslant f(v - w) && \text{a.e. in } \Omega \text{ for all } v \in K, \\ w &> 0 && \text{in } \Omega, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} (-w_{xx} + w_t)(v - w) = 0 \geqslant fv &= f(v - w) && \text{a.e. in } D\backslash\Omega \text{ for all } v \in K, \\ w &= 0 && \text{in } D\backslash\Omega, \end{aligned} \tag{1.10}$$

under the assumption that $f \leqslant 0$ in D.

To introduce the problem we shall treat here, we state some conventions and summarize our hypotheses about the initial data. For a given bounded function $\sigma(t)$, $0 \leqslant t \leqslant R$, $\sigma(0) = s_0$, we define

$$\begin{aligned} f(x, t) = f(x, t; \sigma) &= -\lambda_x(x, t) - \int_x^{\sigma(t)} \lambda_t(\xi, t)\, d\xi \qquad \text{if } 0 \leqslant x \leqslant \sigma(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \leqslant t \leqslant T, \\ &= -\lambda_x(x, t) \qquad \text{if} \quad \sigma(t) \leqslant x \leqslant R, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{1.11}$$

Let $g(x)$, $0 \leqslant x \leqslant R$, be smooth and satisfy

$$\begin{gathered} g(x) > 0 \text{ for } 0 \leqslant x \leqslant s_0, \qquad g(x) = 0 \text{ for } s_0 \leqslant x \leqslant R, \\ g'(x) \leqslant 0 \text{ for } 0 < x < s_0, \end{gathered} \tag{1.12}$$

and let $\psi(t)$, $0 \leqslant t \leqslant T$, be smooth and satisfy

$$\psi(t) \leqslant 0, \qquad \psi'(t) \leqslant 0 \qquad \text{for} \quad 0 \leqslant t \leqslant T, \tag{1.13}$$

$$g'(0) = \psi(0). \tag{1.14}$$

The relations (1.9) and (1.10) lead to the problem below about w.

PROBLEM (*). To find $w \in H^1(D)$ such that

$$\begin{aligned} w \in K\colon\ (-w_{xx} + w_t)(v - w) &\geqslant f(v - w) && \text{a.e. in } D \quad \text{for} \quad v \in K, \\ w(x, 0) &= g(x) && \text{for} \quad 0 < x < R, \\ w_x(0, t) &= \psi(t) && \text{for} \quad 0 < t < T, \\ w(R, t) &= 0 && \text{for} \quad 0 < t < T \end{aligned}$$

where $f = f(x, t; s)$ is defined by (1.11) for

$$s(t) = \inf\{x\colon w(x, t) = 0\}, \quad 0 \leqslant t \leqslant T,$$

and g, ψ satisfy (1.12), (1.13), and (1.14), respectively.

Since f does not depend in a continuous way on $w(x, t)$, Problem (*) is not properly a variational inequality but a form of quasi-variational inequality. Our approach is first to study a variational inequality associated to a given function $s(t)$. We describe this as

PROBLEM 2. To find $w \in H^1(D)$ such that

$$\begin{aligned} w \in K\colon \quad (-w_{xx} + w_t)(v - w) &\geqslant f(v - w) \qquad \text{a.e. in } D \qquad \text{for} \quad v \in K,\\ w(x, 0) &= g(x) \qquad \text{for} \quad 0 < x < R,\\ w_x(0, t) &= \psi(t) \qquad \text{for} \quad 0 < t < T,\\ w(R, t) &= 0 \qquad \text{for} \quad 0 < t < T, \end{aligned}$$

where $f = f(x, t; \sigma(t))$ is defined by (1.11) for a given increasing function $\sigma(t)$, $0 \leqslant t \leqslant T$, $\sigma(0) = s_0$, and g, ψ satisfy (1.12) and (1.13), respectively.

Under some hypotheses about λ, more restrictive than just (1.8) or that $f \leqslant 0$, this latter necessary to obtain a variational inequality under any circumstances, we shall prove that Problem 2 admits a unique solution w for which $s(t) = \inf\{x\colon w(x, t) = 0\}$ is again an increasing function. Let us say, in this situation, that $s = A\sigma$. To solve Problem (*) we show the existence of an s for which $s = As$.

2. The Solution to a Variational Inequality

Let $\sigma(t)$, $0 \leqslant t \leqslant T$, $\sigma(0) = s_0$, be an increasing bounded function and define (cf. (1.11)),

$$\begin{aligned} f(x, t) &= -\lambda_x(x, t) - \int_x^{\sigma(t)} \lambda_t(\xi, t)\, d\xi \qquad \text{if} \quad 0 \leqslant x \leqslant \sigma(t), \quad 0 \leqslant t \leqslant T\\ &= -\lambda_x(x, t) \qquad \text{if} \quad \sigma(t) \leqslant x \leqslant R, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{2.1}$$

About f we assume

$$\begin{aligned} f(x, t) &< 0 \qquad \text{if} \quad (x, t) \in \bar{D},\\ f_t(x, t) &\geqslant 0 \qquad \text{if} \quad (x, t \in D,\\ f_x(x, t) &\leqslant 0 \qquad \text{if} \quad (x, t) \in D, \end{aligned} \tag{2.2}$$

and

$$f(x, 0) + g''(x, 0) \geqslant 0 \qquad \text{if} \quad 0 < x < s_0. \tag{2.3}$$

The conditions (2.2), (2.3) are abstract and lead to the solution of a variational inequality. To understand them in the present context, let us determine their implications for the f defined in (2.1). The condition that f be negative means

$$-\int_x^{\sigma(t)} \lambda_t(\xi, t)\, d\xi < \lambda_x(x, t) \qquad \text{for} \quad 0 < x < \sigma(t), \quad 0 < t < T,$$

and

$$0 < \lambda_x(x, t) \qquad \text{for} \quad \sigma(t) \leqslant x \leqslant R, \qquad 0 < t < T.$$

If we further suppose that $\lambda_t \leqslant 0$, then both of the above will be satisfied if

$$-\int_0^R \lambda_t(\xi, t)\, d\xi < \lambda_x(x, t) \qquad \text{for} \quad (x, t) \in D.$$

Now we compute

$$0 \leqslant \frac{\partial f}{\partial t}(x, t) = -\left\{\lambda_t(\sigma(t), t)\, \sigma'(t) + \lambda_{xt}(x, t) + \int_x^{\sigma(t)} \lambda_{tt}(\xi, t)\, d\xi\right\}$$

$$\text{for} \quad 0 < x < \sigma(t), \quad 0 < t < T,$$

and

$$0 \leqslant \frac{\partial f}{\partial t}(x, t) = -\lambda_{xt}(x, t) \qquad \text{for} \quad \sigma(t) \leqslant x < R, \quad 0 < t < T.$$

Since σ' may be arbitrary, but never negative, and $\sigma \in [0, R]$, that $\partial f/\partial t \geqslant 0$ amounts to assuming

$$\lambda_t(x, t) \leqslant 0, \qquad \lambda_{xt}(x, t) \leqslant 0,$$

and

$$\lambda_{xt}(x, t) + \int_x^y \lambda_{tt}(\xi, t)\, d\xi \leqslant 0 \qquad \text{for} \quad 0 \leqslant y < x \leqslant R, \quad 0 \leqslant t \leqslant T.$$

It is easy to see that the condition $f_x \leqslant 0$ is satisfied if

$$\lambda_t(x, t) \leqslant 0 \qquad \text{and} \qquad \lambda_{xx}(x, t) \geqslant 0.$$

Finally, the boundary condition (2.3) means that

$$g''(x) - \left(\lambda_x(x, 0) + \int_x^{s_0} \lambda_t(\xi, 0)\, d\xi\right) \geqslant 0, \qquad 0 < x < s_0\, .$$

Summarizing, f satisfies (2.2) and (2.3) for any increasing $\sigma(t)$, $\sigma_0(0) = s_0$, $\sigma(T) \leqslant R$, provided that

$$\begin{aligned}
&\lambda_t(x, t) \leqslant 0 && \text{if} \quad (x, t) \in D, \\
&-\int_0^R \lambda_t(\xi, t)\, d\xi < \lambda_x(x, t) && \text{if} \quad (x, t) \in D, \\
&\lambda_{xt}(x, t) \leqslant 0 \quad \text{and} \quad \lambda_{xx}(x, t) \geqslant 0 && \text{if} \quad (x, t) \in D, \\
&\lambda_{xt}(x, t) + \int_x^y \lambda_{tt}(\xi, t)\, d\xi \leqslant 0 && \text{if} \quad 0 \leqslant y < x \leqslant R,\ 0 \leqslant t \leqslant T
\end{aligned} \tag{2.4}$$

and

$$g''(x) - \left(\lambda_x(x, 0) + \int_x^{s_0} \lambda_t(\xi, 0)\, d\xi\right) \geqslant 0, \qquad 0 < x < s_0. \tag{2.5}$$

Problem 2 will be studied by means of an appropriate "penalized" problem. Select a sequence $f_\epsilon(x, t)$, $0 < \epsilon < 1$, of smooth functions which converge pointwise to $f(x, t)$ in $\bar{D}$ with the properties

$$\begin{aligned}
-k \leqslant f_\epsilon &\leqslant 0 && \text{in } D \text{ for a } k > 0 \text{ (independent of } \epsilon), \\
\partial f_\epsilon / \partial t &\geqslant 0 && \text{in } D, \\
\partial f_\epsilon / \partial x &\leqslant 0 && \text{in } D, \\
f_\epsilon(x, 0) + g''(x) &\geqslant 0 && \text{for} \quad 0 < x < s_0.
\end{aligned} \tag{2.6}$$

For example, given $\sigma(t)$ we might choose a family $\sigma_\epsilon(t)$ of smooth increasing functions such that

$$\sigma_\epsilon(t) \leqslant \sigma(t) \qquad \text{if} \quad 0 \leqslant t \leqslant T, \quad \sigma_\epsilon(0) = s_0, \quad 0 < t < 1,$$

and

$$\lim_{\epsilon \to 0} \sigma_\epsilon(t) = \sigma(t) \qquad \text{if} \quad 0 \leqslant t \leqslant T$$

proceeding to define $f_\epsilon(x, t)$ by (2.1) for σ_ϵ. In this case, $f_\epsilon(x, 0) = f(x, 0)$ so the last condition of (2.6) is ensured by (2.5).

Finally, let $\beta_\epsilon(t) \in C^\infty(R^1)$, $0 < \epsilon < 1$, satisfy

$$\begin{aligned}
&\beta_\epsilon(t) = 0 && \text{for} \quad t \geqslant \epsilon, \\
&\beta_\epsilon(0) = -1, \\
&\beta_\epsilon'(t) > 0 && \text{for} \quad -\infty < t < \infty.
\end{aligned}$$

Consider the initial value problem for given $T > 0$ and ϵ, $0 < \epsilon < 1$.

PROBLEM 3. To find $w_\epsilon(x, t)$, $0 < \epsilon < 1$, satisfying

$$\begin{aligned} -w_{\epsilon xx} + w_{\epsilon t} + k\beta_\epsilon(w_\epsilon) &= f_\epsilon && \text{in } D, \\ w_\epsilon &= g && \text{for } t = 0, \quad 0 < x < R, \\ w_{\epsilon x} &= \psi && \text{for } x = 0, \quad 0 < t < T, \\ w_\epsilon &= 0 && \text{for } x = R, \quad 0 < t < T \end{aligned} \tag{2.7}$$

where f_ϵ satisfies (2.6) and g, ψ satisfy (1.12), (1.13), and (1.14). The solution to this problem is known to exist and to be smooth in D. In particular, recall that (1.14) is a continuity condition at $(x, t) = (0, 0)$. We now discuss the properties of the solution.

THEOREM 1. *Let w_ϵ, $0 < \epsilon < 1$, denotes the solution to Problem 3. Then*

$$w_{\epsilon t} \geqslant 0 \qquad \textit{in } D, \tag{2.8}$$

$$w_\epsilon \geqslant 0 \qquad \textit{and} \qquad -1 \leqslant \beta_\epsilon(w_\epsilon) \leqslant 0 \qquad \textit{in } D, \tag{2.9}$$

$$w_{\epsilon x} \leqslant 0 \qquad \textit{in } D, \tag{2.10}$$

and

$$\| w_\epsilon \|_{H^{1,p}(D)} + \| w_{\epsilon xx} \|_{L^p(D)} \leqslant C_p, \tag{2.11}$$

a constant independent of ϵ, $0 < \epsilon < 1$, for $1 \leqslant p < \infty$.

Proof of (2.8). Let $v = \partial w_\epsilon / \partial t$, and suppress the dependence of the various quantities on ϵ. Differentiating (2.7) with respect to t, we obtain that v is a solution to the initial value problem

$$\begin{aligned} -v_{xx} + v_t + k\beta'(w)v &= f_t \geqslant 0 && \text{in } D, \\ v &= f - k\beta(g) + g'' && \text{if } 0 < x < s_0, \quad t = 0 \\ &= f - k\beta(0) && \text{if } s_0 \leqslant x < R, \quad t = 0, \\ v_x &= \psi' && \text{if } x = 0, \quad 0 < t < T, \\ v &= 0 && \text{if } x = R, \quad 0 < t < T. \end{aligned}$$

Since $f_t \geqslant 0$ and $\beta'(w) \geqslant 0$ we know from the maximum principle that

$$v(x, t) \geqslant \min(\min_{\partial' D} v, 0) \qquad \text{for} \quad (x, t) \in D,$$

where $\partial' D$ denotes the parabolic boundary of D. By (1.13), $v_x = \psi' \leqslant 0$ where $x = 0$ so v does not attain its minimum value when $x = 0$. By (2.6) and (2.7),

$$v(x, 0) = f + g'' - k\beta(g) \geqslant f + g'' \geqslant 0 \qquad \text{for} \quad 0 < x < s_0$$

whereas $\beta(0) = -1$ and $f \geqslant -k$ imply that

$$v(x, 0) = f + k \geqslant 0 \quad \text{for} \quad s_0 \leqslant x < R.$$

Hence $\min_{\partial' D} v = 0$.

Proof of (2.9). Since $w_{\epsilon t} \geqslant 0$ and $w_\epsilon(x, 0) \geqslant 0$, $w_\epsilon(x, t) \geqslant 0$ for $(x, t) \in D$. Hence $-1 = \min_{\eta \geqslant 0} \beta_\epsilon(\eta) \leqslant \beta_\epsilon(w_\epsilon(x, t)) \leqslant \sup \beta_\epsilon(\eta) = 0$.

Proof of (2.10). Let $v = \partial w_\epsilon / \partial x$ and again suppress the dependence of the various quantities on ϵ. Differentiating (2.7) with respect to x, we obtain the relations

$$\begin{aligned}
-v_{xx} + v_t + k\beta'(w)v &= f_x \leqslant 0 && \text{in } D, \\
v &= g' && \text{if } \ 0 < x < s_0, \quad t = 0 \\
&= 0 && \text{if } \ s_0 \leqslant x < R, \quad t = 0, \\
v &= \psi && \text{if } \ x = 0, \quad 0 < t < T \\
v &= \partial w / \partial x \leqslant 0 && \text{if } \ x = R, \quad 0 < t < T.
\end{aligned}$$

Here note that $w_x(R, t) \leqslant 0$ since w attains its minimum when $x = R$. Because $f_x \leqslant 0$,

$$v \leqslant \max(0, \max_{\partial' D} v).$$

By (1.12), (1.13), $\psi \leqslant 0$ and $g' \leqslant 0$. Hence $v \leqslant 0$.

Proof of (2.11). This follows from the standard L^p-estimates for the solution of the heat equation in view of (2.9)(cf. [9]).

THEOREM 2. *There exists a unique solution $w(x, t) \in H^1(D)$ to Problem 2. It enjoys the properties*

$w \in H^{1,p}(D)$ *and* $w_{xx} \in L^p(D)$ *for each* p, $1 \leqslant p < \infty$,

$w_t \geqslant 0$ *and* $w_{xx} \geqslant 0$ a.e. *in* D, *and*

$\inf \psi \leqslant w_x \leqslant 0$ a.e. *in* D.

With respect to the approximation w_ϵ, $0 < \epsilon < 1$, $w_\epsilon \to w$ *uniformly in* $\bar{D}$ *and weakly in* $H^{1,p}(D)$, $1 < p < \infty$, *and* $w_{\epsilon xx} \to w_{xx}$ *weakly in* $L^p(D)$, $1 < p < \infty$.

Proof. By Theorem 1, $\| w_\epsilon \|_{H^{1,p}(D)} + \| w_{\epsilon xx} \|_{L^p(D)} \leqslant$ const. independent of ϵ for a fixed p, $2 < p < \infty$. Hence there is a subsequence w_{ϵ_j} such that $w_{\epsilon_j} \to w$ weakly in $H^{1,p}(D)$ and $w_{\epsilon_j xx} \to w_{xx}$ weakly in $L^p(D)$ for a function $w \in H^{1,p}(D)$ with $w_{xx} \in L^p(D)$. By Sobolev's lemma, $w_{\epsilon_j} \to w$ uniformly in $\bar{D}$.

The w so determined satisfies the boundary conditions $w = g$ for $t = 0$, and $w = 0$ for $x = R$. Neglecting the final boundary condition for the moment, we show that w is a solution to the variational inequality of Problem 2. Let $v \in K$ satisfy $v \geqslant \eta > 0$ a.e. for an $\eta > 0$. Then for $\epsilon = \epsilon_j < \eta$, $\beta_\epsilon(v) = 0$, whence $(-w_{\epsilon xx} + w_{\epsilon t})(v - w_\epsilon) - k(\beta_\epsilon(v) - \beta_\epsilon(w_\epsilon))(v - w_\epsilon) = f(v - w_\epsilon)$. Integrating the above over D and recalling that β is an increasing function, namely,

$$(\beta_\epsilon(v) - \beta_\epsilon(w_\epsilon))(v - w_\epsilon) \geqslant 0,$$

we see that

$$\iint_D (-w_{\epsilon xx} + w_{\epsilon t})(v - w_\epsilon)\, dx\, dt \geqslant \iint_D f_\epsilon(v - w_\epsilon)\, dx\, dt.$$

We may pass to the limit as $\epsilon = \epsilon_j \to 0$, since $w_{\epsilon_j} \to w$ uniformly, to obtain

$$\iint_D (-w_{xx} + w_t)(v - w)\, dx\, dt \geqslant \iint_D f(v - w)\, dx\, dt, \qquad v \in K, \quad v \geqslant \eta > 0.$$

Since η is arbitrary, the above holds for all $v \in K$; hence

$$(-w_{xx} + w_t)(v - w) \geqslant f(v - w) \qquad \text{a.e. in } D.$$

It is clear that $w_t \geqslant 0$ and $w_x \leqslant 0$ a.e. in D. Moreover, if $w > 0$, then $-w_{xx} + w_t = f; f < 0$, $w_t > 0$ imply

$$w_{xx} > w_t > 0 \qquad \text{if} \quad w > 0.$$

Since otherwise $w_{xx} = w_t = 0$, we have

$$w_{xx} \geqslant w_t \geqslant 0 \qquad \text{in } D.$$

Also, $D_0 = \{(x, t): 0 < x < s_0, 0 < t < T\} \subset \{(x, t): w(x, t) > 0\}$. Therefore w_x satisfies the equation

$$-w_{xxx} + w_{xt} = f_x \qquad \text{in } D_0,$$

with f_x a smooth function. It follows by a simple argument that $w_{\epsilon_j x} \to w_x$ uniformly in $D_0 \cap \{(x, t): x < s_0 - \delta, t > \delta\}$, $\delta > 0$; hence $w_x(0, t) = \psi(t)$, $0 < t < T$.

It is not difficult to see that w is unique. From this it follows that the entire family $w_\epsilon \to w$ weakly in $H^{1,p}(D)$. The remaining assertions of the theorem now follow easily. Q.E.D.

COROLLARY 2.1. *Let w denote the solution to Problem 2 and define*

$$s(t) = \inf\{x\colon w(x, t) = 0\}, \qquad 0 \leqslant t \leqslant T.$$

Then $s(t)$, $0 \leqslant t \leqslant T$, is a continuous increasing function of t.

Proof. First we observe that by Theorem 2 and the continuity of w, $w(x, t)$ is decreasing in x for fixed t and is increasing in t for fixed x. Hence $s(t)$ is well defined and increasing. It is lower semicontinuous since $\{(x, t)\colon w(x, t) = 0\}$ is closed.

For t given, $0 < t \leqslant T$, $s(\tau) \leqslant s(t)$ for $\tau \leqslant t$. Therefore

$$\limsup_{\substack{\tau\to t\\ \tau<t}} s(\tau) \leqslant s(t).$$

By the lower semicontinuity of s, $s(t_0) = \lim_{\tau<t_0} s(\tau)$.

By the monotonicity of $s(t)$, $s^+ = \lim_{\tau\to t_0+} s(\tau)$ exists. Suppose $s^+ > s(t_0)$. Then $w(x, t)$ satisfies

$$\begin{aligned} -w_{xx} + w_t &= f \qquad \text{in } s(t) < x < s^+, \qquad t_0 < t < T,\\ w &= 0 \qquad \text{if} \quad t = t_0\,, \qquad s(t_0) < x < s^+. \end{aligned}$$

Now f is not a smooth function, but f_x is, so we differentiate this equation with respect to x. Consequently,

$$\begin{aligned} -w_{xxx} + w_{xt} &= f_x \qquad \text{in } s(t) < x < s^+, \qquad t_0 < t < T,\\ w &= 0 \qquad \text{if} \quad s(t_0) < x < s^+, \qquad t = t_0\,. \end{aligned}$$

Now it follows that w_{xx} is smooth in $s(t_0) + \delta < x < s^+ - \delta$ $t_0 \leqslant t < T$ and $w_{xx} = 0$ for $t = t_0$. We assumed $f < 0$; hence

$$w_t = f + w_{xx} < 0 \qquad \text{for} \quad s(t_0) + \delta < x < s^+ - \delta, \quad t_0 \leqslant t \leqslant t_0{}^+$$

for a positive ϵ. Hence $w_t < 0$ on a set of positive measure. This is a contradiction. The argument also shows that

$$s_0 = \lim_{t\to 0} s(t). \qquad \text{Q.E.D.}$$

COROLLARY 2.2. *Let w be a solution to Problem 2 and set $\Omega = \{(x, t\colon w(x, t) > 0\}$. Then $w_x < 0$ in Ω.*

Proof. Since $v = w_x$ satisfies

$$-v_{xx} + v_t = f_x \leqslant 0 \qquad \text{in } \Omega, \qquad v \leqslant 0 \text{ in } \Omega,$$

v cannot assume a maximum in Ω; hence $v < 0$ in Ω.

3. The Solution to Problem (*)

Given two increasing functions σ, σ^* with $\sigma(0) = \sigma^*(0) = s_0$ we define f, f^* according to (2.1). Because $\lambda_t(x, t) \leqslant 0$ (by (2.4)), it follows that if $\sigma(t) \leqslant \sigma^*(t)$, $0 \leqslant t \leqslant T$, then

$$f(x, t) \leqslant f^*(x, t) \qquad (x, t) \in D.$$

Lemma 3.1. *Let σ, σ^* be increasing functions with $\sigma(0) = \sigma^*(0) = s_0$ and let w, w^* denote the corresponding solutions to Problem 2. If $\sigma \leqslant \sigma^*$ in $(0, T)$, then*

$$w \leqslant w^* \qquad \text{in } D.$$

The proof of this lemma is a familiar application of the maximum principle and will be omitted.

Denote by X the set of all increasing functions $\sigma(t)$, $0 \leqslant t \leqslant T$, which satisfy

$$\sigma(0) = s_0 \qquad \text{and} \qquad \sigma(T) \leqslant R.$$

Given $\sigma \in X$, we consider the solution w to Problem 2 defined by σ and the function

$$s(t) = \inf\{x : w(t) = 0\}, \qquad 0 \leqslant t \leqslant T.$$

From Corollary 2.1 we infer that $s \in X$. We say in this circumstance that

$$s = A\sigma.$$

If $\sigma, \sigma^* \in X$ and $\sigma \leqslant \sigma^*$, then Lemma 3.1 implies that

$$A\sigma \leqslant A\sigma^*.$$

Thus $A: X \to X$ is an increasing function.

Theorem 3. *Let (2.4), (2.5) and (1.12)–(1.14) hold. Then there exists a solution w'' to Problem (*) with the property that if w is any solution to Problem (*) then*

$$w \leqslant w'' \qquad \text{in } D.$$

Proof. In this proof we follow Tartar [10]. First let $s_0(t) = s_0$. From Corollary 2.1,

$$s_0(t) = s_0 = As_0(0) \leqslant As_0(t).$$

Now let $s_R(t) = R$, $0 < t \leqslant T$, and $s_R(0) = s_0$. Then $s_R \in X$ and As_R is a function bounded above by R, namely,

$$As_R \leqslant R = s_R \quad \text{for} \quad 0 < t \leqslant T,$$
$$As_R(0) = s_0 = s_R(0).$$

Set $X_- = \{\sigma \in X : \sigma \leqslant A\sigma\}$, which contains s_0, $X_+ = \{\sigma \in X : A\sigma \leqslant \sigma\}$, which contains s_R, and $Y = \{\sigma \in X_+ : \sigma' \leqslant \sigma \text{ for all } \sigma' \in X_-\}$, which contains s_R. Given a family $\{\sigma_i \,;\, i \in I\} \subset Y$,

$$\sigma \equiv \inf \sigma_i \leqslant \sigma_i \quad \text{for each} \quad i$$

and σ is increasing. Hence $\sigma \in X$ and

$$A\sigma \leqslant \inf A\sigma_i \leqslant \inf \sigma_i = \sigma.$$

Therefore $\sigma \in X_+$. Moreover, for $\sigma' \in X_-$,

$$\sigma' \leqslant \sigma_i \quad \text{for all} \quad i,$$

so

$$\sigma' \leqslant \inf \sigma_i = \sigma.$$

Hence $\sigma \in Y$. We have established that every chain in Y has a lower bound; hence, by Zorn's lemma, there are minimal elements in Y, say s. Consequently, $As \leqslant s$.

To show $As = s$, we show that $As \in Y$. Indeed, for any $\sigma \in X_-$, $\sigma \leqslant s$ implies that

$$\sigma \leqslant A\sigma \leqslant As. \tag{3.1}$$

Also, $A^2 s \leqslant As$ since $As \leqslant s$. Hence $As \in X_+$ and, by (3.1), $As \in Y$. Therefore $s \leqslant As$, so $As = s$.

In addition, if σ is any other element of X for which $A\sigma = \sigma$, then σ is necessarily continuous and $\sigma \in X$. Therefore $\sigma \leqslant s$ since $s \in Y$.

If w'' denotes the solution to Problem 2 for this s, then w'' is a solution to Problem (*). The conclusions of Theorem 2 and its corollaries are valid for any solution w to Pıoblem (*). Moreover, if w is any solution of Problem (*) corresponding, let us say, to a curve σ, then $\sigma \leqslant s$ so $w \leqslant w''$ in D. Q.E.D.

Similarly, one proves that there exists a solution w' of Problem (*) with the property that $w \geqslant w'$ for any solution w of Problem (*).

We state the combined result in

COROLLARY 3.2. *Let* (2.4), (2.5) *and* (1.12)–(1.14) *hold. Then there exist unique minimal and maximal solutions w' and w'' of Problem* (*), *such that for any solution w of Problem* (*), $w' \leqslant w \leqslant w''$.

EXAMPLE. The conditions (2.4), (2.5) are satisfied if

$$\lambda(x, t) = x - \frac{t}{R + \epsilon} (\epsilon > 0), \qquad g''(x) > 1 + (s_0 - x) \, (0 < x < s_0).$$

Remark. By (2.4), $f(x, t; s(t)) \leqslant -\eta$ for any monotone function $s(t)$, $0 \leqslant s(t) \leqslant R$, where η is a positive constant independent of R. The results of [4] then show that any free boundary of Problem 2 remains bounded by a positive constant R_0 independent of R. It follows that the solutions s of Problem (*) must satisfy $s(t) \leqslant R_0$ and, thus, are independent of R if $R \geqslant R_0$.

4. PROPERTIES OF THE SOLUTION

In this section we return to the function which corresponds to the solution of Problem 1, primarily to ascertain the manner in which it attains boundary values on the free curve $\Gamma : x = s(t)$, $0 < t < T$. Given a solution $w(x, t)$ of Problem (*) we set, as usual, $\Omega = \{(x, t) : 0 < x < s(t), 0 < t < T\} = \{(x, t) \in D : w(x, t) > 0\}$, and define $u(x, t)$ by the formula, (cf. (1.11)),

$$u(x, t) = \lambda(x, t) - w_x(x, t) \qquad \text{if} \quad (x, t) \in D. \tag{4.1}$$

So defined, u is bounded in D and, in view of Corollary 2.2, $u > \lambda$ in Ω.

Recall that since

$$-w_{xx} + w_t = f \qquad \text{in } \Omega \tag{4.2}$$

and $f_x = -\lambda_{xx} + \lambda_t$ is smooth, w_x is a smooth function in Ω. Now differentiating (4.2) with respect to x reveals that

$$-u_{xx} + u_t = -\lambda_{xx} + \lambda_t - (-w_{xxx} + w_{xt}) = 0 \qquad \text{in } \Omega. \tag{4.3}$$

It is easy to verify the boundary conditions

$$\begin{aligned} u(x, 0) &= u_0(x) && \text{if} \quad 0 < x < s_0, \\ u(0, t) &= h(t) && \text{if} \quad 0 < t < T, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} u_0(x) &= \lambda(x, 0) - g'(x) && \text{if} \quad 0 < x < s_0, \\ h(t) &= \lambda(0, t) - \psi(t) && \text{if} \quad 0 < t < T \end{aligned}$$

(cf. (1.5) and (1.6)).

THEOREM 4. *Let w be a solution of Problem (*) and let u be defined by (4.1). Then*

(i) *there is a subset $\Sigma \subset [0, T]$,* meas$([0, T]\backslash\Sigma) = 0$, *such that $u(s(t) - \epsilon, t) \to \lambda(s(t), t)$ as $\epsilon \to 0$ uniformly for t in compact subsets of Σ and pointwise in Σ;*

(ii) $\lim_{\epsilon \to \infty} \int \int_\Omega u_x(s(t) - \epsilon, t) \, \varphi(x, t) \, dx \, dt = 0$ *for every $\varphi \in C_0^\infty(\Omega)$.*

Proof. Since $w_{xx} \in L^p(D)$, $1 < p < \infty$, $u(x, t)$ is continuous in x, $0 \leqslant x \leqslant R$, for almost every t. For such t,

$$\lim_{x \to s(t)} u(x, t) = \lambda(s(t), t).$$

From Theorem 1, $w_{xx} \geqslant 0 \quad$ in Ω: hence,

$$u_x - \lambda_x = -w_{xx} \leqslant 0 \qquad \text{in } \Omega.$$

So $u - \lambda$ is decreasing and

$$\gamma(t) = \lim_{x \to s(t)} (u(x, t) - \lambda(x, t))$$

exists for every $t \in (0, T)$. Let $\Sigma = \{t : \gamma(t) = 0\}$, which, as we have noted above, satisfies meas $([0, T]\backslash\Sigma) = 0$. Hence the family of continuous functions $\gamma_\epsilon(t) = u(s(t) - \epsilon, t) - \lambda(s(t) - \epsilon, t)$, $t \in \Sigma$, is decreasing in ϵ and converges to 0 as $\epsilon \to 0$. By Dini's theorem, $\gamma_\epsilon(t) \to 0$ uniformly for t in compact subsets of Σ. The result (i) follows from the smoothness assumed of λ.

Since (4.2) is valid in Ω, given $\varphi \in C_0^\infty(\Omega)$, we may compute that

$$\begin{aligned} \iint_\Omega f\varphi \, dx \, dt &= \iint_\Omega (-w_{xx} + w_t)\varphi \, dx \, dt \\ &= \iint_\Omega (-w_{xx}\varphi - w\varphi_t) \, dx \, dt \qquad (4.5) \\ &= -\iint_\Omega (\lambda_x - u_x)\varphi \, dx \, dt - \iint_\Omega \left(\int_x^{s(t)} (u - \lambda) \, d\xi \right) \varphi_t \, dx \, dt. \end{aligned}$$

We consider the second integral in (4.5). Observe that $s'(t) \in L^1(0, T)$ because s is monotone. Therefore,

$$\begin{aligned} \iint_\Omega \int_x^{s(t)} \lambda(\xi, t) \, \varphi_t(x, t) \, d\xi \, dx \, dt = &- \iint_\Omega \left(\int_x^{s(t)} \lambda_t(\xi, t) \, d\xi \right) \varphi(x, t) \, dx \, dt \\ &- \iint_\Omega s'(t) \, \lambda(s(t), t) \, \varphi(x, t) \, dx \, dt. \qquad (4.6) \end{aligned}$$

Similarly, for $\epsilon > 0$ given,

$$\begin{aligned} &\iint_\Omega \int_x^{s(t)-\epsilon} u(\xi, t) \, d\xi \, \varphi_t(x, t) \, dx \, dt \\ &\quad = -\iint_\Omega \left(\int_x^{s(t)-\epsilon} u_t(\xi, t) \, d\xi \right) \varphi(x, t) \, dx \, dt \\ &\qquad - \iint_\Omega s'(t) \, u(s(t) - \epsilon, t) \, \varphi(x, t) \, dx \, dt \\ &\quad = -\iint_\Omega (u_x(s(t) - \epsilon, t) - u_x(x, t)) \, \varphi(x, t) \, dx \, dt \\ &\qquad - \iint_\Omega s'(t) \, u(s(t) - \epsilon, t) \, \varphi(x, t) \, dx \, dt. \end{aligned}$$

Since $u(s(t) - \epsilon, t) \to \lambda(s(t), t)$ a.e. as $\epsilon \to 0$,

$$\iint_\Omega \int_x^{s(t)} u(\xi, t)\, d\xi\, \varphi_t(x, t)\, dx\, dt$$

$$= - \lim_{\epsilon \to 0} \iint_\Omega u_x(s(t) - \epsilon, t)\, \varphi(x, t)\, dx\, dt + \iint_\Omega u_x \varphi\, dx\, dt$$

$$- \iint_\Omega s'(t)\, \lambda(s(t), t)\, \varphi(x, t)\, dx\, dt. \tag{4.7}$$

Combining (4.7) with (4.5) and (4.6), we obtain

$$\iint_\Omega f\varphi\, dx\, dt = - \iint_\Omega \left(\lambda_x \varphi + \int_x^{s(t)} \lambda_t(\xi, t)\, d\xi\, \varphi\right) dx\, dt + \iint_\Omega u_x \varphi\, dx\, dt$$

$$- \iint_\Omega u_x \varphi\, dx\, dt + \lim_{\epsilon \to 0} \iint_\Omega u_x(s(t) - \epsilon, t)\, \varphi(x, t)\, dx\, dt.$$

The result (ii) follows since the first term on the right-hand side equals $\iint_\Omega f\varphi\, dx\, dt$. Q.E.D.

5. Conditions for Uniqueness

According to Section 3, there are a minimal solution w' and a maximal solution w'' to Problem (*) with the property that

$$w' \leqslant w \leqslant w'' \qquad \text{in } D$$

for any solution w to Problem (*). We now impose conditions sufficient to imply uniqueness of the solution. Obviously this amounts to showing that $w' = w''$. These hypotheses will not be the weakest available, but in the interest of simplicity we shall not pursue this question in more detail. A local uniqueness theorem may be proven, for example, assuming only that $w'_{xx} \in C(\Omega')$, $\Omega' = \{(x, t) : w'(x, t) > 0\}$. The theorem of Section 4 is too weak, it would seem, to imply uniqueness in general. Indeed, our proof places in evidence the role of the smoothness of the second derivatives of the solution in determining its uniqueness.

Lemma 5.1. *Let w be a solution to Problem (*) and define u by* (4.1). *Suppose that*

$$\lambda_x(0, 0) - g''(0) < \lambda_x(s_0, 0) - g''(s_0) = 0, \tag{5.1}$$

$$\lambda_{xx}(x, 0) - g'''(x) \geqslant 0 \qquad \textit{for} \quad 0 < x < s_0, \tag{5.2}$$

$$\lambda_t(0, t) - \psi'(t) > 0 \qquad \textit{for} \quad 0 < t < T \tag{5.3}$$

and that $w_{xx} \in C(\bar{\Omega})$. *Then there exists a neighborhood* $U \subset \Omega$ *of* $\Gamma : x = s(t)$, $0 < t < T$, *such that*

$$u_x(x, t) < 0 \qquad \textit{for} \quad (x, t) \in U$$

and

$$u_x(x, t) \leqslant -\epsilon \qquad \textit{for} \quad (x, t) \in \partial U \cap \Omega \qquad \textit{for some} \quad \epsilon > 0.$$

The assertion about U means that $\Gamma \subset \partial U$.

Proof. We first consider $u_x(x, t)$ when $(x, t) \in \partial' \Omega$, the parabolic boundary of Ω. Since

$$u_x(x, t) = \lambda_x(x, t) - w_{xx}(x, t), \qquad (x, t) \in \Omega,$$

and $w_{xx} \in C(\bar{\Omega})$, u_x is continuous in $\bar{\Omega}$. It follows from Theorem 4 that

$$u_x(x, t) = 0 \qquad \text{for} \quad (x, t) \in \Gamma. \tag{5.4}$$

Also

$$u_x(x, 0) = \lambda_x(x, 0) - g''(x) \leqslant 0, \qquad 0 < x < s_0\,, \tag{5.5}$$

by hypothesis (5.1). Consider now the lateral wall $x = 0$. Here $u(0, t) = \lambda(0, t) - \psi(t)$, so

$$u_{xx}(0, t) = u_t(0, t) = \lambda_t(0, t) - \psi'(t) > 0, \qquad 0 < t < T, \tag{5.6}$$

according to (5.3).

Conditions (5.1), (5.2) imply that there is an interval $(0, \epsilon_0)$ for which the equation $u_0'(x) = \lambda_x(x, 0) - g''(x, 0) = -\epsilon$ has a unique solution $x_\epsilon \in (0, s_0)$ when $0 < \epsilon < \epsilon_0$. Consider the level set

$$\alpha = \alpha_\epsilon = \{(x, t) \in \Omega : u_x(x, t) = -\epsilon\}.$$

By Sard's theorem, α is the union of disjoint regular open arcs for almost every ϵ in $(0, \epsilon_0)$ and so we consider a fixed $\alpha = \alpha_\epsilon$ with this property. In particular, observe that $\operatorname{grad} u_x(x, t) \neq (0, 0)$ for $(x, t) \in \alpha$.

Consider a fixed subarc α^0 of α with terminus x_ϵ. Such a subarc must exist by continuity of u_x in $\bar{\Omega}$ and the strict monotonicity of u_x on $(0, s_0)$. Following α^0 we cannot return to x_ϵ for then there would exist a region $V \subset \Omega$ with $u_x = -\epsilon$ on ∂V. By the maximum principle, u_x would be constant in Ω, which is not the case. Nor does α^0 tend to Γ for $u_x(x, t) = 0$ on Γ. If α^0 terminates at a point (y, t), $t < T$, in Ω then the system $dx/d\sigma = -u_{xt}$, $dt/d\sigma = u_{xx}$ has a singular point at (y, t) so $\operatorname{grad} u_x(y, t) = (0, 0)$. But this cannot happen for $(y, t) \in \alpha^0$.

One remaining possibility is that α^0 terminates on $\{(0, t); 0 < t \leqslant T\}$. We exclude this provided ϵ is sufficiently small. Assuming the contrary, let V be the domain bounded by

$$\{(x, 0): 0 \leqslant x \leqslant x_\epsilon\} \cup \alpha^0 \cup \{(0, t): 0 \leqslant t \leqslant \bar{t}\},$$

where

$$\bar{t} = \lim\sup\{t: t = \lim t_i \text{ when } (x_i, t_i) \in \alpha^0, \quad x_i \to 0\}.$$

We search for the maximum of u_x in $\bar{V}$ which must occur in ∂V. Since

$$u_{xx}(x, 0) = \lambda_{xx}(x, 0) - g'''(x) > 0$$

by (5.2), $\max_{0 \leqslant x \leqslant x_\epsilon} u_x(x, 0) = u_x(x_\epsilon, 0) = -\epsilon$. On the other hand $\sup_V u_x$ cannot be attained on $x = 0$, for $u_{xx}(0, t) > 0$ by (5.6). Hence u_x attains its maximum in V on α^0.

Now, α^0 is given by $t = t(\sigma)$, $x = x(\sigma)$, $0 \leqslant \sigma < \bar{\sigma}$. Suppose $t(\sigma)$ increases for $0 < \sigma < \sigma_1$ and decreases for $\sigma > \sigma_1$ [It cannot increase again for $\sigma > \sigma_2$ (with $\sigma_2 > \sigma_1$) for otherwise the maximum principle gives $u_x \equiv -\epsilon$ in some region bounded by α^0 and some line $t = t(\hat{\sigma})$ for some $\hat{\sigma}$ near σ_2.] Since u_x takes its maximum in V on α^0, we must have $u_{xx} \leqslant 0$ on α^0 if $\sigma > \sigma_1$. Hence $u_{xx} \leqslant 0$ at some point $(0, \bar{t})$; contradiction. We have thus proved that $t(\sigma)$ is monotone increasing. It is actually strictly increasing, for otherwise $u_x(x, t) = -\epsilon$ on a segment $x_1 < x < x_2$, $t = t_1$ lying on α^0. Hence $u_{xx}(x, t_1) = 0$ if $x_1 < x < x_2$. By analytic continuation we get $u_{xx}(x, t_1) = 0$ if $0 \leqslant x < x_1$, but this is impossible since $u_{xx}(0, t_1) > 0$ by (5.6).

We conclude that α^0 can be represented in the form $x = x_\epsilon(t)$, $0 \leqslant t \leqslant \bar{t} = \bar{t}_\epsilon$. We have assumed that α^0 intersects $x = 0$ for some sequence $\epsilon = \epsilon' \downarrow 0$, i.e., $x_{\epsilon'}(\bar{t}_{\epsilon'}) = 0$. Define $x_{\epsilon'}(t) = 0$ if $t > \bar{t}_{\epsilon'}$. Then $x_{\epsilon_1}(t) \leqslant x_{\epsilon_2}(t)$ if $\epsilon_1 > \epsilon_2$. Let

$$x_0(t) = \lim_{\epsilon' \to 0} x_{\epsilon'}(t) \qquad (0 \leqslant t \leqslant T). \tag{5.7}$$

If $x_0(t)$ is not continuous at $t = t_0$, then we easily deduce that there is an interval $\bar{x} < x < \bar{\bar{x}}$, $t = t_0$ along which $u_x = 0$. This is impossible. Thus, $x_0(t)$ is continuous, and by Dini's theorem, the convergence in (5.7) is uniform in $0 \leqslant t \leqslant T$. It follows, in particular, that $x = x_0(t)$ intersects $x = 0$, at some time $\leqslant T$.

Denote by W the region bounded by $x = x_0(t)$ and $x = s(t)$ for $0 \leqslant t \leqslant \hat{t}$ where $\hat{t}$ is the first time $x_0(t) = 0$. Since $w_x = 0$ on the parabolic boundary of W, we conclude that $w_x = 0$ in W, which is impossible. This proves that for any small $\epsilon > 0$, the arc α^0 terminates at a point (y, T), $0 < y < s(T)$. The set U bounded by $\overline{\alpha^0} \cup [x_\epsilon, s_0] \cup \Gamma \cup [y, s(T)]$ has the property that

$$u_x(x, t) \leqslant 0 \qquad \text{on } \partial' U$$

by (5.4), (5.5), and the choice of α^0. Hence $u_x(x, t) < 0$ in U and $u_x(x, t) = -\epsilon$ for $(x, t) \in \alpha^0 = \partial U \cap \Omega$. Q.E.D.

THEOREM 5. *Let w' be the minimal solution and w'' be the maximal solution of Problem (*) and set*

$$\Omega' = \{(x, t); w'(x, t) > 0\} \qquad \textit{and} \qquad \Omega'' = \{(x, t); w''(x, t) > 0\}.$$

Suppose that $w'_{xx} \in C(\overline{\Omega'})$, $w''_{xx} \in C(\overline{\Omega''})$, and

$$\lambda_x(0, 0) - g''(0) < \lambda_x(s_0, 0) - g''(s_0) = 0,$$
$$\lambda_{xx}(x, 0) - g'''(x) \geqslant 0 \qquad \textit{for} \quad 0 < x < s_0,$$
$$\lambda_t(0, t) - \psi'(t) \geqslant 0 \qquad \textit{for} \quad 0 < t < T.$$

Then $w' = w''$ and hence the solution to Problem (*) is unique.

Proof. Define u' and u'' corresponding to w' and w'' according to (4.1) and let Γ': $x = s'(t)$, $0 < t < T$, and Γ'': $x = s''(t)$, $0 < t < T$, denote the associated free boundary curves to w' and w'' respectively. To prove the theorem, it suffices to show that $u' = u''$ in D. Clearly $\Omega' \subset \Omega''$.

We observe that $u'' - u' = u'' - \lambda \geqslant 0$ on Γ' and $u'' - u' = 0$ on $\partial'\Omega' - \Gamma'$. Hence

$$u'' - u' \geqslant 0 \qquad \text{in } \Omega'.$$

Let U be the neighborhood of Γ'' in Ω'' such that $u_x(x, t) < 0$ in U and $u_x(x, t) \leqslant -\epsilon$ on $\partial U \cap \Omega''$ which exists by Lemma 5.1. Let $t_1 = \sup\{t$: the arc of Γ' from 0 to t is in $U \cup \Gamma''\}$; $t_1 > 0$ since U is open and ∂U contains a segment $\{(x, 0): x_\epsilon \leqslant x \leqslant s_0\}$.

In the closure of the set $\Omega' \cap \{(x, t); 0 < t < t_1\}$ the maximum of $u'' - u'$ occurs at a point $(x_0, t_0) \in \Gamma'$, $t_0 \leqslant t_1$, because $u'' - u' = 0$ on $\partial'\Omega' - \Gamma'$. Hence

$$u''_x(x_0, t_0) - u_x'(x_0, t_0) \geqslant 0.$$

If $u' \not\equiv u''$ in $\Omega' \cap \{(x, t); 0 < t < t_1\}$ then $(x_0, t_0) \notin \Gamma''$. We conclude that $u_x'(x_0, t_0) \leqslant u''_x(x_0, t_0) < 0$, which is impossible since $u_x' = 0$ on Γ'. Hence $u' \equiv u''$ in $\Omega' \cap \{(x, t); 0 < t < t_1$. By definition of t_1 it then follows that $t_1 = T$, and the proof is complete.

Remark. The conditions (5.1)–(5.3) ensure that the conditions in (4) are satisfied. Consequently, if $\lambda_x > 0$, $\lambda_t \leqslant 0$, then there exists a unique classical solution of (3). Thus, if $s \in C^1$ and w_{xx} is, say, in $C^2(\overline{\Omega})$, then the uniqueness of (w, s) is well known. The novelty of Theorem 5 is in that it requires only that s be continuous and $w_{xx} \in C(\overline{\Omega})$.

6. Other Problems

The previous results extend to other quasi-variational inequalities. For example, they extend to the case where in Problem (*) the function f (which occurs in (1.11)) is replaced by

$$f(x, t; \sigma) = -g(\sigma(t))$$

provided $g > 0$, $g_x \leqslant 0$. Defining

$$u(x, t) = w(x, t) + 1 + \int_0^t g(s(\tau))\, d\tau, \tag{6.1}$$

we find that u, s form a solution of the problem:

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{if } 0 < x < s(t), \quad 0 < t < T, \\ u_x(s(t), t) &= 0 && \text{if } 0 < t < T, \\ u(s(t), t) &= 1 + \int_0^t g(s(\tau))\, d\tau && \text{if } 0 < t < T, \\ u(x, 0) &= g(x) && \text{if } 0 < x < s_0, \\ u_x(0, t) &= \psi(t) && \text{if } 0 < t < T. \end{aligned} \tag{6.2}$$

We can solve quasi-variational inequalities also when the boundary conditions at $x = 0$ depend on $s(t)$. For example, consider a modification of the preceding problem obtained by replacing the condition $w_x(0, t) = \psi(t)$ by

$$w(0, t) = \psi(t) - 1 - \int_0^t g(s(\tau))\, d\tau.$$

Assuming $\psi' > g$ we can establish the existence of solutions w by the methods of this paper. Defining u by (6.1), we find that u satisfies all the equations in (6.2) except for the last one, which is replaced by

$$u(0, t) = \psi(t) \qquad \text{if } 0 < t < T.$$

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