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Viscosity independent numerical errors for Lattice Boltzmann models: From recurrence equations to "magic" collision numbers

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ABSTRACT

We prove for generic steady solutions of the Lattice Boltzmann (LB) models that the variation of the numerical errors is set by specific combinations (called "magic numbers") of the relaxation rates associated with the symmetric and anti-symmetric collision moments. Given the governing dimensionless physical parameters, such as the Reynolds or Peclet numbers, and the geometry of the computational mesh, the numerical errors remain the same for any change of the transport coefficients only when the "free" ("kinetic") anti-symmetric rates and the boundary rules are chosen properly. The single-relaxation-time (BGK) model has no free collision rate and yields viscosity dependent errors with any boundary scheme for hydrodynamic problems. The simplest and most efficient collision operator for invariant errors is the two-relaxation-times (TRT) model. As an example, this model is able to compute viscosity independent permeabilities for any porous structure.

These properties are derived from steady recurrence equations, obtained through linear combinations of the LB evolution equations, in which the equilibrium and non-equilibrium components are directly interconnected via finite-difference link-wise central operators. The explicit dependency of the non-equilibrium solution on the relaxation rates is then obtained. This allows us, first, to confirm the governing role of the "magic" combinations for steady solutions of the Stokes equation, second, to extend this property to steady solutions of the Navier–Stokes and anisotropic advection–diffusion equations, third, to develop a parametrization analysis of the microscopic and macroscopic closure relations prescribed via link-wise boundary schemes.

1. Introduction

Mostly owing to the simplicity of modeling complex solid boundaries via local and mass-conserving Maxwell reflections, the Lattice Boltzmann schemes have rapidly gained some popularity to study flows in *staircase* reconstructions of porous samples (a far to be exhaustive list includes [1–20]). The geometric structure and the effective hydrodynamic properties of porous media are characterized by a series of effective parameters. Among them the permeability tensor $\mathbf{K}$, a measure of the fluid conduction, relates the averaged mass flux $\bar{\vec{j}}$ to the driving pressure drop: ∇P, and forcing: $\vec{F}$, via Darcy's law:

$$\bar{\vec{j}} = \mathbf{K}\frac{\overline{(\vec{F} - \nabla P)}}{\nu}, \quad (1)$$

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assuming a slow steady and incompressible creeping flow governed by the Stokes equation relating the velocity $\vec{u}$ to the pressure P:

$$\nu \Delta \vec{u} = \frac{1}{\rho_0}(\nabla P - \vec{F}), \qquad \vec{j} = \rho_0 \vec{u}. \tag{2}$$

Although the LB schemes are not especially efficient for steady problems (see Ref. [19] for an alternative), their convergence rate toward the steady state can be improved by using rather large values for the kinematic viscosity ν. However, prescribing the same forcing, e.g., across a periodic sample of the porous media, but using different values of ν, the solutions obtained for $\nu\vec{u}$ can differ, contrary to what is expected from the linearity of the Stokes equation. This leads to obvious non-physical numerical artifacts, as the dependency of the computed permeability on the viscosity. Based on the exact LB solution [21] for a Poiseuille flow in a straight channel, the simplest example is the deviation from the expected permeability value: $k = H^2/12$, equal to $(48\nu^2 - 1)/12$ [22] for the single-relaxation-time (BGK) model [23] with the bounce-back boundary condition. One should find therefore a trade-off between efficiency (using the BGK with large viscosities) and accuracy (using sufficiently fine resolutions and/or small viscosities), or try to rescue the permeability measurements with the help of more accurate boundary rules, such as those recently elaborated for a better description of solid shapes [24,25] through the use of more complicated but still directional rules [22,26]. However, as we show in this paper, the permeability dependency on the viscosity is unavoidable when using the BGK model, whatever boundary scheme is involved.

At the same time, there were strong numerical evidences that this numerical artifact disappears completely and the permeability values do not change when the viscosity varies, provided that some specific combinations of the relaxation rates are kept constant. This was first observed using the *FCHC* and $d3Q19$ velocity sets with the bounce-back boundary rule for flows around square arrays of cylinders and inside different shaped tubes. These combinations, named "magic" at that time [7,8], relate the collision rates of the "symmetric" and "anti-symmetric" moments of the multiple-relaxation-times (MRT) models [27–29]. The two-relaxation-times (TRT) model [30,31] has the minimal sufficient number of collision rates: one for the symmetric and one for the anti-symmetric modes, while retaining the same computational simplicity and efficiency as the BGK model. A viscosity independent permeability is then obtained, e.g., for cubic arrays of spheres and reconstructed fiber materials, using the $d3Q15$ velocity set in Ref. [22], and for body-centered cubic arrays of spheres and random-size sphere-packed porous media, using $d3Q19$ velocity in Ref. [25]. For any value of the magic combination, these results were obtained not only with the bounce-back rule but also with the parabolic (third-order accurate) MR1 scheme, Ref. [22]. At the same time, while the linear interpolations [26,32] reduce the dependency on the viscosity, they do not remove it completely. Until now no rigorous explanation has been found except for simple solutions [8,21,22], such as Poiseuille flow or for truncated (at third, fourth, ..., orders) steady expansions of the populations.

In this paper we prove these results using the following methodology. First, we write the collision operator in a link-wise form [30,31,33], the TRT-operator being the common sub-class of the link-wise and MRT ones. Based on a parity argument, the link-wise form enables very simple manipulations of the symmetric/anti-symmetric components of the evolution equation and its solutions. Second, we derive some recurrence equations for the link-wise operators. Independently of the conservation constraints, they express each individual non-equilibrium component as a solution of a link-wise finite-difference type equation, governed (along a given link) by the equilibrium components. Then we substitute the solution of the recurrence equation into the *exact* microscopic steady mass and momentum conservation equations. Prescribing the Stokes equilibrium, we show that the macroscopic solution for $\nu\vec{u}$ varies linearly with the driving forcing only when the "magic" combination Λ_{eo} of the two relaxation rates is kept constant. This proves that the collision number Λ_{eo} controls the Stokes solution in the bulk and raises the more general question of finding relevant parameters for the Lattice Boltzmann macroscopic solutions (called in the sequel its parametrization).

It is well understood that the solutions of the second-order macroscopic mass and momentum conservation laws are governed by several non-dimensional numbers, such as the Peclet number for linear advection–diffusion equations or the Reynolds and Froude numbers for incompressible hydrodynamic equations. However, due to the neglected higher-order corrections, the numerical schemes for solving these equations are not guaranteed to follow exactly the "physical" scalings. These corrections depend usually on the free parameters of the chosen numerical scheme, such as the weights of the finite-difference or the equilibrium stencils, resulting in different solutions for different choices of these parameters, e.g., two incompressible Navier–Stokes steady solutions on the same grid, computed with different characteristic velocities and viscosities, may differ for the same grid Reynolds number.

In LB schemes the second- and higher-order terms in the expansion of the populations, then the third- and higher-order corrections to the derived macroscopic equations, depend on the values of the free ("kinetic") collision rates. Based on the solutions of the recurrence equations, we show that on a given grid the dimensionless steady pressure and velocity distributions, obtained for the TRT model with the standard Navier–Stokes polynomial equilibrium, are controlled by the hydrodynamic numbers and collision number Λ_{eo}. We extend this analysis to the anisotropic, advection–diffusion equations (AADE), in the frame of more general link-wise operators [30,34], and show that the dimensionless steady solution of the AADE on a given mesh is controlled by the grid Peclet number and all the possible link combinations of "symmetric" and "anti-symmetric" rates, a rescaling not available for the BGK model for which the error is related to the powers of the transport coefficient.

A specific property of the LB schemes originates from the kinetic nature of their boundary conditions. Namely, in contrast with the bulk populations, the incoming populations do not obey the evolution equation. Instead, they are computed through microscopic closure relations prescribed by the boundary rules. For suitable boundary schemes, these microscopic closure relations approximate the desired macroscopic boundary conditions, but only up to some order. It follows that the macroscopic closure relations are not guaranteed *a priori* to share the parametrization of the *exact* macroscopic bulk equations, even at *second order*. This question is examined in parallel works [35,36]. Additional constraints on the coefficients of generic multi-reflection link-wise boundary schemes have been derived there which enable the closure relations to share the parametrization of the bulk solutions. The bounce-back and multi-reflection MR1 schemes obey naturally these constraints and yields viscosity independent permeabilities, but not the linear interpolation schemes in [26,32]. *Infinite* classes of exactly parametrized linear and parabolic rules have been designed for Dirichlet velocity, pressure, or mixed boundary conditions in [35,36]. In this note we explain this analysis in more detail, but only for the simplest Dirichlet velocity or pressure conditions.

Along these lines, the paper is organized as follows. Section 2 derives the recurrence equations for the link-wise operators and presents their steady solutions. Section 3 examines the parametrization of the exact, conservation and closure, steady relations, with a focus on the Stokes, Navier–Stokes, and anisotropic advection–diffusion equilibrium distributions. We show that the bounce-back and anti-bounce-back boundary rules share the bulk parametrization and we improve the linear schemes of Ref. [26] for this property. Section 4 summarizes the paper. Appendix A gives a second pair of recurrence relations and discusses its relation with the pair derived in Section 2. In Appendix B we build the solution of the recurrence equations as infinite series where all the coefficients are *explicit functions* of the collision rates. An extension of all the results for MRT-L-models is given in Appendix C.

2. Recurrence equations and solutions

2.1. Link-based evolution equation

The unknown variable of the scheme at the node $\vec{r}$ and time t is the population vector $\boldsymbol{f}(\vec{r}, t) = \{f_q, q = 0, \ldots, Q-1\}$. We assume the equidistant d-dimensional computational mesh $\{\vec{r}\}$ where the velocity vectors $\{\vec{c}_q\}$ interconnect grid nodes. The velocity set contains Q vectors: one zero, $\vec{c}_0 = \vec{0}$, for the rest population, and $Q-1$ nonzero ones, $\vec{c}_q = \{c_{q\alpha}, \alpha = 1, \ldots, d\}$, $q = 1, \ldots, Q-1$, for the moving populations. Cubic velocity sets, Ref. [23], with two moving classes are mostly assumed, e.g., $d2Q9$, $d3Q15$, $d3Q19$. They allow formulating a single scheme for the hydrodynamic (Navier–Stokes) equation and for the anisotropic, linear or nonlinear, advection–diffusion equation (AADE). The models with one class of nonzero velocities are only sufficient for diagonal diffusion tensors.

Each nonzero velocity $\vec{c}_q$ has an opposite one $\vec{c}_{\bar{q}} = -\vec{c}_q$ and below such a pair of anti-parallel velocities $(\vec{c}_q, \vec{c}_{\bar{q}})$ is called *a link*. The link-wise basis of Ref. [30] is such that the projections of any Q-vector $\boldsymbol{\phi}$ on the pair of basis vectors associated with the q^{th}-link are equal to its symmetric (even), ϕ_q^+, and anti-symmetric (odd), ϕ_q^-, components: $\phi_q = \phi_q^+ + \phi_q^-$. The even parts are equal: $\phi_q^+ = \phi_{\bar{q}}^+ = \frac{1}{2}(\phi_q + \phi_{\bar{q}})$, $\phi_0^+ = \phi^0$, and the odd parts have opposite signs: $\phi_q^- = -\phi_{\bar{q}}^- = \frac{1}{2}(\phi_q - \phi_{\bar{q}})$, $\phi_0^- = 0$. Specifying the collision operator (3) by an equilibrium distribution $\{e_q^\pm\}$ and a pair of collision rates $\{\lambda_q^+, \lambda_q^-\}$ for each link, the evolution of the populations obeys the following update rule:

$$
\begin{aligned}
& f_q(\vec{r} + \vec{c}_q, t+1) = \tilde{f}_q(\vec{r}, t) \equiv f_q(\vec{r}, t) + g_q^+ + g_q^-, \quad q = 0, \ldots, Q-1, \\
& g_q^\pm = \lambda_q^\pm n_q^\pm, \qquad n_q^\pm = (f_q^\pm - e_q^\pm), \qquad f_q^\pm = \frac{1}{2}(f_q \pm f_{\bar{q}}), \qquad e_q^\pm = \pm e_{\bar{q}}^\pm, \\
& \lambda_q^\pm = \lambda_{\bar{q}}^\pm, \qquad -2 < \lambda_q^\pm < 0\,.
\end{aligned} \tag{3}
$$

Here and below, any relation with "±" and/or "∓" addresses *two* different relations: one with the "upper" sign and another one with the "lower" sign, for all the variables. The population $f_q = e_q^+ + e_q^- + n_q^+ + n_q^-$ and the outgoing (post-collision) one $\tilde{f}_q$ can also be expressed via the equilibrium and post-collision components:

$$
f_q(\vec{r}, t) = \left[e_q^+ + e_q^- - \left(\frac{1}{2} + \Lambda_q^+\right) g_q^+ - \left(\frac{1}{2} + \Lambda_q^-\right) g_q^-\right](\vec{r}, t), \tag{4}
$$

$$
\tilde{f}_q(\vec{r}, t) = \left[e_q^+ + e_q^- + \left(\frac{1}{2} - \Lambda_q^+\right) g_q^+ + \left(\frac{1}{2} - \Lambda_q^-\right) g_q^-\right](\vec{r}, t), \tag{5}
$$

$$
\Lambda_q^+ = -\left(\frac{1}{2} + \frac{1}{\lambda_q^+}\right) > 0, \qquad \Lambda_q^- = -\left(\frac{1}{2} + \frac{1}{\lambda_q^-}\right) > 0, \quad \forall\, q.
$$

The link-wise operator has at most $(Q-1)/2$ distinct "magic" values Λ_q^{eo} given by the products of the eigenvalue functions Λ_q^+ and Λ_q^-:

$$
\Lambda_q^{eo} = \Lambda_q^+ \Lambda_q^-, \qquad \Lambda_q^{eo} > 0, \quad q = 1, \ldots, Q-1. \tag{6}
$$

Locally prescribed mass M and/or momentum $\vec{F}$ source terms can be split into arbitrary proportions between the equilibrium and the outgoing populations via suitable modifications of the equilibrium mass and momentum variables (see for instance relation (28) and Ref. [35]). Here, for the sake of simplicity, the source terms have been hidden in the equilibrium and do not appear in the evolution equation (3). Hence the links are related by the following *conservation* relations:

$$\sum_{q=0}^{Q-1} g_q^+(\vec{r},t) = M(\vec{r},t), \qquad \sum_{q=1}^{Q-1} g_q^-(\vec{r},t)\vec{c}_q = \vec{F}(\vec{r},t). \tag{7}$$

2.2. Steady recurrence equations

In the sequel we assume, first, that at least one steady state exists as a solution of the LB evolution equation (3) with given boundary conditions and source terms. Although this is probably not necessary, we assume also that this steady state is locally stable, i.e., it is an attractor of the time evolution for any starting point in a compact neighborhood. Second, we assume that the steady state is unique for linear LB models and that there is a finite number of them for nonlinear LB models, an assumption true for most boundary schemes used in practical applications, hence referred to as "generic". Possible extensions (or counter-examples) of the present results to models having "spurious" conserved quantities coming from the bulk evolution or from pathological boundary conditions (see Ref. [35] and the references therein) is left for future work. Finally, the collision rates are assumed link-wise constant.

At steady state, the immobile population takes its equilibrium value then $g_0 = g_0^+ \equiv 0$. The subsequent developments are then relevant only for moving populations, $q = 1, \ldots, Q-1$. Keeping in mind that $g_q^+ = g_{\bar{q}}^+$ and $g_q^- = -g_{\bar{q}}^-$, we first write down two steady-state evolution equations (3) from $\vec{r}$ to $\vec{r} \pm \vec{c}_q$:

$$\left[e_q^+ + e_q^- - \left(\frac{1}{2} + \Lambda_q^+\right) g_q^+ - \left(\frac{1}{2} + \Lambda_q^-\right) g_q^-\right](\vec{r}+\vec{c}_q) = \left[e_q^+ + e_q^- + \left(\frac{1}{2} - \Lambda_q^+\right) g_q^+ + \left(\frac{1}{2} - \Lambda_q^-\right) g_q^-\right](\vec{r}), \tag{8}$$

$$\left[e_q^+ - e_q^- - \left(\frac{1}{2} + \Lambda_q^+\right) g_q^+ + \left(\frac{1}{2} + \Lambda_q^-\right) g_q^-\right](\vec{r}-\vec{c}_q) = \left[e_q^+ - e_q^- + \left(\frac{1}{2} - \Lambda_q^+\right) g_q^+ - \left(\frac{1}{2} - \Lambda_q^-\right) g_q^-\right](\vec{r}). \tag{9}$$

With the help of the following link-wise finite-difference operators,

$$\bar{\Delta}_q\phi(\vec{r}) = \frac{1}{2}(\phi(\vec{r}+\vec{c}_q) - \phi(\vec{r}-\vec{c}_q)),$$
$$\Delta_q^2\phi(\vec{r}) = \phi(\vec{r}+\vec{c}_q) - 2\phi(\vec{r}) + \phi(\vec{r}-\vec{c}_q), \tag{10}$$

the sum and the difference of Eqs. (8) and (9) become:

$$\left[\Delta_q^2 e_q^\pm + 2\bar{\Delta}_q e_q^\mp - \left(\frac{1}{2} + \Lambda_q^\pm\right)\Delta_q^2 g_q^\pm - 2\left(\frac{1}{2} + \Lambda_q^\mp\right)\bar{\Delta}_q g_q^\mp\right](\vec{r}) = 2g_q^\pm(\vec{r}). \tag{11}$$

A second pair of the evolution equations describes the propagation from $\vec{r} \pm \vec{c}_q$ to $\vec{r}$:

$$\left[e_q^+ - e_q^- + \left(\frac{1}{2} - \Lambda_q^+\right) g_q^+ - \left(\frac{1}{2} - \Lambda_q^-\right) g_q^-\right](\vec{r}+\vec{c}_q) = \left[e_q^+ - e_q^- - \left(\frac{1}{2} + \Lambda_q^+\right) g_q^+ + \left(\frac{1}{2} + \Lambda_q^-\right) g_q^-\right](\vec{r}), \tag{12}$$

$$\left[e_q^+ + e_q^- + \left(\frac{1}{2} - \Lambda_q^+\right) g_q^+ + \left(\frac{1}{2} - \Lambda_q^-\right) g_q^-\right](\vec{r}-\vec{c}_q) = \left[e_q^+ + e_q^- - \left(\frac{1}{2} + \Lambda_q^+\right) g_q^+ - \left(\frac{1}{2} + \Lambda_q^-\right) g_q^-\right](\vec{r}). \tag{13}$$

Their sum and difference give two recurrence equations, one for $g_q^+(\vec{r})$ and one for $g_q^-(\vec{r})$:

$$\left[\Delta_q^2 e_q^\pm - 2\bar{\Delta}_q e_q^\mp + \left(\frac{1}{2} - \Lambda_q^\pm\right)\Delta_q^2 g_q^\pm - 2\left(\frac{1}{2} - \Lambda_q^\mp\right)\bar{\Delta}_q g_q^\mp\right](\vec{r}) = -2g_q^\pm(\vec{r}). \tag{14}$$

Let the four equations (11) and (14) be split into two pairs: one with the "upper" superscript and one with the "lower" superscript. Eliminating $\bar{\Delta}_q g_q^\mp$ yields two new recurrence relations for $g_q^+(\vec{r})$ and $g_q^-(\vec{r})$:

$$g_q^\pm(\vec{r}) = \left[\bar{\Delta}_q e_q^\mp - \Lambda_q^\mp \Delta_q^2 e_q^\pm + \left(\Lambda_q^{eo} - \frac{1}{4}\right)\Delta_q^2 g_q^\pm\right](\vec{r}), \tag{15}$$

for any node $\vec{r} \in \Omega_q$, called q-bulk node, having its two neighbors along the link q in the computational domain Ω. The nodes in Ω, but not in Ω_q (at least one neighbor along the link q not in Ω), are called q-boundary nodes and denoted $\vec{r}_b$.

Let us now associate to any quantity $\phi(\vec{r})$ defined in Ω the auxiliary quantity $\Gamma_q(\phi)$ defined on Ω_q by the following recurrence equation:

$$2\Gamma_q(\phi) = \Delta_q^2\phi + 2\left(\Lambda_q^{eo} - \frac{1}{4}\right)\Delta_q^2\Gamma_q(\phi), \tag{16}$$

with $[\Gamma_q(\phi)](\vec{r}_b) = 0$ for all the q-boundary nodes, where the value zero has been chosen to close the system in the simplest way, but this does not rule out possible other choices. The quantity $\Gamma_q(\phi)$ is related to $\Delta_q^2\phi/2$ by a diagonal dominant, hence nonsingular, tridiagonal matrix, since $1 + 2((\Lambda_q^{eo} - \frac{1}{4}) - |\Lambda_q^{eo} - \frac{1}{4}|)$ is equal to one for $\Lambda_q^{eo} \geq 1/4$ and to $4\Lambda_q^{eo}$ for $0 < \Lambda_q^{eo} \leq 1/4$. Thus $\Gamma_q(\phi)$ exists, is unique and related to ϕ through the eigenvalue combination Λ_q^{eo} only.

Let us now associate to $\phi(\vec{r})$ and $\Gamma_q(\phi)$ a second auxiliary quantity $\gamma_q(\phi)$ defined on Ω_q by

$$\gamma_q(\phi) = \bar{\Delta}_q\phi + 2\left(\Lambda_q^{eo} - \frac{1}{4}\right)\bar{\Delta}_q\Gamma_q(\phi), \tag{17}$$

and on the boundary nodes $\vec{r}_{b_1}$ and $\vec{r}_{b_2} = \vec{r}_{b_1} + N_q\vec{c}_q$ (ends of a segment of $N_q + 1$ consecutive nodes all in Ω) by

$$\begin{aligned}\left[\gamma_q(\phi)\right](\vec{r}_{b_1}) &= [\gamma_q(\phi)](\vec{r}_{b_1} + \vec{c}_q) - [\Gamma_q(\phi)](\vec{r}_{b_1} + \vec{c}_q), \quad \vec{r}_{b_1} - \vec{c}_q \notin \Omega,\\ \left[\gamma_q(\phi)\right](\vec{r}_{b_2}) &= [\gamma_q(\phi)](\vec{r}_{b_2} - \vec{c}_q) + [\Gamma_q(\phi)](\vec{r}_{b_2} - \vec{c}_q), \quad \vec{r}_{b_2} + \vec{c}_q \notin \Omega.\end{aligned} \tag{18}$$

With these choices, we show in Appendix A that $\gamma_q(\phi)$ is solution of the recurrence equation:

$$\gamma_q(\phi) = \bar{\Delta}_q\phi + \left(\Lambda_q^{eo} - \frac{1}{4}\right)\Delta_q^2\gamma_q(\phi). \tag{19}$$

Then, for any conservation relation and any equilibrium function, the steady solution of the evolution equation (3) can be written as

$$g_q^{\pm}(\vec{r}) = \gamma_q(e_q^{\mp}) - 2\Lambda_q^{\mp}\Gamma_q(e_q^{\pm}) + \delta g_q^{\pm}(\vec{r}), \tag{20}$$

where the terms $\delta g_q^{\pm}(\vec{r})$ are required to accommodate any mismatch between the values of $g_q^{\pm}(\vec{r}_b)$ and $[\gamma_q(e_q^{\mp})](\vec{r}_b)$ (since $\Gamma_q(e_q^{\pm})(\vec{r}_b) = 0$):

$$\delta g_q^{\pm}(\vec{r}_b) = \left[g_q^{\pm} - \gamma_q(e_q^{\mp})\right](\vec{r}_b), \tag{21}$$

and are solutions of

$$\delta g_q^{\pm} = \left(\Lambda_q^{eo} - \frac{1}{4}\right)\Delta_q^2\delta g_q^{\pm}. \tag{22}$$

Note that the actual values of $\delta g_q^{\pm}(\vec{r})$ depend on the boundary scheme used to get the steady state. For instance $\delta g_q^{\pm}(\vec{r}) = 0$ along any segment with periodic boundary conditions since the systems (15), (16), and (19) are defined for all the nodes. Other properties of the $\delta g_q^{\pm}$ and their relation with other recurrence equations derived from Eqs. (11) and (14) are discussed in Appendix A. The post-collision components, as infinite series around the equilibrium, are given in Appendix B for arbitrary $\Lambda_q^{eo} > 0$.

For the particular value:

$$\Lambda_q^{eo} \equiv \frac{1}{4}, \quad \forall q, \tag{23}$$

the recurrence part of Eqs. (15), (16) and (22) vanishes, $\delta g_q^{\pm}(\vec{r}) = 0$ for the q-bulk nodes, and the post-collision components are expressed via the gradients of the equilibrium components. This value of Λ_q^{eo}, available for both hydrodynamic and advection–diffusion problems with the TRT evolution operator (25), has also several particular stability properties (Ref. [34]).

At equilibrium, the mass and momentum variables are related to the moments of the populations and the source quantities such that relations (7) are satisfied. They close the system of equations for the bulk. Substituting solution in the form (20), the steady conservation relations (7) become:

$$\begin{aligned}\sum_{q=1}^{Q-1}\gamma_q(e_q^-) - 2\sum_{q=1}^{Q-1}\Lambda_q^-\Gamma_q(e_q^+) + \sum_{q=1}^{Q-1}\delta g_q^+(\vec{r}) &= M(\vec{r}),\\ \sum_{q=1}^{Q-1}\gamma_q(e_q^+)\vec{c}_q - 2\sum_{q=1}^{Q-1}\Lambda_q^+\Gamma_q(e_q^-)\vec{c}_q + \sum_{q=1}^{Q-1}\delta g_q^-(\vec{r})\vec{c}_q &= \vec{F}(\vec{r}).\end{aligned} \tag{24}$$

The conserved quantities and the $\delta g_q^{\pm}$ are given by the solutions of the implicit system built from (24) and the boundary conditions (see next sections). For each solution, the procedure outlined above and in Appendix A can be used to compute the associated $g_q^{\pm}$, solutions of the recurrence equations (15) and (A.1), hence the corresponding steady solution of the LBE. This implies that the numbers of steady solutions of the LBE and of system (24) have to be the same: one for the linear cases and a finite number of them otherwise, as assumed at the beginning of this section.

In the next sections we will make an extensive use of the linear properties of γ_q and Γ_q: $\gamma_q(\phi_1 + \phi_2) = \gamma_q(\phi_1) + \gamma_q(\phi_2)$ and $\gamma_q(\mu\phi_1) = \mu\gamma_q(\phi_1)$, where μ is a constant.

3. Exact conservation and closure relations

We first restrict the LB framework to the link-wise collision operator (3). Specifying equal symmetric eigenvalues ($\lambda_q^+ = \lambda^+, \forall\, q$) but using distinct anti-symmetric eigenvalues $\{\lambda_q^-\}$, the so-called L- model, Ref. [30,34], keeps the equilibrium mass variable equal to a microscopic mass of population, in contrast to the BGK-type anisotropic models, Ref. [37]. Extending this property to the momentum equilibrium variable, for the hydrodynamic equations, the link-wise operator reduces to the two-relaxation-time model (TRT). The polynomial MRT vectors [27–29,38–40] and the link-wise ones represent two alternative collision bases, with the TRT as the only common sub-class. However, the bases can be combined, e.g., the link-wise anti-symmetric vectors and the MRT polynomial symmetric vectors, the MRT-L-model [30,33]. This combination has Q (the highest possible value) distinct eigenvalues for the AADE. For the hydrodynamic equations, it reduces to the commonly used MRT collision operator, with one relaxation rate for all the non-conserved anti-symmetric modes, $\lambda_q^- = \lambda^-, \forall\, q$.

Below, the conservation and closure relations, based on the Stokes and Navier–Stokes equilibrium distributions, are first parametrized for the TRT-operator, and the AADE is modeled with the L-operator. The results are extended for MRT-L-operators in Appendix C.

3.1. The TRT model

The two-relaxation-time model assigns one value for all the λ_q^+ and one value for all the λ_q^-:

$$\lambda_q^+ = \lambda^+, \qquad \lambda_q^- = \lambda^-, \quad q = 0, \dots, Q-1, \quad \text{then}$$
$$\Lambda_q^- = \Lambda_o = -\left(\frac{1}{2} + \frac{1}{\lambda^-}\right), \qquad \Lambda_q^+ = \Lambda_e = -\left(\frac{1}{2} + \frac{1}{\lambda^+}\right),$$
$$\Lambda_q^{eo} = \Lambda_{eo} = \Lambda_o \Lambda_e, \quad \forall\, q. \tag{25}$$

The BGK model is a special sub-class of the TRT with $\lambda^- = \lambda^+ = -1/\tau$, i.e., with $\Lambda_o = \Lambda_e = \tau - 1/2$, $\nu = \Lambda_e/3$, and $\Lambda_{eo} = 9\nu^2$. Assuming the local conservation laws (7), the equilibrium mass ρ^{eq} and momentum $\vec{J}^{\,\mathrm{eq}}$ are related to microscopic population mass ρ and momentum $\vec{J}$ as follows:

$$\rho = \sum_{q=0}^{Q-1} f_q = \sum_{q=0}^{Q-1} f_q^+, \qquad \rho^{\mathrm{eq}} = \sum_{q=0}^{Q-1} e_q^+ = \rho - \frac{M}{\lambda^+}, \tag{26}$$

$$\vec{J} = \sum_{q=1}^{Q-1} f_q \vec{c}_{\bar{q}} = \sum_{q=1}^{Q-1} f_q^- \vec{c}_{\bar{q}}, \qquad \vec{J}^{\,\mathrm{eq}} = \sum_{q=1}^{Q-1} e_q^- \vec{c}_q = \vec{J} - \frac{\vec{F}}{\lambda^-}. \tag{27}$$

In the presence of sources, the macroscopic variables ρ^m and $\vec{j}$ differ from the microscopic variables ρ and $\vec{J}$:

$$\rho^m = \rho + \frac{1}{2} M, \qquad \vec{j} = \vec{J} + \frac{1}{2}\vec{F}, \tag{28}$$

$$\text{then } \rho^{\mathrm{eq}} = \rho^m + \Lambda_e M, \qquad \vec{J}^{\,\mathrm{eq}} = \vec{j} + \Lambda_o \vec{F}. \tag{29}$$

Following the Chapman–Enskog method, the macroscopic TRT hydrodynamic equations are derived in Ref. [35] for ρ^m and $\vec{j}$. At second order, *and for uniform sources*, they depend neither on the free eigenvalue λ^- nor on Λ_{eo}. However, mass and momentum steady conservation relations (24) depend on Λ_{eo} for general flows, owing to the truncated corrections and/or the variation of the source distributions. Let us examine this dependency for exact steady Stokes and Navier–Stokes equations.

3.1.1. Viscosity independent permeability

The equilibrium distribution for the Stokes equation is a *linear* function of the pressure P and the momentum $\vec{j}$:

$$e_q^- = t_q^\star j_q^{\mathrm{eq}} = t_q^\star (j_q + \Lambda_o F_q), \quad j_q = (\vec{j} \cdot \vec{c}_q), \quad F_q = (\vec{F} \cdot \vec{c}_q),$$
$$e_q^+ = t_q^\star (P + c_s^2 \Lambda_e M), \quad q = 1, \dots, Q-1, \quad e_0^+ = e_0 = \rho^{\mathrm{eq}} - \sum_{q=1}^{Q-1} e_q^+, \tag{30}$$

$$P = c_s^2 \rho^m, \quad \forall c_s^2, \qquad \sum_{q=1}^{Q-1} t_q^\star c_{q\alpha} c_{q\beta} = \delta_{\alpha\beta}, \qquad 3\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha}^2 c_{q\beta}^2 = 1 + 2\delta_{\alpha\beta}, \quad \forall \alpha, \beta. \tag{31}$$

Multiplying the mass equation by Λ_e, the exact mass and momentum conservation equations (24) become:

$$\sum_{q=1}^{Q-1}\gamma_q(\Lambda_e j_q^\star) - \Lambda_{eo}\sum_{q=1}^{Q-1}(2\Gamma_q(P_q^\star) - \gamma_q(F_q^\star)) + \Lambda_e\sum_{q=1}^{Q-1}\delta g_q^+ = \Lambda_e M + 2\Lambda_{eo}\sum_{q=1}^{Q-1}\Gamma_q(\Lambda_e M_q^\star), \tag{32}$$

$$\sum_{q=1}^{Q-1}\gamma_q(P_q^\star)\vec{c}_q - \vec{F} + \sum_{q=1}^{Q-1}\delta g_q^-\vec{c}_q = 2\sum_{q=1}^{Q-1}\Gamma_q(\Lambda_e j_q^\star)\vec{c}_q - \sum_{q=1}^{Q-1}\gamma_q(\Lambda_e M_q^\star)\vec{c}_q + 2\Lambda_{eo}\sum_{q=1}^{Q-1}\Gamma_q(F_q^\star)\vec{c}_q, \tag{33}$$

with $j_q^\star = t_q^\star j_q, \quad P_q^\star = t_q^\star P, \quad F_q^\star = t_q^\star F_q, \quad M_q^\star = c_s^2 t_q^\star M.$

Since the LB equation is assumed to have a unique steady state for the given boundary conditions (beginning of Section 2.2), the solution of Eqs. (32) and (33) exists and is unique (see the end of Section 2.2). According to Eq. (17), any non-equilibrium component γ_q or Γ_q in Eqs. (32) and (33) depends on the relaxation parameters *only via the combination* Λ_{eo}. Therefore, if the contributions of the source quantities: $\Lambda_e M$ and $\vec{F}$, and the boundary terms: $\Lambda_e\sum_{q=1}^{Q-1}\delta g_q^+$ and $\sum_{q=1}^{Q-1}\delta g_q^-\vec{c}_q$, vanish or depend on Λ_{eo} only, both equations, written for the variables $\Lambda_e\vec{j}$ and P, are controlled by Λ_{eo} only, independently of Λ_e and Λ_o separately. Note that the last two terms in Eqs. (32) and (33) take into account the variations of the source terms. Their impact for Brinkman models is investigated in Ref. [41].

Assuming then Darcy's law (1) for the mean velocity value $\bar{\bar{\vec{u}}}$ (averaged over the whole volume V_s), the components of the permeability tensor $\mathbf{K}$ are derived from the solution for $\nu\vec{j}(\vec{r})$:

$$\vec{j}^{(\nu)} = \mathbf{K}\overline{(\vec{F} - \nabla P)}, \qquad \vec{j}^{(\nu)} = \frac{1}{V_s}\sum_{\vec{r}}\nu\vec{j}(\vec{r}) = \frac{1}{3V_s}\sum_{\vec{r}}\Lambda_e\vec{j}(\vec{r}), \quad \nu = \frac{\Lambda_e}{3}. \tag{34}$$

Since the linear relation between ∇P and $\vec{j}^{(\nu)}$ is controlled by Λ_{eo}, the components of the permeability tensor stay constant when the viscosity varies but Λ_{eo} is kept constant. When the boundary scheme obeys the parametrization properly, the derived permeability value is then independent of the used viscosity for any porous structure and can be computed efficiently. Let us first examine the bounce-back rule with respect to this property.

3.1.2. Using the bounce-back rule

The simplest and mass-conserving boundary rule for incoming populations $f_{\bar{q}}$ in the boundary grid node $\vec{r}_{\rm b}$, the bounce-back scheme is commonly applied for no-slip velocity in porous media:

$$f_{\bar{q}}(\vec{r}_{\rm b}, t+1) = \tilde{f}_q(\vec{r}_{\rm b}, t), \tag{35}$$

where q is chosen such that $\vec{r}_{\rm b} + \vec{c}_q$ is outside Ω. Early analysis (Ref. [21]) of the bounce-back rule in the frame of the MRT FCHC model has shown that *an effective location* of the no-slip wall for Poiseuille flow depends, at second order, on the "magic" combination Λ_{eo} of the "symmetric" (stress) rate defining the kinematic viscosity and the free "anti-symmetric" one (energy fluxes). It was then understood that fixing Λ_{eo} uniquely defines the effective channel width H, therefore the permeability $H^2/12$ of a straight channel as a function of Λ_{eo} (see exact formulas for $H(\Lambda_{eo})$ in [22,36]). Let us extend now these results to an arbitrary geometry.

Replacing $f_{\bar{q}}$ and $\tilde{f}_q$ with relations (4) and (5), the steady closure relation takes the form:

$$\left[e_q^- + \frac{1}{2}g_q^+ - \Lambda_o g_q^-\right](\vec{r}_{\rm b}) = 0. \tag{36}$$

Using relation (20) for $g_q^\pm$ and the fact that $\Gamma_q(\phi) = 0$ on the boundary nodes, this closure relation multiplied by Λ_e becomes:

$$\left[\Lambda_e j_q^\star + \Lambda_{eo}F_q^\star + \frac{1}{2}\left(\gamma_q(\Lambda_e j_q^\star) + \Lambda_{eo}\gamma_q(F_q^\star)\right) - \Lambda_{eo}\gamma_q(P_q^\star) + \frac{1}{2}\Lambda_e\delta g_q^+ - \Lambda_{eo}\delta g_q^-\right](\vec{r}_{\rm b}) = 0, \tag{37}$$

for any values of $e_q^\pm$ given in Eq. (30). Replacing $\delta g_q^-(\vec{r}_{\rm b})$ in Eq. (37) by its value given by relation (A.8) (for $\vec{r}_{{\rm b}_2}$) allows one to compute $\Lambda_e\delta g_q^+(\vec{r}_{\rm b})$ (at both ends of the link segment with the proper choice of q), hence $\delta g_q^-(\vec{r}_{\rm b})$, as unique functions of $\Lambda_e j_q^\star + \Lambda_{eo}F_q^\star + \gamma_q(\Lambda_e j_q^\star)/2 - \Lambda_{eo}\gamma_q(P_q^\star)$ and Λ_{eo} only. Again, since $\gamma_q(\phi)$ and $\Gamma_q(\phi)$ depend on $\{\lambda^+, \lambda^-\}$ only via Λ_{eo} for any ϕ, it follows that, with the bounce-back rule, Eqs. (32) and (33), written for the variables $\Lambda_e\vec{j}$, $\vec{F}$, and P, are controlled by Λ_{eo} only. This explains why this boundary rule yields viscosity independent permeabilities for any porous media when Λ_{eo} is kept constant. The MR1 scheme has similar properties (as shown in Ref. [35]).

3.1.3. Example of inexact parametrization, BFL-rule

Let us consider the linear interpolation introduced in Ref. [26] (BFL-rule for Bouzidi, Firdaouss, and Lallemand, or Upward/Downward-Linear-Interpolation (ULI/DLI) in [22,35]). This boundary scheme extends the bounce-back to arbitrary link distance $\delta_q \vec{c}_q$, $0 \le \delta_q \le 1$, between $\vec{r}_b$ and the point where velocity $\vec{c}_q$ cuts the wall. For no-slip condition, these rules are:

$$\begin{aligned}
&\text{if } \delta_q \le \frac{1}{2} : f_{\bar{q}}(\vec{r}_b, t+1) = \alpha^{(u)} \left[\delta_q \tilde{f}_q(\vec{r}_b, t) + \left(\frac{1}{2} - \delta_q\right) \tilde{f}_q(\vec{r}_b - \vec{c}_q, t) \right], \quad \alpha^{(u)} = 2,\\
&\text{if } \delta_q \ge \frac{1}{2} : f_{\bar{q}}(\vec{r}_b, t+1) = \alpha^{(u)} \left[\frac{1}{2} \tilde{f}_q(\vec{r}_b, t) + \left(\delta_q - \frac{1}{2}\right) \tilde{f}_{\bar{q}}(\vec{r}_b, t) \right], \quad \alpha^{(u)} = \frac{1}{\delta_q}.
\end{aligned} \tag{38}$$

The DLI-rule ($\delta_q \ge \frac{1}{2}$) is local and its exact steady closure relation can be obtained exactly as for the bounce-back. The analysis of the ULI-rule ($\delta_q \le \frac{1}{2}$) becomes local for steady solutions when $\tilde{f}_q(\vec{r}_b - \vec{c}_q)$ is replaced with $f_q(\vec{r}_b)$. For both ULI and DLI schemes, the exact steady state closure relation is:

$$\alpha^{(u)} \left[\Lambda_e j_q^\star + \Lambda_{eo} F_q^\star + \delta_q \left(\gamma_q(\Lambda_e j_q^\star) + \Lambda_{eo}\gamma_q(F_q^\star)\right) + \delta_q \Lambda_e \delta g_q^+ - \left(\Lambda_{eo} + \Lambda_e \left|\frac{1}{2} - \delta_q\right|\right) \left(\gamma_q(P_q^\star) + \delta g_q^-\right) \right](\vec{r}_b) = 0. \tag{39}$$

Except for $\delta_q = 1/2$ when both ULI/DLI reduce to the bounce-back rule, the term $\Lambda_e|\delta_q - \frac{1}{2}|$ breaks the reasoning following Eq. (37). This explains the dependency of the permeability upon the viscosity for these schemes, reported in [22,24,25]. Since this term appears first together with the second-order term $\partial_q P_q^\star - \partial_q^2 \Lambda_e j_q^\star$ (see Appendix B), the microscopic closure condition violates the exact bulk parametrization *already at second order*. The "magic" linear interpolations MGULI/MGDLI in [35,36] improve the linear schemes [26,32] with respect to this property by adding a local correction $\alpha^{(u)}|\frac{1}{2} - \delta_q| g_q^-(\vec{r}_b, t)$ to the RHS of the boundary conditions (38):

$$f_{\bar{q}}(\vec{r}_b, t+1) \to f_{\bar{q}}(\vec{r}_b, t+1) + \alpha^{(u)} \left|\frac{1}{2} - \delta_q\right| g_q^-(\vec{r}_b, t). \tag{40}$$

Ref. [36] gives a particular value: $\Lambda_{eo} = 3\delta^2/4$, for which the TRT-operator, with these and other "magic" linear (second-order accurate) boundary conditions, gives Poiseuille flows exactly in straight channels ($\delta_q \equiv \delta$), even for channels (pores) only one or two lattice nodes wide. This extends the previous bounce-back solution [21,22] ($\Lambda_{eo} = 3/16$ for $\delta = 1/2$) to $0 < \delta \le 1$. The dependency of the measured permeability on the selected collision number is studied in Ref. [35] for the bounce-back, linear and "magic" boundary schemes, this dependency being drastically reduced for third-order accurate parabolic schemes.

3.1.4. Navier–Stokes equilibrium

For the sake of completeness, we summarize here the developments of Ref. [35]. Let us introduce the dimensionless macroscopic velocity $\vec{j}'$, pressure P', and force $\vec{F}'$, computed with the reference density ρ_0 and the characteristic physical velocity U and acceleration constant g:

$$\begin{aligned}
&\vec{j}' = \frac{\vec{j}}{\rho_0 U}\frac{\delta x}{\delta t}, \qquad P' = \frac{P - P_0}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2}, \qquad \vec{F}' = \frac{\vec{F}}{\rho_0 g}\frac{\delta x}{\delta t^2},\\
&\rho^{m\prime} = \frac{\rho^m}{\rho_0} = 1 + Ma^2 P', \quad \text{where } P = c_s^2 \rho^m, \quad Ma = \frac{U}{c_s}\frac{\delta t}{\delta x},
\end{aligned} \tag{41}$$

where δx and δt are the mesh size and the time step in physical units. For the sake of simplicity, we assume the standard form [23,42] for the nonlinear Navier–Stokes equilibrium term $E_q^+(\vec{j}, \widehat{\rho})$ and no mass sources ($M = 0$, $\rho^m = \rho$):

$$\begin{aligned}
&e_q^+ = \Pi_q^\star = t_q^\star \Pi_q, \qquad \Pi_q(\rho, \vec{j}, \widehat{\rho}) = P + E_q^+(\vec{j}, \widehat{\rho}), \quad q = 1, \dots, Q-1,\\
&e_0^+ = e_0 = \rho - \sum_{q=1}^{Q-1} e_q^+, \qquad E_q^+(\vec{j}, \widehat{\rho}) = \frac{3j_q^2 - \|\vec{j}\|^2}{2\widehat{\rho}}.
\end{aligned} \tag{42}$$

Expressing the equilibrium and the post-collision components (20) via the dimensionless variables and substituting them into the exact mass and momentum steady conservation relations (24), the latter become:

$$3\sum_{q=1}^{Q-1} \gamma_q(j_q^{\prime\star}) + \frac{Re_g}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2} \sum_{q=1}^{Q-1} \Lambda_e \delta g_q^+(\vec{r}) = 2\Lambda_{eo} Re_g \sum_{q=1}^{Q-1} \Gamma_q(\Pi_q^{\prime\star}) + \Lambda_{eo}\frac{Re_g}{Fr_g^2} \sum_{q=1}^{Q-1} \gamma_q(F_q^{\prime\star}), \tag{43}$$

$$\sum_{q=1}^{Q-1} \gamma_q(\Pi_q^{\prime\star})\vec{c}_q + \frac{1}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2} \sum_{q=1}^{Q-1} \delta g_q^-(\vec{r}) = \frac{\vec{F}'}{Fr_g^2} + \frac{6}{Re_g} \sum_{q=1}^{Q-1} \Gamma_q(j_q^{\prime\star})\vec{c}_q + \frac{2\Lambda_{eo}}{Fr_g^2} \sum_{q=1}^{Q-1} \Gamma_q(F_q^{\prime\star})\vec{c}_q, \tag{44}$$

$$\text{where } Re_g = \frac{U\delta x}{\nu_p}, \quad Fr_g^2 = \frac{U^2}{g\delta x}, \quad \nu_p = \frac{1}{3}\frac{\Lambda_e \delta x^2}{\delta t},$$

$$\Pi_q'^{\star}(\rho', \vec{j}', \widehat{\rho}') = \frac{\Pi_q^{\star}(\rho, \vec{j}, \widehat{\rho})}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2} = t_q^{\star}(Ma^{-2} + P' + E_q^{+}(\vec{j}', \widehat{\rho}')), \qquad \widehat{\rho}' = \frac{\widehat{\rho}}{\rho_0},$$

where ν_p is the physical kinematic viscosity. When $\Lambda_e \delta g_q^+$ and δg_q^-, through the exact microscopic closure relations set by the boundary schemes, do not depend on the individual values of Λ_e and Λ_o, it follows from relations (43) and (44) that the *steady* solutions for $\vec{j}'$ and P' are *exactly* the same on a given grid provided that Λ_{eo} and the grid Reynolds (Re_g) and Froude (Fr_g) numbers are kept constant when the viscosity varies. Additionally, in compressible regimes, the Mach (Ma) number should also be kept constant, e.g., via c_s^2. The *exact* (without second-order truncation) macroscopic relations (43) and (44) depend however on Λ_{eo} through the finite-difference Laplacian of $\gamma_q(e_q^{\mp})$ and $\Gamma_q(e_q^{\pm})$ in Eqs. (16) and (19) and the nonzero $\gamma_q(F_q'^{\star})$ and $\Gamma_q(F_q'^{\star})$ in the presence of an inhomogeneous forcing. Note that the existence of a solution for relations (43) and (44) in terms of $\vec{j}'$ and ρ' is assumed at the beginning of Section 2.2. Since the system of equations is nonlinear, it has also been assumed that it exists only a finite number of stable steady states (see the end of Section 2.2), each of them is then parametrized by the macroscopic dimensionless numbers and Λ_{eo}.

The microscopic, then the macroscopic, exact steady closure relations can be obtained in a generic form, in terms of the coefficients of all the involved populations. For the multi-reflection-type schemes the generic closure relations are worked out in Ref. [35], for both the Dirichlet velocity and pressure conditions. They allow one to derive sufficient constraints on the multi-reflection coefficients which enforce exact bulk parametrization of the closure relations. These conditions are the same for the Stokes and Navier–Stokes equations. The bounce-back and the MR1 scheme of Ref. [22] obey them, but not the linear interpolations [26,32], unless special local corrections, as relations (40), improve them from this deficiency, giving rise to the "magic" linear schemes mentioned above. These schemes, as well as the new parabolic ones [35,36] allow exact parametrization of the steady Stokes and Navier–Stokes solutions.

3.1.5. Using again the bounce-back rule

As the simplest example, we demonstrate this property for the bounce-back closure relation (35), multiplying it by $\delta x/(\rho_0 U \delta t)$ and reorganizing it in terms of $\vec{j}'$ and Π_q':

$$\left[3j_q^{\star\prime} + \frac{3}{2}\gamma_q(j_q^{\star\prime}) - \Lambda_{eo} Re_g \gamma_q(\Pi_q') + \frac{\Lambda_{eo} Re_g}{Fr_g^2}\left(F_q^{\star\prime} + \frac{1}{2}\gamma_q(F_q^{\star\prime})\right) + \frac{Re_g}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2}\left(\frac{1}{2}\Lambda_e \delta g_q^+ - \Lambda_{eo}\delta g_q^-\right)\right](\vec{r}_b) = 0. \quad (45)$$

For the same reasons as in Section 3.1.2, the bounce-back closure relation does not depend on the individual eigenvalues. Once the hydrodynamic numbers (Re_g, Fr_g, and, in compressible regime, Ma) and the "magic" collision one Λ_{eo} have been chosen, any subset of parameters $\{U, \nu, g, c_s^2, \lambda^+, \lambda^-\}$ will then yield the same steady dimensionless solution on a given grid.

3.2. The AADE

For modeling anisotropic advection–diffusion equations (AADE), we keep only one conserved quantity (mass, i.e., no momentum conservation) and use the link-wise collision (3) with the same eigenvalue λ^+ for all the symmetric components. The equilibrium distribution for a "diffusion" scalar variable $\bar{\mathcal{D}}(\rho)$ and a d-dimensional "advection vector" $\vec{j}^{\,\mathrm{eq}}(\rho)$ is chosen as:

$$e_q^+ = E_q \bar{\mathcal{D}}(\rho) + C t_q^{\star} \Lambda_e M, \quad \Lambda_e = -\left(\frac{1}{2} + \frac{1}{\lambda^+}\right), \quad q = 1, \ldots, Q-1,$$

$$e_q^- = t_q^{\star} j_q^{\mathrm{eq}}, \qquad j_q^{\mathrm{eq}} = (\vec{j}^{\,\mathrm{eq}} \cdot \vec{c}_q), \qquad \sum_{q=1}^{Q-1} t_q^{\star} c_{q\alpha} c_{q\beta} = \delta_{\alpha\beta}, \quad \forall \alpha, \beta,$$

$$e_0^+ = e_0 = \rho^m + \Lambda_e M - \sum_{q=1}^{Q-1} e_q^+, \quad \rho^m = \rho + \frac{1}{2}M, \quad \rho = \sum_{q=0}^{Q-1} f_q, \quad (46)$$

where C is some positive constant restricted by linear stability analysis (see Ref. [34]). All the equilibrium components are given as functions of the local microscopic mass quantity ρ. The equilibrium weights $\{t_q^{\star}\}$ and $\{E_q\}$ are symmetric (their values are the same for opposite velocities). The second-order approximation of the mass conservation relation (7) takes the form of the advection–dispersion equation [30,34]:

$$\partial_t \rho^m + \nabla \cdot \vec{j}^{\,\mathrm{eq}} = M + \sum_{\alpha,\beta} \partial_\alpha \left[D_{\alpha\beta} \partial_\beta \bar{\mathcal{D}}(\rho)\right],$$

$$D_{\alpha\beta} = 2 \sum_{q=1}^{(Q-1)/2} \mathcal{T}_q c_{q\alpha} c_{q\beta}, \quad \mathcal{T}_q = \Lambda_q^- E_q. \quad (47)$$

The linear advection–diffusion equation is comprised with $\bar{\mathcal{D}}(\rho) = \rho^m$ and $\vec{j}^{\,\mathrm{eq}} = \rho^m \vec{U}$. The components of the diffusion tensor $D_{\alpha\beta}$ represent the linear combination of the link-wise parameters $\mathcal{T}_q$. The velocity sets without "diagonal links" ($c_{q\alpha}c_{q\beta} = 0$ for all α and $\beta \neq \alpha$), e.g., $d2Q5$ and $d3Q7$, cannot yield nonzero off-diagonal elements. For the $d3Q13$ velocity set with 6 moving links, the mapping from $\{D_{\alpha\beta}\}$ to $\{\mathcal{T}_q\}$ is unique. The $d2Q9$ and $d3Q15$ sets have one free parameter and the $d3Q19$ set has three free parameters for this linear transformation. The available range of the off-diagonal elements with respect to the diagonal ones is examined in Ref. [34] for positive weights $\{E_q\}$.

Once the products $\mathcal{T}_q$ are set from the diffusion tensor, different strategies are developed to get Λ_q^- and E_q. Following Ref. [30], two "extreme" strategies are the TRT-E and L-models. The first one uses the TRT-operator ($\Lambda_q^- \equiv \Lambda_o, \forall\, q$), and it gets the anisotropic diffusion elements via the *anisotropic* weights $\{E_q\}$. The anisotropic weights may differ for any two non-parallel velocities from the same class. In contrast, the L-model maintains isotropic weights, e.g., $E_q = t_q^\star$, but the set of its eigenvalue functions $\{\Lambda_q^-\}$ is anisotropic for anisotropic tensors.

Different combinations of the two techniques are possible, e.g., the *mixed* M-model of Ref. [34]. Distinct configurations have different stability and, especially, continuity properties for problems where the diffusion elements are not uniform, e.g., in heterogeneous soils (see Ref. [34]). Also, for the transient problems, the second-order tensor of the numerical diffusion should be canceled. For the TRT-E anisotropic model this can be achieved simply, via a suitable "diffusive" correction of the equilibrium distribution (see Ref. [30]).

Let us address the parametrization properties of the exact mass conservation. For steady states with the equilibrium (46), Eq. (24) becomes:

$$\sum_{q=1}^{Q-1} \gamma_q(j_q'^\star) + \frac{1}{\rho_0 U}\frac{\delta x}{\delta t}\sum_{q=1}^{Q-1} \delta g_q^+(\vec{r}) = \frac{2}{Pe_g}\sum_{q=1}^{Q-1} \mathcal{T}_q' \Gamma_q(\bar{\mathcal{D}}') + 2\sum_{q=1}^{Q-1} \Gamma_q(\Lambda_q^{eo} M_q^{\star\prime}) + M',$$

$$\text{where } \Lambda_q^{eo} = \Lambda_q^- \Lambda_e, \quad \bar{\mathcal{D}}' = \frac{\bar{\mathcal{D}}(\rho)}{\rho_0}, \quad Pe_g = \frac{U\delta x}{D_0}, \quad M' = \frac{M}{\rho_0}\frac{\delta x}{U\delta t},$$

$$\mathcal{T}_q' = \frac{\Lambda_q^- E_q}{D_0}\frac{\delta x^2}{\delta t}, \qquad j_q'^\star = t_q^\star \frac{j_q^{\mathrm{eq}}}{\rho_0 U}\frac{\delta x}{\delta t}, \qquad M_q^{\star\prime} = C t_q^\star M', \tag{48}$$

where D_0 and ρ_0 are reference values for respectively the components of the physical diffusion tensor and the mass variable. When the characteristic velocity is changed on a given grid for a given *grid Peclet number* Pe_g, the solution will stay the same if all the combinations $\Lambda_q^- E_q$ vary proportionally to the characteristic velocity U. This condition is sufficient for the second-order mass conservation relations, at least when the mass source is uniform ($\Gamma_q(\Lambda_q^{eo} M_q^{\star\prime}) = 0$). But again, the higher-order corrections to AADE, hidden in $\sum_{q=1}^{Q-1} \gamma_q(j_q'^\star)$ and $\sum_{q=1}^{Q-1} \mathcal{T}_q' \Gamma_q(\bar{\mathcal{D}}')$, depend on $\{\Lambda_q^{eo}\}$. The proper rescaling of these terms requires to set all the values $\{\Lambda_q^{eo}\}$, with the help of the free eigenvalue function Λ_e. This becomes possible only if the whole set $\{\Lambda_q^-\}$, then the whole set $\{E_q\}$, varies similarly with U.

It is noted that these conditions constrain the free equilibrium and collision parameters. When they are rescaled properly, the solution of the link-wise AADE model is fully controlled, on a given grid, by Pe_g and $\{\Lambda_q^{eo}\}$. A distinguished property of the TRT-E model is that it yields equal collision number for all the links, reducing the disparity in the distribution of the bulk and boundary errors.

3.3. Using the anti-bounce-back rule

The anti-bounce-back rule prescribes the boundary value e_q^{+b} for the symmetric equilibrium component:

$$f_{\bar{q}}(\vec{r}_{\mathrm{b}}, t+1) = -\tilde{f}_q(\vec{r}_{\mathrm{b}}, t) + 2e_q^{+b}. \tag{49}$$

Its steady closure relation is (cf. relation (37)):

$$\left[e_q^+ + \frac{1}{2}\gamma_q(e_q^+) + \frac{1}{2}\delta g_q^- - \Lambda_e \gamma_q(e_q^-) - \Lambda_e \delta g_q^+\right](\vec{r}_{\mathrm{b}}) = e_q^{+b}. \tag{50}$$

Multiplying for each cut link the relation (50) by $\frac{\Lambda_q^-}{\rho_0 D_0}\frac{\delta x^2}{\delta t}$, and assuming the advection–diffusion equilibrium (46):

$$\left[\mathcal{T}_q' \bar{\mathcal{D}}' + \frac{1}{2}\mathcal{T}_q' \gamma_q(\bar{\mathcal{D}}') - \Lambda_q^{eo} Pe_g \gamma_q(j_q'^\star) + \frac{Pe_g}{\rho_0 U}\frac{\delta x}{\delta t}\left(\frac{1}{2}\Lambda_q^- \delta g_q^- - \Lambda_q^{eo} \delta g_q^+\right)\right](\vec{r}_{\mathrm{b}}) = \mathcal{T}_q' \bar{\mathcal{D}}'^b. \tag{51}$$

Using now the fact that $\Lambda_q^- \delta g_q^-$ is a function of δg_q^+ and Λ_q^{eo} only, it follows that, for the anti-bounce-back rule, $\frac{1}{\rho_0 U}\frac{\delta x}{\delta t}\delta g_q^+$ is a function of $\mathcal{T}_q'(\bar{\mathcal{D}}' + \gamma_q(\bar{\mathcal{D}}')/2) - \Lambda_q^{eo} Pe_g \gamma_q(j_q'^\star)$, $\{\Lambda_q^{eo}\}$, and Pe_g only. Then the solution of Eq. (48) is controlled by the grid Peclet number Pe_g and $\{\Lambda_q^{eo}\}$ only.

When e_q^+ contains a mass source component, the boundary value e_q^{+b} should be computed similarly. Otherwise, the source term should be removed "by hand" from the closure relation (see the anti-bounce-back rule in Ref. [43]). According to solutions (B.3) with (B.4), $g_q^-(\bar{\mathcal{D}}') = \partial_q \bar{\mathcal{D}}' + O(\varepsilon^3)$, $g_q^+(\bar{\mathcal{D}}') = -\Lambda_q^- \partial_q^2 \bar{\mathcal{D}}' + O(\varepsilon^4)$. The boundary value is prescribed in the middle of the link with a second-order accuracy. When $\Lambda_q^{eo} = 1/8$ and without mass source, the anti-bounce-back becomes exact for a parabolic distribution $\bar{\mathcal{D}}'(\vec{r})$.

The anti-bounce-back rule can also be used to prescribe the hydrodynamic pressure values, e.g., on the free interface in Ref. [44]. Like for the bounce-back, multiplying the exact relation (50) by $\frac{1}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2}$ and restricting the link-wise operator to the TRT model, the dimensionless pressure condition becomes:

$$\left[\Pi_q'^{\star} + \frac{1}{2}\gamma_q(\Pi_q'^{\star}) - \frac{3}{Re_g}\gamma_q(j_q') + \frac{1}{\rho_0 U^2}\frac{\delta x^2}{\delta t^2}\left(\frac{1}{2}\delta g_q^- - \Lambda_e \delta g_q^+\right)\right](\vec{r}_b) = \Pi_q'^{\star b}. \tag{52}$$

The anti-bounce-back scheme with the correction $-\Lambda_e g_q^+(\vec{r}_b, t)$ is called PAB in [35,36]. Similar to the bounce-back, the anti-bounce-back and the PAB yield the parametrization properties of the bulk solutions for the hydrodynamic equations. The second- and third-order accurate, linear and parabolic, respectively, multi-reflection pressure schemes [35,36] extend the anti-bounce-back for arbitrary distance to the boundary. A coupling of the Dirichlet velocity and pressure schemes is called the "mixed" scheme. It is suitable to prescribe pressure/tangential velocity mixed condition, which is sufficient to fix the Navier–Stokes solution, e.g., at the inlet/outlet.

4. Concluding remarks

This paper is focused on the derivation of steady recurrence equations, given by Eq. (15) for the link-wise collision operator (3) and by Eqs. (C.6) and (C.8) for the MRT-L-operator (C.1). Due to the linearity of the evolution equation, the post-collision component $g_q^{\pm}(\vec{r})$ of the link-wise operator is split into a sum of two "bulk" components: $\gamma_q(e_q^{\mp})$ and $-2\Lambda_q^{\mp}\Gamma_q(e_q^{\pm})$, each of them being a solution of the recurrence equations (16) and (19), plus an additional term $\delta g_q^{\pm}$ handling arbitrary boundary conditions. These recurrence equations, then their solutions, depend on the collision rates only via their combinations $(\Lambda_q^{eo} - 1/4)$. This property explains the parametrization role of the TRT eigenvalue combination Λ_{eo} ($\Lambda_q^{eo} \equiv \Lambda_{eo}, \forall q$) for generic steady Stokes and Navier–Stokes bulk solutions. It follows that the measured permeability value depends on the relaxation rates only via Λ_{eo}, provided that the boundary scheme is also parametrized exactly. This dependency is structure-dependent and individual for each boundary scheme, such that there is no universal ("most accurate") collision number. Using third-order accurate, "parabolic" boundary schemes (e.g., [22,35,36]), one can shift the dependency on Λ_{eo} beyond second order. Both bulk and boundary truncated errors become then insignificant when Λ_{eo}, typically, stays inside the interval $]0, 1]$ (we use mostly $\Lambda_{eo} \in [\frac{1}{8}, \frac{1}{4}]$). These schemes are based on expansions of the "bulk" components of the post-collision components and, as a result, the "boundary" terms $\delta g_q^{\pm}$ are of the order of the neglected corrections. The linear interpolations MGULI/MGDLI in [35,36]: relations (38) and (40), are exactly parametrized, robust, and sufficient for computations in porous media.

When the governing physical numbers and the "magic" collision ones are given, one can find an efficient (fast) choice of the computational parameters (characteristic length, velocity and transport coefficients) for the same physical solution. Also, owing to the improved accuracy for suitable values of Λ_{eo}, very modest grid resolutions become sufficient, a crucial property for realistic computations in porous media. Moreover, when the "magic" collision numbers are kept constant, the explicit expansion of the steady solutions given by Eq. (B.3) with (B.4) allows one to estimate the errors on the transport coefficients with the grid refining at arbitrary order. Finally, all these properties are valid for any equilibrium/source distribution and any kind of conservation and boundary constraints. They are worthwhile also for "meso-scale" modeling, combining the Stokes/Navier–Stokes and Brinkman equations, where suitable values of Λ_{eo} can help to reduce the leading-order corrections due to the force variations (see Ref. [41]). We emphasize that all these improvements are not possible with the BGK model.

We believe that the TRT model is the simplest one sufficiently efficient for creeping flow modeling. However one can also measure the permeability independently of the assigned collision rates and rescale the truncated errors correctly with the "full" MRT-operators, at least when they have only one eigenvalue λ^- for all non-conserved anti-symmetric modes. Their symmetric eigenvalues can take distinct values, one value for stress flux mode, say $\{\lambda_\nu\}$, and other ones, say $\{\lambda^{+\,(i)}\}$, for "kinetic energy" and "kinetic energy squared" modes [29,38,40]. Even if their values are irrelevant for the second-order incompressible equations, the "free" symmetric eigenvalues have an impact, not yet completely understood, on the effective stability (see Ref. [38]). In particular, high bulk viscosity values, defined via the rate of "kinetic energy" mode, can be used to damp transient acoustic waves. The preliminary results of Ref. [7] and our later computations show that it is again sufficient to set $\Lambda_{eo}(\lambda_\nu, \lambda^-)$ along with the additional combinations $\Lambda^{eo\,(i)}(\lambda^{+\,(i)}, \lambda^-)$ (when the MRT model does reduce to the TRT one). Then specifying λ^- exactly as for the TRT model: $\Lambda_o(\lambda^-) = \Lambda_{eo}/\Lambda_e(\lambda_\nu)$, the free even eigenvalues are given by $\Lambda_e(\lambda^{+\,(i)}) = \Lambda^{eo\,(i)}/\Lambda_o(\lambda^-)$. These properties have motivated us to extend the TRT analysis to MRT-L-operators, which include the MRT models as a special sub-class suitable for hydrodynamic problems.

An extension of the recurrence equations (15) to transient regime is straightforward:

$$\begin{aligned} g_q^{\pm}(\vec{r}, t) \;=\; & \left[\bar{\Delta}_t e_q^{\pm} + \bar{\Delta}_q e_q^{\mp} - \Lambda_q^{\mp}(\Delta_q^2 - \Delta_t^2)e_q^{\pm} + \left(\Lambda_q^{eo} - \frac{1}{4}\right)(\Delta_q^2 - \Delta_t^2)g_q^{\pm}\right](\vec{r}, t) \\ & - \left[\frac{1}{2}\Delta_t^2 + (\Lambda_q^{\pm} + \Lambda_q^{\mp})\bar{\Delta}_t\right] g_q^{\pm}(\vec{r}, t), \end{aligned}$$

$$\text{where } \bar{\Delta}_t\phi(t) = (\phi(t+1) - \phi(t-1))/2,$$
$$\text{and} \quad \Delta_t^2\phi(t) = \phi(t+1) - 2\phi(t) + \phi(t-1), \quad \forall\phi. \tag{53}$$

As far as we can see, only the spatial components (then the spatial truncated errors) are controlled by $\{\Lambda_q^{eo}\}$, whereas the higher-order corrections in time may depend on Λ_q^+ and Λ_q^-. Future works are needed to represent, if possible, the coefficients of time-dependent solutions as explicit functions of the eigenvalues and find their most efficient parametrization.

Acknowledgments

The "magic" parameters for porous media computations have been found during the winter 1994–1995 while working with P. Lallemand and L. Giraud at the ASCI laboratory (CNRS). Turning back to these results from time to time, we finally proved them. We dedicate this proof to Professor Pierre Lallemand with our most sincere gratitude for his constant attention and help in all our efforts.

Appendix A. "Mixed" recurrence equations: solution for $\delta g_q^\pm(\vec{r})$.

Four link-wise recurrence equations (11) and (14) are derived from the steady evolution equation. Two linear combinations of these equations, relations (15), relate independently the symmetric and the anti-symmetric post-collision components to the equilibrium distribution. Two other linear combinations are given by the sum of the "upper" and "lower" relations (11) and (14):

$$[\Delta_q^2 e_q^\pm - \Lambda_q^\pm \Delta_q^2 g_q^\pm - \bar{\Delta}_q g_q^\mp](\vec{r}) = 0. \tag{A.1}$$

These recurrence relations involve both components, g_q^+ and g_q^-, and any steady solution of the evolution equation (3) is also solution of equations (15) and (A.1). In addition, the sequence of linear combinations leading to these equations from Eqs. (8), (9), (12) and (13) can be reverted to show that any solution of equations (15) and (A.1) is also a steady solution of the LBE. However, as it is discussed below, the recurrence equations (15) (or (A.1)) alone have solutions that are not solutions of the other pair. For instance, the state $g_q^\pm = \Delta_q^2 e_q^\pm = 0$, but at least one $\bar{\Delta}_q e_q^\pm \neq 0$, is solution of the recurrence equations (A.1), but not of (15).

Taking the sum of $\Gamma_q(\phi)$ at $\vec{r} - \vec{c}_q$ and $\vec{r}$ plus $\gamma_q(\phi)$ at $\vec{r} - \vec{c}_q$, where $\vec{r} - \vec{c}_q$ and $\vec{r}$ are both q-bulk nodes, it comes from relations (16) and (17):

$$\begin{aligned}
&\left[\Gamma_q(\phi)\right](\vec{r}) + [\Gamma_q(\phi) + \gamma_q(\phi)](\vec{r} - \vec{c}_q) \\
&= \frac{1}{2}\left(\phi(\vec{r} + \vec{c}_q) - \phi(\vec{r} - \vec{c}_q)\right) + \left(\Lambda_q^{eo} - \frac{1}{4}\right)\left([\Gamma_q(\phi)](\vec{r} + \vec{c}_q) - [\Gamma_q(\phi)](\vec{r} - \vec{c}_q)\right) \\
&= [\gamma_q(\phi)](\vec{r}).
\end{aligned} \tag{A.2}$$

The definitions of $[\gamma_q(\phi)](\vec{r}_{b_1})$ and $[\gamma_q(\phi)](\vec{r}_{b_2})$ in Eqs. (18) have been chosen to extend relation (A.2) for any pair of adjacent nodes in the computational domain Ω. The difference and half the sum of relation (A.2) taken at $\vec{r}$ and $\vec{r} - \vec{c}_q$ give respectively

$$\Delta_q^2 \gamma_q(\phi) = 2\bar{\Delta}_q \Gamma_q(\phi), \tag{A.3}$$

$$\bar{\Delta}_q \gamma_q(\phi) = \frac{1}{2}\Delta_q^2 \Gamma_q(\phi) + 2\Gamma_q(\phi) = \Delta_q^2 \phi + 2\Lambda_q^{eo}\Delta_q^2 \Gamma_q(\phi). \tag{A.4}$$

Using relation (A.3) in Eq. (17) yields

$$\gamma_q(\phi) = \bar{\Delta}_q \phi + \left(\Lambda_q^{eo} - \frac{1}{4}\right)\Delta_q^2 \gamma_q(\phi), \tag{A.5}$$

for any q-bulk node. Using Eqs. (A.3), (A.4) and (20), it comes

$$[\Lambda_q^\pm \Delta_q^2 \delta g_q^\pm + \bar{\Delta}_q \delta g_q^\mp](\vec{r}) = 0. \tag{A.6}$$

Although the solutions of the recurrence equations (15) and (A.1) are the same if $\delta g_q^\pm(\vec{r}) \equiv 0$, this is not true in general unless the terms $g_q^\pm$ are related as shown below.

When $\Lambda_q^{eo} = 1/4$ then $\delta g_q^\pm(\vec{r}) = 0$ (see Eq. (22)) except for the boundary nodes. Then the finite-difference operators in Eq. (A.6) reduce to their components at both ends $\vec{r}_{b_1}$ and $\vec{r}_{b_2} = \vec{r}_{b_1} + N_q\vec{c}_q$ of any segment in Ω (as defined in Eqs. (18)). The first equation (A.6) gives: $\delta g_q^-(\vec{r}_{b_1}) = 2\Lambda_q^+ \delta g_q^+(\vec{r}_{b_1})$ and $\delta g_q^-(\vec{r}_{b_2}) = -2\Lambda_q^+ \delta g_q^+(\vec{r}_{b_2})$, and the second one: $\delta g_q^+(\vec{r}_{b_1}) = 2\Lambda_q^- \delta g_q^-(\vec{r}_{b_1})$ and $\delta g_q^+(\vec{r}_{b_2}) = -2\Lambda_q^- \delta g_q^-(\vec{r}_{b_2})$. Since $4\Lambda_q^-\Lambda_q^+ = 1$, the two sets of conditions are the same and it follows that any solution of Eq. (15) and one of Eq. (A.1) is also solution of the other.

When $\Lambda_q^{eo} \neq 1/4$, the solutions of Eq. (22) are given by $g_q^{\pm}(\vec{r}) = g_{0q}^{\pm}K^n$, where n is the node index along the direction $\vec{c}_q$, the $g_{0q}^{\pm}$ are link-wise constants, and K is a root of $k = (\Lambda_q^{eo} - 1/4)(k-1)^2$ (or $4\Lambda_q^{eo}(k-1)^2 = (k+1)^2$), i.e., $K = K_\sigma = (2\sqrt{\Lambda_q^{eo}} - \sigma)/(2\sqrt{\Lambda_q^{eo}} + \sigma)$ for $\sigma = \pm 1$. For these solutions, the right-hand side of Eq. (A.1) is equal to $(\Lambda_q^{\pm}(K_\sigma - 1)^2 g_{0q}^{\pm} + (K_\sigma^2 - 1)g_{0q}^{\mp}/2)K_\sigma^{n-1}$ and is nonzero unless $\sqrt{\Lambda_q^+}g_{0q}^+ = \sigma_q\sqrt{\Lambda_q^-}g_{0q}^-$ with $\sigma_q = -\sigma_{\bar{q}} = \sigma$. This condition being obtained independently for each Eq. (A.6), any solution of Eq. (15) and one of Eq. (A.1) is also solution of the other as for $\Lambda_q^{eo} = 1/4$.

It follows that the solution of the recurrence equation (22) for given values of δg_q^+ at boundary points $\vec{r}_{b_1}$ and $\vec{r}_{b_2}$, also steady solution of the evolution equation (3), hence (A.6), is given by:

$$\delta g_q^+(\vec{r}) = \frac{K_1^n - K_1^{-n}}{K_1^{N_q} - K_1^{-N_q}}\delta g_q^+(\vec{r}_{b_2}) + \frac{K_1^{(N_q-n)} - K_1^{-(N_q-n)}}{K_1^{N_q} - K_1^{-N_q}}\delta g_q^+(\vec{r}_{b_1}), \tag{A.7}$$

$$\sqrt{\frac{\Lambda_q^-}{\Lambda_q^+}}\delta g_q^-(\vec{r}) = \delta g_q^+(\vec{r}) + \frac{2\delta g_q^+(\vec{r}_{b_2})K_1^{-n}}{K_1^{N_q} - K_1^{-N_q}} - \frac{2\delta g_q^+(\vec{r}_{b_1})K_1^{(N_q-n)}}{K_1^{N_q} - K_1^{-N_q}}, \tag{A.8}$$

where K_1 is the value of K_σ for $\sigma = 1$ and $\vec{r} = \vec{r}_{b_1} + n\vec{c}_q$. Multiplying relation (A.8) by $\Lambda_q^+/\sqrt{\Lambda_q^{eo}}$ and since K_1 is a function of Λ_q^{eo} only, $\delta g_q^-(\vec{r}_{b_1})$ and $\delta g_q^-(\vec{r}_{b_2})$ are unique functions of $\Lambda_q^+\delta g_q^+(\vec{r}_{b_1})$, $\Lambda_q^+\delta g_q^+(\vec{r}_{b_2})$, and Λ_q^{eo} (for $N_q \gg 1$, the influence on $\vec{r}_{b_1}$ of $\vec{r}_{b_2}$ can be neglected and conversely). The closure relations of the boundary schemes set two boundary values, here $\delta g_q^+(\vec{r}_{b_1})$ and $\delta g_q^+(\vec{r}_{b_2})$.

Note that here K does not depend on $\vec{c}_q$. This is not the case when one looks for solutions $f_q(\vec{r}) \sim \exp(\vec{k}\cdot\vec{r})$, as in the general Knudsen problem, unless $\vec{k}$ is aligned along one of the main axes, as in the Knudsen layer studied in Ref. [36].

Appendix B. Solutions of the steady recurrence equations

Assuming that the transforms of $\phi(\vec{r})$ by the link-wise finite-difference operators $\bar{\Delta}_q$ and Δ_q^2 have infinite Taylor expansions:

$$\bar{\Delta}_q\phi = \sum_{k\geq 1}\frac{\partial_q^{2k-1}\phi}{(2k-1)!}, \qquad \Delta_q^2\phi = 2\sum_{k\geq 1}\frac{\partial_q^{2k}\phi}{(2k)!}, \tag{B.1}$$

$$\partial_q^k\phi = (\vec{c}_q\cdot\nabla)^k\phi, \quad \forall\, k \geq 1, \tag{B.2}$$

the solutions of equations (17) can be written

$$\gamma_q(\phi) = \sum_{k\geq 1}\frac{a_{2k-1}\partial_q^{2k-1}\phi}{(2k-1)!}, \qquad \Gamma_q(\phi) = \sum_{k\geq 1}\frac{a_{2k}\partial_q^{2k}\phi}{(2k)!}, \tag{B.3}$$

where the coefficients $\{a_{2k-1}, a_{2k}\}$ are given by

$$a_1 = 1, \qquad a_2 = 1, \qquad X = 2\left(\Lambda_q^{eo} - \frac{1}{4}\right),$$
$$a_{2k-1} = 1 + X\sum_{n=1}^{k-1}\binom{2k-1}{2n-1}a_{2n-1}, \quad k \geq 2,$$
$$a_{2k} = 1 + X\sum_{n=1}^{k-1}\binom{2k}{2n}a_{2n}, \quad k \geq 2. \tag{B.4}$$

The coefficients a_{2k-1} and a_{2k} are polynomials of degree $k-1$ in X, hence they are related to the relaxation parameters through Λ_q^{eo} *only*.

If the expansion (B.3) is substituted into Eqs. (A.3) and (A.4) for $\phi = e_q^{\pm}$, it comes

$$2\bar{\Delta}_q\sum_{k\geq 1}\frac{a_{2k}\partial_q^{2k}e_q^{\mp}}{(2k)!} = \Delta_q^2\sum_{k\geq 1}\frac{a_{2k-1}\partial_q^{2k-1}e_q^{\mp}}{(2k-1)!},$$
$$\bar{\Delta}_q\sum_{k\geq 1}\frac{a_{2k-1}\partial_q^{2k-1}e_q^{\pm}}{(2k-1)!} = \Delta_q^2 e_q^{\pm} + 2\Lambda_q^{eo}\Delta_q^2\sum_{k\geq 1}\frac{a_{2k}\partial_q^{2k}e_q^{\pm}}{(2k)!}. \tag{B.5}$$

Expanding the finite-difference link-wise operators using the series (B.1) and equating the coefficients of equal derivatives, it comes:

$$a_1 = 1,$$
$$\sum_{n=1}^{k-1} \binom{2k-1}{2n} a_{2n} = \sum_{n=1}^{k-1} \binom{2k-1}{2n-1} a_{2n-1}, \quad k \geq 2, \tag{B.6}$$
$$\sum_{n=1}^{k} \binom{2k}{2n-1} a_{2n-1} = 2 + (2X+1) \sum_{n=1}^{k-1} \binom{2k}{2n} a_{2n}, \quad k \geq 2. \tag{B.7}$$

Relations (B.6) and (B.7) have also been obtained from an infinite steady Chapman–Enskog expansion in Ref. [41]. This confirms, with the following proof, that the series (B.3), with (B.4), and the steady form of the Chapman–Enskog expansion give equivalent bulk solutions.

Let us now prove that the solution of the recurrence equations (15) given by (B.3) with (B.4) is also solution of Eq. (A.1), i.e., the values of the coefficients a_{2k-1} and a_{2k} computed from relations (B.4) or from relations (B.6) and (B.7) are the same.

First, let us assume that the coefficients a_{2n-1} and a_{2n} given by Eq. (B.4) satisfy relation (B.6) for $1 \leq n \leq k-1$. Inserting Eq. (B.4) for a_{2k-1} into the RHS of Eq. (B.6) gives:

$$\begin{aligned}
\sum_{n=1}^{k} \binom{2k+1}{2n-1} a_{2n-1} &= \sum_{n=1}^{k} \binom{2k+1}{2n-1} + X \sum_{n=2}^{k} \left(\binom{2k+1}{2n-1} \sum_{l=1}^{n-1} \binom{2n-1}{2l-1} a_{2l-1} \right) \\
&= \sum_{n=1}^{k} \binom{2k+1}{2n-1} + X \sum_{l=1}^{k-1} \left(\sum_{n=l+1}^{k} \binom{2k+1}{2n-1} \binom{2n-1}{2l} \right) a_{2l} \\
&= \sum_{n=1}^{k} \binom{2k+1}{2n} + X \sum_{n=2}^{k} \left(\binom{2k+1}{2n} \sum_{l=1}^{n-1} \binom{2n}{2l} a_{2l} \right) \\
&= \sum_{n=1}^{k} \binom{2k+1}{2n} a_{2n},
\end{aligned} \tag{B.8}$$

where the second line is derived from the first one by using Eq. (B.6) for $n = \{2, \dots, k\}$ and rearranging the summations. Using

$$\begin{aligned}
\sum_{n=l+1}^{k} \binom{2k+1}{2n-1} \binom{2n-1}{2l} &= \sum_{n=l+1}^{k} \binom{2k+1}{2(k+l-n+1)-1} \binom{2(k+l-n+1)-1}{2l} \\
&= \sum_{n=l+1}^{k} \binom{2k+1}{2n} \binom{2n}{2l},
\end{aligned} \tag{B.9}$$

in the second line gives the third one after rearranging the summations, then the last line using Eq. (B.4) for a_{2k}.

Then, if the coefficients a_{2n-1} and a_{2n} given by Eq. (B.4) satisfy relation (B.6) for $1 \leq n \leq k-1$, this is also true for $1 \leq n \leq k$. Since $a_1 = a_2 = 1$, a_{2n-1} and a_{2n} satisfy relation (B.6) for $1 \leq n \leq k-1$ with $k = 2$, then with $k = 3$, and, by induction, for all the values of k.

Let us now assume that the coefficients a_{2n-1} and a_{2n} given by Eq. (B.4) satisfy relation (B.7) for respectively $1 \leq n \leq k-1$ and $1 \leq n \leq k-2$. Inserting Eq. (B.4) for a_{2k-1} into the LHS of (B.7) gives

$$\begin{aligned}
\sum_{n=1}^{k} \binom{2k}{2n-1} a_{2n-1} &= \sum_{n=1}^{k} \binom{2k}{2n-1} + X \sum_{n=2}^{k} \binom{2k}{2n-1} \left(\sum_{l=1}^{n-1} \binom{2n-1}{2l-1} a_{2l-1} \right) \\
&= \sum_{n=1}^{k} \binom{2k}{2n-1} + X \sum_{l=1}^{k-1} \left(\sum_{n=l+1}^{k} \binom{2k}{2n-1} \binom{2n-1}{2l-1} \right) a_{2l-1} \\
&= \sum_{n=1}^{k} \binom{2k}{2n-1} + X \sum_{l=1}^{k-1} \left(\sum_{n=l}^{k-1} \binom{2k}{2n} \binom{2n}{2l-1} \right) a_{2l-1} \\
&= 2 + (2X+1) \left(\binom{2k}{2} + \sum_{n=2}^{k-1} \binom{2k}{2n} \left(1 + X \sum_{l=1}^{n-1} \binom{2n}{2l} a_{2l} \right) \right) \\
&= 2 + (2X+1) \sum_{n=1}^{k-1} \binom{2k}{2n} a_{2n},
\end{aligned} \tag{B.10}$$

where the third line is derived from the second one by using

$$\sum_{n=l+1}^{k-1}\binom{2k}{2n-1}\binom{2n-1}{2l-1}=\sum_{n=l+1}^{k-1}\binom{2k}{2(k+l-n)-1}\binom{2(k+l-n)-1}{2l-1}$$
$$=\sum_{n=l+1}^{k-1}\binom{2k}{2n}\binom{2n}{2l-1}. \tag{B.11}$$

Rearranging the summations in the third line, with (B.7) for $n=\{2,\dots,k-1\}$ and $\sum_{n=1}^{k}\binom{2k}{2n-1}=2+\sum_{n=1}^{k-1}\binom{2k}{2n}=2^{2k-1}$, gives the fourth line, then the last line using Eq. (B.4) for a_{2k}.

Then, if the coefficients a_{2n-1} and a_{2n} given by Eq. (B.4) satisfy relation (B.7) for respectively $1\le n\le k-1$ and $1\le n\le k-2$, this is also true for respectively $1\le n\le k$ and $1\le n\le k-1$. Since $a_1=a_2=1$ and $a_3=1+3X$, the coefficients a_{2n-1} and a_{2n} given by Eq. (B.4) satisfy relation (B.7) for respectively $1\le n\le k-1$ and $1\le n\le k-2$ with $k=3$, then with $k=4$, and by induction for all the values of k.

Appendix C. MRT-L-model

Let us combine the link-wise anti-symmetric vectors and the MRT polynomial symmetric vectors (MRT-L-operator in [30, 33]). Denoting $\{\mathbf{v}^{+\,(i)}\}$ the $N^+=1+Q/2$ vectors of the symmetric part of the MRT-L-basis and $\{\lambda^{+\,(i)}\}$ the corresponding eigenvalues, the evolution equation (3) remains valid with the following changes:

$$n_q^+=(f_q^+-e_q^+)=\sum_{i=1}^{N^+}n_q^{+\,(i)},\qquad n_q^{+\,(i)}=\frac{(\mathbf{n}^+\cdot\mathbf{v}^{+\,(i)})}{\|\mathbf{v}^{+\,(i)}\|^2}v_q^{+\,(i)},$$
$$g_q^+=\sum_{i=1}^{N^+}g_q^{+\,(i)},\qquad g_q^{+\,(i)}=\lambda^{+\,(i)}n_q^{+\,(i)}. \tag{C.1}$$

For the MRT-L-operators, the relations (4) and (5) take respectively the form

$$f_q(\vec r,t)=\left[e_q^++e_q^--\sum_{i=1}^{N^+}\left(\frac12+\Lambda^{+\,(i)}\right)g_q^{+\,(i)}-\left(\frac12+\Lambda_q^-\right)g_q^-\right](\vec r,t),$$
$$\tilde f_q(\vec r,t)=\left[e_q^++e_q^-+\sum_{i=1}^{N^+}\left(\frac12-\Lambda^{+\,(i)}\right)g_q^{+\,(i)}+\left(\frac12-\Lambda_q^-\right)g_q^-\right](\vec r,t),$$
$$\Lambda^{+\,(i)}=-\left(\frac12+\frac{1}{\lambda^{+\,(i)}}\right)>0,\qquad \Lambda_q^-=-\left(\frac12+\frac{1}{\lambda_q^-}\right)>0,\quad\forall q,\quad\forall i. \tag{C.2}$$

The MRT-L-operator has at most $(Q-1)^2/4$ distinct "magic" values $\Lambda_q^{eo\,(i)}$:

$$\Lambda_q^{eo\,(i)}=\Lambda^{+\,(i)}\Lambda_q^-,\qquad \Lambda_q^{eo\,(i)}>0,\quad q=1,\dots,Q-1,\quad i=1,\dots,N^+, \tag{C.3}$$

all of them being available for the AADE. The eigenvalues related to the conserved "moments", such as the density and momentum, are not relevant and can take any values in the MRT-basis. Taking $\lambda_q^-=\lambda^-,\forall\,q$, it follows that the evolution equation (C.1) is equivalent to the usual MRT models with one relaxation rate λ^- for all anti-symmetric non-conserved modes. Eqs. (11) and (14) become, respectively:

$$\left[\Delta_q^2e_q^++2\bar\Delta_qe_q^--\sum_{i=1}^{N^+}\left(\frac12+\Lambda^{+\,(i)}\right)\Delta_q^2g_q^{+\,(i)}-2\left(\frac12+\Lambda_q^-\right)\bar\Delta_qg_q^-\right](\vec r)=2g_q^+(\vec r),$$
$$\left[\Delta_q^2e_q^-+2\bar\Delta_qe_q^+-\left(\frac12+\Lambda_q^-\right)\Delta_q^2g_q^--2\sum_{i=1}^{N^+}\left(\frac12+\Lambda^{+\,(i)}\right)\bar\Delta_qg_q^{+\,(i)}\right](\vec r)=2g_q^-(\vec r). \tag{C.4}$$

and

$$\left[\Delta_q^2e_q^+-2\bar\Delta_qe_q^-+\sum_{i=1}^{N^+}\left(\frac12-\Lambda^{+\,(i)}\right)\Delta_q^2g_q^{+\,(i)}-2\left(\frac12-\Lambda_q^-\right)\bar\Delta_qg_q^-\right](\vec r)=-2g_q^+(\vec r),$$
$$\left[\Delta_q^2e_q^--2\bar\Delta_qe_q^++\left(\frac12-\Lambda_q^-\right)\Delta_q^2g_q^--2\sum_{i=1}^{N^+}\left(\frac12-\Lambda^{+\,(i)}\right)\bar\Delta_qg_q^{+\,(i)}\right](\vec r)=-2g_q^-(\vec r). \tag{C.5}$$

Again, splitting the four equations (C.4) and (C.5) into two pairs: one with the "upper" superscript and one with the "lower" superscript, and eliminating $\bar{\Delta}_q g_q^-$ and $\bar{\Delta}_q g_q^{+\,(k)}$ from respectively the first and second pairs yields:

$$g_q^+(\vec{r}) = \left[\bar{\Delta}_q e_q^- - \Lambda_q^- \Delta_q^2 e_q^+ + \sum_{i=1}^{N^+}\left(\Lambda_q^{eo\,(i)} - \frac{1}{4}\right)\Delta_q^2 g_q^{+\,(i)}\right](\vec{r}), \tag{C.6}$$

$$g_q^-(\vec{r}) = \left[\bar{\Delta}_q e_q^+ - \Lambda^{+\,(k)}\Delta_q^2 e_q^- + \Lambda^{+\,(k)}\bar{\Delta}_q g_q^+ - \sum_{i=1}^{N^+}\Lambda^{+\,(i)}\bar{\Delta}_q g_q^{+\,(i)}\right](\vec{r}) + \left(\Lambda_q^{eo\,(k)} - \frac{1}{4}\right)\Delta_q^2 g_q^-(\vec{r}),$$
$$k = 1,\ldots,N^+. \tag{C.7}$$

The last equation reduces to "lower" Eq. (15) for the TRT-operator with $\Lambda^{+\,(i)} \equiv \Lambda_e, \forall\, i$. The sum of relations (C.7) for all k, divided by N^+, is:

$$g_q^-(\vec{r}) = \left[\bar{\Delta}_q e_q^+ - \bar{\Lambda}^+\Delta_q^2 e_q^- + \bar{\Lambda}^+\bar{\Delta}_q g_q^+ - \sum_{i=1}^{N^+}\Lambda^{+\,(i)}\bar{\Delta}_q g_q^{+\,(i)}\right](\vec{r}) + \left(\bar{\Lambda}_q^{eo} - \frac{1}{4}\right)\Delta_q^2 g_q^-(\vec{r}),$$
$$\bar{\Lambda}^+ = \frac{1}{N^+}\sum_{k=1}^{N^+}\Lambda^{+\,(k)}, \qquad \bar{\Lambda}_q^{eo} = \frac{1}{N^+}\sum_{k=1}^{N^+}\Lambda_q^{eo\,(k)} = \bar{\Lambda}^+\Lambda_q^-. \tag{C.8}$$

The role of terms $\delta g_q^\pm$ does not change compared to the TRT model and they will be omitted in the sequel in order to keep the algebra as simple as possible. Thus we look for solutions in the form

$$g_q^-(\vec{r}) = \gamma_q(\mathbf{e}^+) - 2\bar{\Lambda}^+\Gamma_q(\mathbf{e}^-),$$
$$g_q^+(\vec{r}) = \gamma_q(\mathbf{e}^-) - 2\Lambda_q^-\Gamma_q(\mathbf{e}^+),$$
$$\gamma_q(\mathbf{e}^-) = \sum_{i=1}^{N^+}\gamma_q^{(i)}(\mathbf{e}^-), \qquad \gamma_q^{(i)}(\mathbf{e}^-) = \frac{(\gamma(\mathbf{e}^-)\cdot\mathbf{v}^{+\,(i)})}{\|\mathbf{v}^{+\,(i)}\|^2}v_q^{+\,(i)},$$
$$\Gamma_q(\mathbf{e}^+) = \sum_{i=1}^{N^+}\Gamma_q^{(i)}(\mathbf{e}^+), \qquad \Gamma_q^{(i)}(\mathbf{e}^+) = \frac{(\Gamma(\mathbf{e}^+)\cdot\mathbf{v}^{+\,(i)})}{\|\mathbf{v}^{+\,(i)}\|^2}v_q^{+\,(i)},$$
$$\text{for } i = 1,\ldots,N^+. \tag{C.9}$$

In contrast with solutions (15), the link-wise components γ_q and Γ_q depend now on the whole equilibrium vectors $\mathbf{e}^+$ and $\mathbf{e}^-$, since the MRT-collision "couples" the symmetric link components. Eqs. (C.6)–(C.8) yield the recurrence relations:

$$\gamma_q(\mathbf{e}^-) = \bar{\Delta}_q e_q^- + \sum_{i=1}^{N^+}\left(\Lambda_q^{eo\,(i)} - \frac{1}{4}\right)\Delta_q^2\gamma_q^{(i)}(\mathbf{e}^-),$$
$$2\Gamma_q(\mathbf{e}^+) = \Delta_q^2 e_q^+ + 2\sum_{i=1}^{N^+}\left(\Lambda_q^{eo\,(i)} - \frac{1}{4}\right)\Delta_q^2\Gamma_q^{(i)}(\mathbf{e}^+),$$
$$\gamma_q(\mathbf{e}^+) = \bar{\Delta}_q e_q^+ - 2\bar{\Lambda}_q^{eo}\bar{\Delta}_q\Gamma_q(\mathbf{e}^+) + 2\sum_{i=1}^{N^+}\Lambda_q^{eo\,(i)}\bar{\Delta}_q\Gamma_q^{(i)}(\mathbf{e}^+) + \left(\bar{\Lambda}_q^{eo} - \frac{1}{4}\right)\Delta_q^2\gamma_q(\mathbf{e}^+),$$
$$2\Gamma_q(\mathbf{e}^-) = \Delta_q^2 e_q^- - \bar{\Delta}_q\gamma_q(\mathbf{e}^-) + \frac{1}{\bar{\Lambda}_q^{eo}}\sum_{i=1}^{N^+}\Lambda_q^{eo\,(i)}\bar{\Delta}_q\gamma_q^{(i)}(\mathbf{e}^-) + 2\left(\bar{\Lambda}_q^{eo} - \frac{1}{4}\right)\Delta_q^2\Gamma_q(\mathbf{e}^-). \tag{C.10}$$

"Mixed" recurrence equations similar to Eq. (A.1):

$$\left[\Delta_q^2 e_q^+ - \sum_{i=1}^{N^+}\Lambda^{+\,(i)}\Delta_q^2 g_q^{+\,(i)} - \bar{\Delta}_q g_q^-\right](\vec{r}) = 0. \tag{C.11}$$

$$\left[\Delta_q^2 e_q^- - \Lambda_q^-\Delta_q^2 g_q^- - \bar{\Delta}_q g_q^+\right](\vec{r}) = 0, \tag{C.12}$$

come from the sum of Eqs. (C.4) and (C.5) or, for Eq. (C.12), from the difference between any pair of relations (C.7). Using relations (C.9) the microscopic conservation relations (7) become:

$$\sum_{q=1}^{Q-1}\gamma_q(\mathbf{e}^-) - 2\sum_{q=1}^{Q-1}\Lambda_q^-\Gamma_q(\mathbf{e}^+) = M(\vec{r}),$$

$$\sum_{q=1}^{Q-1} \gamma_q(\mathbf{e}^+)\vec{c}_q - 2\sum_{q=1}^{Q-1} \bar{\Lambda}^+ \Gamma_q(\mathbf{e}^-)\vec{c}_q = \vec{F}(\vec{r}). \tag{C.13}$$

Expanding again the solution for $\gamma_q(\mathbf{e}^{\mp})$ and $\Gamma_q(\mathbf{e}^{\pm})$ around the equilibrium, and substituting the projections $\gamma_q^{(i)}(\mathbf{e}^-)$ and $\Gamma_q^{(i)}(\mathbf{e}^+)$ given in (C.9), one can derive the solutions for the coefficients of the series, which depend now on the choice of the basis vectors $\{\mathbf{v}^{+\,(i)}\}$. Dropping the third and higher orders in Eqs. (C.10) (they correspond then to $k = 1$ in series (B.3)), the solution is:

$$\gamma_q{}^{[1]}(\mathbf{e}^-) = \bar{\Delta}_q e_q^-, \qquad \gamma_q{}^{[1]}(\mathbf{e}^+) = \bar{\Delta}_q e_q^+, \qquad \Gamma_q{}^{[1]}(\mathbf{e}^+) = \frac{1}{2}\Delta_q^2 e_q^+,$$

$$\begin{aligned} -2\bar{\Lambda}^+\Gamma_q{}^{[1]}(\mathbf{e}^-) &= -\bar{\Lambda}^+\Delta_q^2 e_q^- + \bar{\Lambda}^+\bar{\Delta}_q\gamma_q{}^{[1]}(\mathbf{e}^-) - \sum_{i=1}^{N^+}\Lambda^{+\,(i)}\bar{\Delta}_q\gamma_q^{(i)\,[1]}(\mathbf{e}^-) \\ &= \frac{1}{2}\Delta_q^2 e_q^- + \sum_{i=1}^{N^+}\frac{\bar{\Delta}_q(\bar{\Delta}\mathbf{e}^-\cdot\mathbf{v}^{+\,(i)})}{\lambda^{+\,(i)}\|\mathbf{v}^{+\,(i)}\|^2}v_q^{+\,(i)}, \end{aligned} \tag{C.14}$$

where $\bar{\Delta}\mathbf{e}^- = (\bar{\Delta}_q e_q^-)$ and the first relation has been injected into the last one as: $\bar{\Delta}_q\gamma_q{}^{[1]}(\mathbf{e}^-) = \bar{\Delta}_q\bar{\Delta}_q e_q^- = \Delta_q^2 e_q^- + \Delta_q^2\Delta_q^2 e_q^-/4$, and the double Laplacian neglected. The second-order solution (C.9) with (C.14) could have also been obtained by a steady Chapman–Enskog expansion performed in the MRT-L-basis.

The recurrence equations (C.10) or (C.11) and (C.12) confirm that the bulk solution has the form (C.9) where all the components $\gamma_q(\mathbf{e}^{\mp})$ and $\Gamma_q(\mathbf{e}^{\pm})$ depend on the eigenvalues only via the combinations $\Lambda_q^{eo\,(i)}$. The analysis of the exact conservation constraints (C.13) and boundary closure relations follow the same lines as for the link-wise operators. The higher-order truncated errors are rescaled properly on a given grid provided that all the collision numbers $\Lambda_q^{eo\,(i)}$, then their mean values $\bar{\Lambda}_q^{eo}$, are kept constant. Under these conditions, the closure relation (37) of the bounce-back and the closure relation (50) of the anti-bounce-back keep their parametrization properties. This can be proven by replacing $2\Lambda_{eo}\Gamma_q(\Lambda_e j_q^\star)$ in relation (20) with $2\bar{\Lambda}_q^{eo}\Gamma_q(\Lambda_e j_q^\star)$, for Stokes solution, and similarly for $2\Lambda_{eo}\Gamma_q(j_q^{\star\prime})$ in relation (45), for Navier–Stokes solution. Then $\Lambda_e\gamma_q$ is replaced with $\sum_{i=1}^{N^+}\Lambda^{+\,(i)}\gamma_q^{(i)}$ and $\Lambda_e\Gamma_q(e_q^-)$ with $\bar{\Lambda}^+\Gamma_q(\mathbf{e}^-)$ for the anti-bounce-back closure relations (50). Finally, the sufficient conditions of Ref. [35] on the coefficients of the boundary schemes, derived for the TRT-operator, are valid for the MRT- and MRT-L-operators, replacing Λ_o with Λ_q^-, if necessary. Altogether, the macroscopic dimensionless solutions on a given mesh, obtained with properly parametrized boundary schemes, are controlled by the physical dimensionless numbers when all the combinations $\Lambda_q^{eo\,(i)}$ are kept constant.

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