Topology Vol. 27, No. 2, pp. 245–248, 1988
Printed in Great Britain.
0040-9383/88 \$03.00+.00

ON A PROBLEM OF J. MILNOR

QI-MING WANG

(*Received* 27 *October* 1986)

LET $f_1, \ldots f_r$ $(r \geqslant 2)$ be real polynomials in n variables defining a real polynomial map f: $\mathbb{R}^n \to \mathbb{R}^r$. In his book [3], Milnor studied real polynomial maps f: $\mathbb{R}^n \to \mathbb{R}^r$ under the following

HYPOTHESIS (H). There should exist a neighborhood U of the origin in $\mathbb{R}^n$ so that the matrix $(\partial f_i/\partial x_j)$ has rank r for all x in U other than the origin.

The hypothesis (H), as Milnor puts it, "is so strong that examples are very difficult to find". He raised the following

PROBLEM. ([3], p. 100) For which dimensions $n \geqslant r \geqslant 2$, do non-trivial examples exist?

As was mentioned in [3], non-trivial (to be defined below) examples were known to exist only for $r = 2, 3, 5, 9$. For $r = 2$, they are essentially the complex polynomials. For $r = 3, 5, 9$, they were constructed by N. H. Kuiper using the multiplications of complex, quaternionic and octavionic numbers.

In this paper, we are going to exhibit new non-trivial examples for every $r \geqslant 3$ which generalise the Kuiper examples.

Moreover, for $r \equiv 1 \pmod 4$, our construction gives rise to topologically distinct germs of maps within the same dimensions n and r. We verified Milnor's conjecture ([3], p. 100) on the homeomorphic types of fibers for our new maps. As a by-product, we find new examples of families of smooth manifolds F_i with boundary (in fact, disk bundles over spheres) which are mutually non-homeomorphic but the products $F_i \times I$ are all homeomorphic. These F_i's are moreover semi-algebraic in that they are complete intersections of $r+1$ real quadrics restricted to the unit ball in $\mathbb{R}^n$. The number of these F_i's can be arbitrarily large if r is chosen properly.

1. THE CLIFFORD MAPS

We first recall some terminology from [3] and [6]. Let f: $\mathbb{R}^n \to \mathbb{R}^r$ $(r \geqslant 2)$ be a polynomial map satisfying (H). The equations $f_1(x) = \ldots = f_r(x) = 0$ define an algebraic set V which is a smooth manifold of dimension $n-r$ in $U - \{0\}$. The intersection K of V with a small sphere S_ε^{n-1} centered at $0 \in \mathbb{R}^n$ is a smooth manifold of dimension $n-r-1$. Milnor showed that the complement of an open tubular neighborhood of K in S_ε^{n-1} is the total space of a smooth fiber bundle over the sphere S^{r-1}, the fiber F being a smooth compact $(n-r)$-dimensional manifold bounded by a copy of K.

DEFINITION. F is called the *Milnor fiber* of the map f. f is said to be *non-trivial* if F is not diffeomorphic to the disk D^{n-r}.

Let $E_1, \ldots, E_{m-1}$ be an orthogonal representation of C_{m-1} on $\mathbf{R}^l$, i.e. $E_i \in \mathrm{O}(l)$ and

$$E_iE_j + E_jE_i = -2\delta_{ij}I. \tag{1.1}$$

For $x \in S^{l-1}$, $(x, E_1x, \ldots, E_{m-1}x)$ is clearly an orthonormal m-frame in $\mathbb{R}^l$. The map $\sigma(x) = (x, E_1x, \ldots, E_{m-1}x)$ of S^{l-1} into $V_{l,m}$ is a cross-section of $V_{l,m}$, called the *Clifford cross-section* induced by $E_1, \ldots, E_{m-1}$. Let ζ be the rank $l-m$ vector bundle over S^{l-1}, the fiber over x being the orthogonal complement of $\{x, E_1x, \ldots, E_{m-1}x\}$ in $\mathbb{R}^l$.

DEFINITION. ζ is called the *Clifford bundle* induced by (1.1).

Note that l can not be arbitrary. In fact, $l = k\delta(m)$, where k is an integer and $\delta(m)$ is the dimension of irreducible C_{m-1} modules. As is well-known (cf. e.g. [2]) there is a 1–1 correspondence between systems $\{E_1, \ldots, E_{m-1}\}$ satisfying (1.1) and systems $\{P_0, \ldots, P_m\}$ satisfying

$$P_iP_j + P_jP_i = 2\delta_{ij}I, \qquad P_i \in \mathrm{O}(2l). \tag{1.2}$$

For every such system $\{P_0, \ldots, P_m\}$ we define a *Clifford map* $f\colon \mathbb{R}^{2l} \to \mathbb{R}^{m+1}$ by

$$f(x) = (\langle P_0x, x\rangle, \ldots, \langle P_mx, x\rangle). \tag{1.3}$$

THEOREM 1. *For every $m \geqslant 2$, the Clifford map $f\colon \mathbb{R}^{2l} \to \mathbb{R}^{m+1}$ satisfies (H) and is non-trivial. The Milnor fiber of f is $D(\zeta)$, the disk bundle of the Clifford bundle ζ induced by $E_1 = P_1P_2, \ldots, E_{m-1} = P_1P_m$ restricted to $E_+(P_0) \cong \mathbb{R}^l$, the $+1$ eigenspace of P_0.*

Proof. (1.2) implies that for any $x \in S^{2l-1}$, $P_0x, \ldots, P_mx$ are orthonormal in $\mathbb{R}^{2l}$, hence (H) is trivially satisfied. It is also clear that $E_1 = P_1P_2, \ldots, E_{m-1} = P_1P_m$ restricted to $E_+(P_0)$ is a system satisfying (1.1).

By definition, the Milnor fiber of f is $F = D_\varepsilon^{2l} \cap f^{-1}(y)$, where D_ε^{2l} is a small round ball centered at $0 \in \mathbb{R}^{2l}$ and y is a regular value of $f|S_\varepsilon^{2l-1}$ such that $\|y\| > 0$ is sufficiently small. Since f is homogeneous, we may take $\varepsilon = 1$ and $y = (a, 0, 0, \ldots, 0)$. $\mathbb{R}^{2l}$ is the orthogonal direct sum of $E_+(P_0)$ and $E_-(P_0)$, hence we can write $x = (x_+, x_-)$ with $x_+ \in E_+(P_0)$, $x_- \in E_-(P_0)$. Therefore $F = D^{2l} \cap f^{-1}(y_0) = \{(x_+, x_-)|\ \|x_+\|^2 + \|x_-\|^2 = t,\ \|x_+\|^2 - \|x_-\|^2 = a,\ \langle P_1x_-, x_+\rangle = 0, \ldots, \langle P_mx_-, x_+\rangle = 0, a \leqslant t \leqslant 1\} = \{(x_+, x_-)|\ \|x_+\|^2 = \tfrac{1}{2}(a+t),\ \|x_-\|^2 = \tfrac{1}{2}(t-a),\ \langle P_1x_-, P_1P_2x_+\rangle = 0, \ldots, \langle P_1x_-, P_1P_mx_+\rangle = 0, a \leqslant t \leqslant 1\}$. Hence the map $\pi\colon F \to S^{l-1}$ given by $\pi(x) = x_+/\|x_+\|$ is a bundle map, the fiber over z is the P_1-image of the $\sqrt{(t-a)/2}$ disk in the fiber of ζ over z, where ζ is the Clifford bundle induced by $E_1 = P_1P_2, \ldots, E_{m-1} = P_1P_m$ on $\mathbb{R}^l \cong E_+(P_0)$. F is clearly not contractible, f is therefore non-trivial.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^r$ be a C^∞ map. The germ of f at $0 \in R$ is denoted by $(f, 0)$.

DEFINITION. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}^r$ be C^∞ maps. $(f, 0)$ and $(g, 0)$ are *topologically equivalent* if there exist open neighborhoods U and U' of 0 in $\mathbb{R}^n$ and V and V' of 0 in $\mathbb{R}^r$ and homeomorphisms $h_1\colon U \to U'$ and $h_2\colon V \to V'$ such that $h_2 \circ f = g \circ h_1$.

We would like to classify the germs of Clifford maps with respect to this equivalent relation. When $m \not\equiv 0 \pmod 4$, all Clifford maps $f\colon \mathbb{R}^{2l} \to \mathbb{R}^{m+1}$ are globally equivalent, as any two systems $\{P_0, \ldots, P_m\}$ and $\{P'_0, \ldots, P'_m\}$ satisfying (1.2) are orthogonally equivalent, i.e. there exists $A \in \mathrm{O}(2l)$ such that $P'_i = AP_iA^{-1}$.

When $m \equiv 0 \pmod 4$, however, there are orthogonally inequivalent systems. These are distinguished by an integer q, defined by

$$tr(P_0 \ldots P_m) = 2\delta(m)q. \tag{1.4}$$

The Clifford map defined by a system satisfying (1.4) will be denoted by f_q. The Milnor fiber of f_q will be denoted by F_q. The change from q to $-q$ amounts to changing the sign of P_0, say. Clearly $(f_q, 0)$ is equivalent to $(f_{-q}, 0)$ and $F_q \cong F_{-q}$.

THEOREM 2. *Suppose that $m \equiv 0$ (mod 4), let d_m be the denominator of $B_{m/4}/m$, $B_{m/4}$ being the $m/4$th Bernoulli number.* (a) *If $q \equiv \pm q'$ (mod $2d_m$), F_q and $F_{q'}$ are diffeomorphic.* (b) *If $(f_q, 0)$ and $(f_{q'}, 0)$ are equivalent, then $q \equiv \pm q'$ (mod d_m).*

Proof. (a) We may suppose that q and q' are non-negative. It is shown in [6] that when $q \equiv \pm q'$ (mod $2d_m$), the Clifford bundles ζ_q and $\zeta_{q'}$ are isomorphic as vector bundles, hence

$$F_q \cong D(\zeta_q) \cong D(\zeta_{q'}) \cong F_{q'}.$$

(b) Suppose $(f_q, 0)$ and $(f_{q'}, 0)$ are equivalent. There is then a homeomorphism of a neighborhood of 0 in $V_q = f_q^{-1}(0)$ onto a neighborhood of O in $V_{q'} = f_{q'}^{-1}(0)$. V_q and $V_{q'}$ have cone neighborhoods with vertices O and base K and K' respectively. It follows (e.g. [5], p. 125) that K and K' are homotopically equivalent. On the other hand, K and K' are diffeomorphic to the manifolds denoted in [6] by $M_+(m, l, q)$ and $M_+(m, l, q')$ respectively. It follows from Theorem 2b in [6] that this implies in turn $q \equiv \pm q'$ (mod d_m).

2. EXAMPLES OF HOMEOMORPHIC PRODUCTS WITH [0, 1]

It is well-known that there exist smooth manifolds X and Y such that X and Y are not homeomorphic but $X \times I$ and $Y \times I$ are. The first examples were shown by J. H. C. Whitehead [7]: X is the torus with a hole and Y is the disk with two holes. Examples for every dimension $n \geqslant 5$ were constructed ([1]) which is contractible and occurs as a factor of I^{n+1}. In the following, we will show that the Milnor fiber F_q of the Clifford maps provide us with infinitely many new series of such examples. The main feature of these new examples lies in that they are semi-algebraic sets and that the number of examples within the same dimension can be arbitrarily large.

LEMMA 1. *If $m \not\equiv 0$ (mod 4), any two Clifford cross-sections of $V_{l,m}$ are homotopic.*

Proof. Let $\{E_1, \ldots, E_{m-1}\}$ and $\{E'_1, \ldots, E'_{m-1}\}$ be two systems satisfying (1.1). Since $m \not\equiv 0$ (mod 4), there exists ([2]) $A \in O(l)$ such that $E_i = AE_iA^{-1}$. The endomorphisms E_1 and E'_1 define complex structures on $\mathbb{R}^l$ and A is a complex linear map with respect to E_1 and E'_1 hence has determinant $+1$. In the following commutative diagram

$$\begin{array}{ccc} S^{l-1} & \xrightarrow{\sigma} & V_{l,m} \\ \Big\downarrow A & & \Big\downarrow \tilde{A} \\ S^{l-1} & \xrightarrow{\sigma'} & V_{l,m} \end{array}$$

$A \simeq Id$ and $\tilde{A} \simeq \mathrm{Id}$, where $\tilde{A}(e_1, \ldots, e_m) = (Ae_1, \ldots, Ae_m)$. Hence $\sigma \simeq \sigma'$.

THEOREM 3. *Let $m \equiv 0$ (mod 4).*
(a) *For any q and q', $F_q \times I$ is homeomorphic to $F_{q'} \times I$.*
(b) *F_q and $F_{q'}$ are not homeomorphic if $q \not\equiv \pm q'$ (mod d_m)*

Proof. Denote by ζ (resp. ξ) the Clifford bundle induced by $\{E_1, \ldots, E_{m-1}\}$ (resp. $\{E_1, \ldots, E_{m-2}\}$). We have

$$\xi = \zeta \oplus \theta^1. \tag{2.1}$$

On the other hand, for any vector bundle ζ, $D(\zeta \oplus \theta^1) \cong D(\zeta) \times I$, (cf. e.g. [4]). Hence $F_q \times I \cong F_{q'} \times I$ in view of Lemma 1. (b) If F_q is homeomorphic to $F_{q'}$, it follows from the invariance of boundary that $K = \partial F_q$ and $K' = \partial F_{q'}$ are homeomorphic. Hence $q \equiv \pm q'$ as in the proof of Theorem 2b.

REMARK 1. For big l, the number of mutually non-homeomorphic F_q's is at least $\frac{1}{4} d_m$. The number d_m can be arbitrarily large. For example, $d_{24} = 65520$. On the other hand, the dimension of F_q grows very fast too.

REMARK 2. Let f: $\mathbb{R}^n \to \mathbb{R}^r$ be a real polynomial map satisfying (H), so is $p \circ f$: $\mathbb{R}^n \to \mathbb{R}^{r-1}$, where p is the linear projection map of $\mathbb{R}^r$ onto $\mathbb{R}^{r-1}$. Let F and F' be the Milnor fibers of f and $p \circ f$ respectively. Milnor conjectured ([3], p. 100) that F' and $F \times I$ are homeomorphic. Theorem 3 shows that the conjecture is true for the Clifford maps.

Acknowledgements—This work was done during the author's visit to the Max-Planck-Institut für Mathematik at Bonn. The author is grateful to Professor F. Hirzebruch for his hospitality.

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Institute of Mathematics
Academia Sinica, Beijing
The Peoples Republic of China