Abstract

We survey the main results of the theory of graphs with least eigenvalue $-2$ starting from late 1950s (papers by A.J. Hoffman et al.), via important results (P.J. Cameron et al., J. Algebra 43 (1976) 305) involving root systems, to the recent approach by the star complement technique which culminated in finding and characterizing maximal exceptional graphs. Some novel results on maximal exceptional graphs are included as well. In particular, we show that all exceptional graphs, except for the cone over $L(K_8)$, can be obtained by the star complement technique starting from a unique (exceptional) star complement for the eigenvalue $-2$.

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0. Introduction

Graphs with least eigenvalue $-2$ can be represented by sets of vectors at $60^\circ$ or $90^\circ$ via the corresponding Gram matrices. Maximal sets of lines through the origin with such mutual angles are closely related to the root systems known from the theory of Lie algebras. Using such a geometrical characterization one can show that graphs in question are either generalized line graphs (representable in the root system $D_n$ for some $n$) or exceptional graphs (representable in the exceptional root system $E_8$).

We present several results on generalized line graphs and on exceptional graphs; some are new and some are old but from new view points. In particular, we describe how maximal exceptional graphs, 473 in number, have recently been determined...
independently by a Serbian–British and a Japanese group of researchers using computers and then constructed theoretically by the first group. The recent star complement technique approach to the study of graphs with least eigenvalue \(-2\) can be used as an alternative to the root system technique, but the construction was achieved by combining the two techniques. We prove several theorems describing properties of maximal exceptional graphs. In particular, we show that all exceptional graphs, except for the cone over \(L(K_8)\), can be obtained by the star complement technique starting from a unique (exceptional) star complement for the eigenvalue \(-2\).

The plan of the paper is as follows: Section 1 contains definitions from general graph theory and from the theory of graph spectra, including some basic theorems, needed for the presentation of results. A history of the research on graphs with least eigenvalue \(-2\) is given in Section 2 which includes a presentation of the main theorems and some observations on various ways of proving the main results. Section 3 is devoted to maximal exceptional graphs.

1. Basic notions and results

The characteristic polynomial \(\det(xI - A)\) of the adjacency matrix \(A\) of \(G\) is called the characteristic polynomial of \(G\) and denoted by \(P_G(x)\). The eigenvalues of \(A\) (i.e. the zeros of \(\det(xI - A)\)) and the spectrum of \(A\) (which consists of the \(n\) eigenvalues) are also called the eigenvalues and the spectrum of \(G\), respectively. The eigenvalues of \(G\) are usually denoted by \(\lambda_1, \ldots, \lambda_n\); they are real because \(A\) is symmetric. Unless we indicate otherwise, we shall assume that \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) and use the notation \(\lambda_i = \lambda_i(G)\) for \(i = 1, 2, \ldots, n\).

The eigenvalues of \(A\) are the numbers \(\lambda\) satisfying \(Ax = \lambda x\) for some non-zero vector \(x \in \mathbb{R}^n\). Each such vector \(x\) is called an eigenvector of the matrix \(A\) (or of the labelled graph \(G\)) belonging to the eigenvalue \(\lambda\). The relation \(Ax = \lambda x\) can be interpreted in the following way: if \(x = (x_1, x_2, \ldots, x_n)^T\) then \(\lambda x_u = \sum_{v \sim u} x_v\) where the summation is over all neighbours \(v\) of the vertex \(u\).

If \(\lambda\) is an eigenvalue of \(A\) then the set \(\{x \in \mathbb{R}^n : Ax = \lambda x\}\) is a subspace of \(\mathbb{R}^n\), called the eigenspace of \(\lambda\) and denoted by \(E(\lambda)\). Such eigenspaces are called eigenspaces of \(G\).

The largest eigenvalue (\(\mu_1 = \lambda_1\)) of a graph \(G\) is called the index of \(G\); since adjacency matrices are non-negative there is a corresponding eigenvector whose entries are all non-negative.

Cosines of the angles between all-1 vector \(j\) and eigenspaces \(E(\lambda_i)\) are called main angles of the graph. An eigenvalue is called main if the corresponding main angle is different from 0. The concept of a main angle and of a main eigenvalue can be introduced for other kinds of graph spectra in the same manner.

Next we present certain notation, definitions and results from graph theory.

As usual, \(K_n\), \(C_n\) and \(P_n\) denote respectively the complete graph, the cycle and the path on \(n\) vertices. Further, \(K_{m,n}\) denotes the complete bipartite graph on \(m + n\)
vertices. The cocktail-party graph \( \text{CP}(n) \) is the unique regular graph with \( 2n \) vertices of degree \( 2n - 2 \); it is obtained from \( K_{2n} \) by deleting \( n \) mutually non-adjacent edges.

A connected graph with \( n \) vertices is said to be \textit{unicyclic} if it has \( n \) edges.

The \textit{complement} of a graph \( G \) is denoted by \( \overline{G} \), while \( mG \) denotes the union of \( m \) disjoint copies of \( G \). We write \( V(G) \) for the vertex set of \( G \), and \( E(G) \) for the edge set of \( G \).

For \( v \in V(G) \), \( G - v \) denotes the graph obtained from \( G \) by deleting the vertex \( v \) and all edges incident with \( v \). More generally, for \( U \subseteq V(G) \), \( G - U \) is the subgraph of \( G \) induced by \( V(G) \setminus U \).

The join \( G \vee H \) of (disjoint) graphs \( G \) and \( H \) is the graph obtained from \( G \) and \( H \) by joining each vertex of \( G \) with each vertex of \( H \). For any graph \( G \), the \textit{cone} over \( G \) is the graph \( K_1 \vee G \) obtained from \( G \) by adding a new vertex adjacent to all vertices of \( G \).

The \textit{line graph} \( L(H) \) of any graph \( H \) is defined as follows. The vertices of \( L(H) \) are the edges of \( H \) and two vertices of \( L(H) \) are adjacent whenever the corresponding edges of \( H \) have a vertex of \( H \) in common. Let \( N \) denote the vertex–edge \((0, 1)\)-incidence matrix of \( H \). Then the \((0, 1)\)-adjacency matrices \( B \) of \( H \) and \( A \) of \( L(H) \) satisfy

\[
NN^T = D + B, \quad N^T N = 2I + A,
\]

where now \( D \) is the diagonal matrix whose diagonal entries are the vertex degrees of \( H \).

A \textit{generalized line graph} \( L(H; a_1, \ldots, a_n) \) is defined for graphs \( H \) with vertex set \( \{1, \ldots, n\} \) and non-negative integers \( a_1, \ldots, a_n \) by taking the graphs \( L(H) \) and \( \text{CP}(a_i) \) \( (i = 1, \ldots, n) \) and adding extra edges: a vertex \( e \) in \( L(H) \) is joined to all vertices in \( \text{CP}(a_i) \) if \( i \) is an endvertex of \( e \) as an edge of \( H \). We include as special cases an ordinary line graph \((a_1 = a_2 = \cdots = a_n = 0)\) and the cocktail-party graph \( \text{CP}(n) \) \((n = 1 \text{ and } a_1 = n)\).

An exceptional graph is a connected graph with least eigenvalue greater than or equal to \(-2\) which is not a generalized line graph.

Given a subset \( U \) of vertices of the graph \( G \), the graph \( G' \) obtained from \( G \) by \textit{switching} with respect to \( U \) differs from \( G \) as follows: for \( u \in U, v \notin U \) the vertices \( u, v \) are adjacent in \( G' \) if and only if they are non-adjacent in \( G \). Switching is described easily in terms of the \textit{Seidel matrix} \( S \) of \( G \) defined as follows: the \((i, j)\)-entry of \( S \) is \( 0 \) if \( i = j \), \(-1\) if \( i \) is adjacent to \( j \), and \( 1 \) otherwise. The Seidel matrix of \( G' \) is \( D^{-1}SD \) where \( D \) is the (involutory) diagonal matrix whose \( i \)th diagonal entry is \( 1 \) if \( i \in U \), \(-1\) if \( i \notin U \). Switching determines an equivalence relation on graphs; moreover, switching-equivalent graphs have similar Seidel matrices and hence the same Seidel spectrum.

\textbf{Example 1.1.} Let \( S_1, S_2, S_3 \) be sets of vertices of \( L(K_8) \) which induce subgraphs isomorphic to \( 4K_1 \), \( C_5 \cup C_3 \) and \( C_8 \), respectively. The graphs \( Ch_1, Ch_2, Ch_3 \) obtained from \( L(K_8) \) by switching with respect to \( S_1, S_2, S_3 \) respectively are
called the Chang graphs. The graphs $L(K_8), Ch_1, Ch_2, Ch_3$ are regular of degree 12, cospectral and mutually non-isomorphic (see, for example, [4, p. 105], and also [51]). If we switch $L(K_8)$ with respect to the set of neighbours of a vertex $v$, we obtain a graph $H$ in which $v$ is an isolated vertex. If we delete $v$ from $H$ we obtain a graph which is called the Schläfli graph.

Example 1.2. Let $S_1, S_2$ be sets of vertices of $L(K_{4,4})$ which induce subgraphs isomorphic to $4K_1, L(K_2,4)$, respectively. The graph obtained by switching with respect to $S_1$ is called the Shrikhande graph while switching with respect to $S_2$ yields the Clebsch graph.

We take $G$ to be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$, and with $(0,1)$-adjacency matrix $A$. Let $P$ denote the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$ of $A$, and let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Since $\mathcal{E}(\mu)$ is spanned by the vectors $Pe_j$ ($j = 1, \ldots, n$) there exists $X \subseteq V(G)$ such that the vectors $P e_j$ ($j \in X$) form a basis for $\mathcal{E}(\mu)$. Such a subset $X$ of $V(G)$ is called a star set for $\mu$ in $G$. (The terminology reflects the fact that the vectors $P e_1, \ldots, P e_n$ form a eutactic star (cf. [27, p. 151]). In the context of star partitions [27, Section 7.1], star sets are called star cells.) An equivalent definition which is useful in a computational context is the following: if $\mu$ has multiplicity $k$ then a star set for $\mu$ in $G$ is a set of $k$ vertices of $G$ such that $\mu$ is not an eigenvalue of $G - X$ [27, Theorem 7.2.9]. Here $G - X$ is the subgraph of $G$ induced by $\overline{X}$, the complement of $X$ in $V(G)$. Accordingly, the graph $G - X$ is called the star complement for $\mu$ corresponding to $X$. (Such graphs are called $\mu$-basic subgraphs in [36].)

If $G$ has $H$ as a star complement for $\mu$ with corresponding star set $X$ then each induced subgraph $G - Y$ ($Y \subseteq X$) also has $H$ as a star complement for $\mu$. Moreover any graph with $H$ as a star complement for $\mu$ is an induced subgraph of such a graph $G$ for which $X$ is maximal, because $H$-neighbourhoods determine adjacencies among vertices in a star set (cf. [27]). Accordingly, in determining all the graphs with $H$ as a star complement for $\mu$, it suffices to describe those for which a star set $X$ is maximal. We call such graphs $H$-maximal; they always exist when $\mu \neq -1, 0$ because then distinct vertices of $X$ have distinct $H$-neighbourhoods [27, Corollary 7.3.6], and it follows that $|X|$ is bounded by $2^t$, where $t = |V(H)|$. In fact, $|X| \leq \frac{1}{2}(t - 1)(t + 4)$ when $t > 1$ and $\mu \neq -1, 0$ [47]. This bound of order $\frac{1}{2}t^2$ is asymptotically best possible as $t \to \infty$ because in $L(K_t)$ the eigenspace of $-2$ has codimension $t$. In any case there are only finitely many graphs with a prescribed star complement for an eigenvalue $\mu \neq -1, 0$, and this is the basis for several characterizations of graphs by star complements (cf. [28,48]).

The set of procedures for constructing and characterizing graphs by their star complements is called the star complement technique and it is based on independent discoveries by Rowlinson [46] and Ellingham [36].
2. A history of the research on graphs with least eigenvalue $-2$

Interest in the topic began with an elementary observation that line graphs have the least eigenvalue greater than or equal to $-2$ (see the second equality in (1)). A natural problem arose to characterize the graphs with such a remarkable property. It appeared that line graphs share this property with generalized line graphs and with some exceptional graphs.

Exceptional graphs first appeared in spectral characterizations of some classes of line graphs by Hoffman [38] and others in the 1960s (cf. e.g. [27, pp. 12–14]). In particular, the three Chang graphs on 28 vertices, the Schlafli graph on 27 vertices, the Shrikhande and Clebsch graphs on 16 vertices and the ubiquitous Petersen graph were among the exceptional graphs first identified. The most important particular results of that time include the spectral characterization of the line graph of a complete graph (along with the three Chang graphs cospectral with $L(K_8)$) by Hoffman and others (cf. [12–14,37–39,52]) and a complete enumeration of strongly regular graphs with least eigenvalue $-2$ by Seidel [49]. The techniques used were the method of forbidden subgraphs and switching. However, a kind of general characterization theorem for graphs with least eigenvalue $-2$ by Hoffman and Ray-Chaudhuri [42] remained unpublished since the authors were not pleased with the length proof based on the forbidden subgraph technique.

In 1970 Beineke [1] characterized line graphs by means of forbidden induced subgraphs (nine in number). Although this result does not mention eigenvalues explicitly, it is relevant to the study of graphs with least eigenvalue $-2$.

In 1976 the key paper [9] by Cameron et al. introduced root systems into the study of graphs with least eigenvalue $-2$. The main result is that an exceptional graph can be represented in the exceptional root system $E_8$. In particular, it is proved in this way that an exceptional graph has at most 36 vertices and each vertex has degree at most 28.

The regular exceptional graphs, 187 in number, were found in [5,6] in 1976 by a mixture of mathematical reasoning and a computer search, while the problem of a suitable description of all exceptional graphs remained open. Enumeration of these graphs made it possible to improve characterization theorems for several classes of line graphs. In the papers [17,26] some effort was made to eliminate the computer search from proofs of various characterization theorems for line graphs. The book [4] also contains some simplified proofs and a computer-free proof for a part of these results.

In 1979 Doob and Cvetković [35] classified graphs with least eigenvalue greater than $-2$; in particular there are 573 exceptional such graphs (20 on six vertices, 110 on seven and 443 on eight vertices; see Theorem 2.10 below).

Generalized line graphs were introduced by Hoffman [41] in 1970, and studied extensively by Cvetković et al. [20,21] in 1980. Generalized line graphs were characterized by a collection of 31 forbidden induced subgraphs in [20,21], and independently by Rao et al. [45].
Minimal forbidden graphs for the property of having the least eigenvalue $-2$ have been found in [7,43]. These forbidden graphs have at most 10 vertices and there are 1812 of them.

In 1984 Vijayakumar [54] published new proofs of the last two results as well as a proof of the main result of [9].

The subject attracted less attention from researchers for a while, the book [4] and the paper [7] being most the important publications for almost two decades.

Much information on these problems can be found in the books [4,10,18,19,27] and in the expository paper [7].

We shall now present some of the main results of [9] and related works.

Let $G$ be a graph with $n$ vertices, adjacency matrix $A$, and least eigenvalue greater than or equal to $-2$.

Eigenvalues greater than $-2$ are called principal eigenvalues. Let $m$ be the multiplicity of $-2$ as an eigenvalue, and let $r = n - m$, so that $r$ is the number of principal eigenvalues.

The symmetric matrix $I + \frac{1}{2}A$ is positive semi-definite of rank $r$. Since $I + \frac{1}{2}A$ is orthogonally diagonalizable, it follows that $I + \frac{1}{2}A = B^T B$, where $B$ is an $r \times n$ matrix of rank $r$, with real entries. Thus $I + \frac{1}{2}A$ is the Gram matrix (the matrix of inner products) of $n$ unit vectors which span the Euclidean space $\mathbb{R}^n$. The angle between any two of these vectors is $60^\circ$ or $90^\circ$ according as the corresponding vertices of $G$ are adjacent or non-adjacent. The vectors span a system of lines at $60^\circ$ and $90^\circ$. Maximal sets of lines through the origin with such mutual angles are closely related to the root systems known from the theory of Lie algebras (cf. e.g. [11,44]).

**Theorem 2.1.** The only indecomposable maximal systems of lines at $60^\circ$ and $90^\circ$ in $\mathbb{R}^n$ ($n > 1$) are those determined by the root systems:

1. $A_n$ for $n = 2, 3$;
2. $A_n, D_n$ for $n = 4, 5$;
3. $A_6, D_6$ and $E_6$ for $n = 6$;
4. $D_7$ and $E_7$ for $n = 7$;
5. $E_8$ for $n = 8$;
6. $A_n, D_n$ and $E_8$ for $n > 8$.

Here we omit the definitions of root systems $A_n$ and $D_n$. However, two representations of the exceptional root system $E_8$ will be given and used in Section 3.

The theory of maximal line systems makes it possible to formulate a general characterization theorem as follows.

**Theorem 2.2.** A connected graph $G$ has least eigenvalue $\geq -2$ if and only if it can be represented in the root system $D_n$, for some $n$, or in the exceptional root system $E_8$. 

We have also the following more specific results.

**Theorem 2.3.** A graph can be represented in $A_n$ if and only if it is the line graph of a bipartite graph.

**Theorem 2.4.** A graph can be represented in $D_n$ if and only if it is a generalized line graph.

**Corollary.** A graph is exceptional if and only if it is not a generalized line graph but can be represented in the root system $E_8$.

Next we quote the following observation.

**Proposition 2.1** [35]. There are no regular exceptional graphs with least eigenvalue greater than $-2$.

In view of Proposition 2.1 we introduce the following definition.

**Definition 2.1.** $\mathcal{G}$ is the set of all connected regular graphs which have least eigenvalue $-2$, and which are neither line graphs nor cocktail-party graphs.

Hoffman [40] posed the problem of determining $\mathcal{G}$ in 1969. All of the graphs in $\mathcal{G}$ were found by Bussemaker et al. [5,6] in 1975, by a mixture of mathematical reasoning and a computer search. The report [5] contains a table of all 187 graphs from $\mathcal{G}$, while the paper [6] is an announcement of results.

Hoffman and Ray-Chaudhuri [42] showed that graphs in $\mathcal{G}$ cannot have degree $\geq 17$. This result also appears as Theorem 4.4 of [9].

There were some efforts to formulate related theorems which could be proved without a computer search.

**Theorem 2.5.** The spectrum of a graph $G$ determines whether or not it is a regular connected line graph except for 17 cases. The exceptional cases are those in which $G$ has the spectrum of $L(H)$ where $H$ is one of the three-connected regular graphs on eight vertices or $H$ is a connected semi-regular bipartite graph on $6 + 3$ vertices.

This theorem was first announced in [16]. Its proof from [17] does not require a computer search.

It is still possible, of course, for two non-isomorphic cospectral regular line graphs to arise from non-isomorphic root graphs. Theorem 1.3.26 from [27] specifies the possibilities.

It turns out that there are exactly 68 regular exceptional graphs which are cospectral with the 17 line graphs from Theorem 2.5. They are easily identified from the list of all 187 regular exceptional graphs obtained by the computer search mentioned above.
In order to clarify the situation let us adopt the following terminology. A graph from \( G \) which is cospectral to a connected regular line graph is called a mate. If a mate is switching equivalent to any regular connected line graph to which it is cospectral we say that it is a mate of type (a); otherwise it is of type (b). It follows from the computer search that all (the 68) mates are of type (a).

Cvetković and Radosavljević [26] produced a table of the aforementioned 68 graphs which can convince the reader without recourse to a computer that these graphs are the only mates of type (a). However, arguments from [26] do not demonstrate the non-existence of mates of type (b).

The following theorem was first established as a consequence of the computer search mentioned above; the computer-free proof, given in [4], uses arguments from the theory of integral lattices.

**Theorem 2.6.** If \( G \) is a graph in \( G \) with \( n \) vertices and degree \( r \) then one of the following holds:

\[
\begin{align*}
(a) & \quad n = 2(r + 2) \leq 28, \\
(b) & \quad n = \frac{5}{2}(r + 2) \leq 27 \text{ and } G \text{ is an induced subgraph of the Schläfli graph}, \\
(c) & \quad n = \frac{4}{3}(r + 2) \leq 16 \text{ and } G \text{ is an induced subgraph of the Clebsch graph}.
\end{align*}
\]

The proof of this theorem further simplifies the proof of Theorem 2.5.

In conclusion we can say that Theorems 2.5 and 2.6, together with the table of mates from [26], represent the major general results related to the original work of Bussemaker et al. [5,6], which can be proved without recourse to a computer.

The above mentioned forbidden subgraph characterizations of line graphs and generalized line graphs can be expressed by the next two theorems.

**Theorem 2.7** [1]. A graph is a line graph if and only if it does not contain as an induced subgraph any of the nine graphs of Fig. 1.

**Theorem 2.8** [20,21,45]. A graph is a generalized line graph if and only if it does not contain as an induced subgraph any of 31 graphs of Fig. 2.

(The least eigenvalue is given in parentheses for each graph in Figs. 1 and 2.)

Consider a generalized line graph \( L(G; a) \), where \( G \) is connected and \( \sum a_i > 0 \). The root graph of \( L(G; a) \) is defined in [20] as the multigraph \( H \) obtained from \( G \) by adding \( a_i \) pendant double edges at vertex \( v_i \) for each \( i = 1, \ldots, n \). Then \( L(G; a) = L(H) \) if we understand that in \( L(H) \) two vertices are adjacent if and only if the corresponding edges in \( H \) have exactly one vertex in common.

Papers [20,21] contain also the following result.

**Theorem 2.9.** Let \( G \) be a generalized line graph of a connected graph and let \( H \) be any generalized line graph. If \( G \) and \( H \) are isomorphic then the corresponding root graphs are isomorphic except for the pairs in Fig. 3.
Fig. 1.

Fig. 2.
Theorem 2.8 is not the only result on graphs with least eigenvalue $-2$ which has been obtained simultaneously by different researchers. We conclude this presentation of main results by pointing out to other two independent discoveries in the field.

Papers [8,28] both contain the same assertion on the relations between automorphism groups of a generalized line graph and its root graph. Theorem 2.9 appeared both in [8,20] in the same year (1980) but was also included in the doctoral thesis [53] of the third author of [20,21] in 1979!

Cvetković [15] and Doob [33] gave independently a theorem on the spectral characterization of line graphs of complete bipartite graphs.

Now we turn to recent developments.

In 1998 the paper [29] introduced the star complement technique into the study of graphs with the least eigenvalue $-2$.

The following result of Doob and Cvetković [35] was the starting point. (It appears as Theorem 1.3 of [18] with a misprint in part (v).)

**Theorem 2.10.** If $G$ is a connected graph with least eigenvalue greater than $-2$ then one of the following holds:

(i) $G = L(T; 1, 0, \ldots, 0)$ where $T$ is a tree;
(ii) $G = L(H)$ where $H$ is a tree or an odd unicyclic graph;
(iii) $G$ is one of 20 graphs on six vertices represented in the root system $E_6$;

![Fig. 3.](image-url)
(iv) \( G \) is one of 110 graphs on seven vertices represented in the root system \( E_7 \);
(v) \( G \) is one of 443 graphs on eight vertices represented in the root system \( E_8 \).

If \( G \) is a connected graph with least eigenvalue \(-2\) then a connected star complement for \(-2\) is necessarily a graph of one of the five types described above.

It is shown in [29] (a) that exceptional graphs can always be constructed from exceptional star complements, and (b) that consequently it is possible for the exceptional graphs to be constructed independently of root systems. The exceptional graphs with least eigenvalue greater than \(-2\) are those appearing in parts (iii)--(v) of Theorem 2.10. Those of type (v) are one-vertex extensions of graphs of type (iv), which are in turn one-vertex extensions of graphs of type (iii). The 443 graphs of type (v) are tabulated in [7]. The 110 graphs of type (iv) are identified in [22] by means of the list of seven-vertex graphs in [18]. The 20 six-vertex graphs of type (iii) are identified in [25], and are here denoted by \( F_1, \ldots, F_{20} \). They belong to the family \( \mathcal{F} \) of 31 minimal forbidden subgraphs which characterize generalized line graphs, the other 11 having least eigenvalue less than \(-2\) (cf. Theorem 2.8 and Fig. 2). Accordingly we can make the following assertion.

**Theorem 2.11.** A connected graph is exceptional if and only if its least eigenvalue is greater than or equal to \(-2\) and it contains as an induced subgraph one of the graphs \( F_1, \ldots, F_{20} \).

This characterization of exceptional graphs is contained implicitly in Theorem 2.8, but in practice was not recognized until 1998 when [29] was being prepared.

As already mentioned, it was proved in [29] that a graph is exceptional if and only if it has an exceptional star complement. Based on this observation, the maximal exceptional graphs, 473 in number, were first found by computer and then theoretically derived in [23,31]. The computer search was completed in June 1999 and the paper [23] was submitted to the *Journal of Combinatorial Theory, Series B*, in November 1999.

Independently, Akihiro Munemasa and Masaaki Kitazume found the maximal exceptional graphs by computer. According to an e-mail message of Munemasa to Rowlinson in April 2000, these Japanese colleagues also completed their search in summer 1999. The principal results of the two groups of researchers are the same but we do not know in which way the other computer search was organized or whether the results were followed by theoretical considerations.

The rest of the paper is devoted to these recent developments, and includes some new results.
3. Maximal exceptional graphs

In this section we shall describe and construct the maximal exceptional graphs.

The basic results are obtained in [23,30,31]. Detailed results of the computer search are given in [24]. Some new results are also presented in this section.

By the Corollary to Theorem 2.4, an exceptional graph $G$ is representable in the root system $E_8$. This means that if $G$ has $A$ as a $(0, 1)$-adjacency matrix then $I + \frac{1}{2}A$ is the Gram matrix of a set of normalized vectors in $E_8$.

The problem of finding all the maximal exceptional graphs, well known from the literature, is posed explicitly in [4, p. 107]. It is equivalent to the problem [7] of finding all the graphs which are maximal with respect to the property of being representable in $E_8$.

If $\{e_1, \ldots, e_8\}$ is an orthonormal basis for $\mathbb{R}^8$ then the root system $E_8$ can be represented by the following 240 vectors (cf. [5]):

- **Type a**: 28 vectors of the form $a_{ij} = 2e_i + 2e_j$; $i, j = 1, \ldots, 8$, $i < j$;
- **Type a’**: 28 vectors opposite to those of type $a$;
- **Type b**: 28 vectors of the form $b_{ij} = -2e_i - 2e_j + \sum_{k=1}^{8} e_k$;
- **Type b’**: 28 vectors opposite to those of type $b$;
- **Type c**: 56 vectors of the form $c_{ij} = 2e_i - 2e_j$; $i, j = 1, \ldots, 8$, $i \neq j$;
- **Type d**: 70 vectors of the form $d_{ijkl} = -2e_i - 2e_j - 2e_k - 2e_l + \sum_{s=1}^{8} e_s$ with distinct $i, j, k, l \in \{1, \ldots, 8\}$;
- **Type e**: 2 vectors $e$ and $-e$, where $e = \sum_{i=1}^{8} e_i$.

These 240 vectors determine 120 lines at 60° or 90°. Let $O^- (8, 2)$ denote the graph which has these lines as vertices, with two vertices adjacent if and only if the corresponding lines are at 60°. It is known from [50] that the automorphism group of $O^- (8, 2)$ is transitive on vertices and edges. Thus the root system $E_7$ can then be realized as the set of 126 vectors which determine the 63 lines orthogonal to a given line, and $E_6$ as the set of 72 vectors which determine the 36 lines orthogonal to a pair of lines at 60°.

It is useful to note that the line graph $L(K_8)$ can be represented by all vectors of type $a$, or by all vectors of type $b$: in both cases the Gram matrix of the vectors is $8I + 4A$, where $A$ is an adjacency matrix of $L(K_8)$. Replacing some vectors of type $a$ by the corresponding vectors of type $b$ (or vice versa) is equivalent to switching with respect to those vectors (i.e. vertices).

As explained at the end of Section 2, in order to construct maximal exceptional graphs by the star complement technique, one should start with one of the 573 exceptional graphs of Theorem 2.10:

(i) one of 20 graphs on six vertices representable in $E_6$;
(ii) one of 110 graphs on seven vertices representable in $E_7$ (but not $E_6$);
(iii) one of 443 graphs on eight vertices representable in $E_8$ (but not $E_7$).
If \( H \) is of type (iii), the \( H \)-maximal graphs are precisely the maximal exceptional graphs. (Note that since \( E_8 \) is an extension of \( E_6 \) and \( E_7 \), each \( H \)-maximal graph of type (i) or (ii) is an induced subgraph of an \( H \)-maximal graph of type (iii).) The \( H \)-maximal graphs in question have been obtained by Lepović (Kragujevac) using a program (called 'Star') which implements the algorithm for finding maximal star sets from maximal cliques in an extendability graph (cf. [22]). We obtain 10 \( H \)-maximal graphs when \( H \) is of type (i), and they are described in [22, Example 5]. When \( H \) is of type (ii) 39 \( H \)-maximal graphs arise [29, Section 3].

It was an enormous task to generate all the maximal graphs starting from the 443 star complements of type (iii). For example, in a particularly difficult case a PC-586 computer took about 24 h to produce all 1,048,580 maximal graphs, which fall into 457 isomorphism classes. There are 473 maximal exceptional graphs in all, and some data concerning them are given below.

We introduce the following notation:

(a) the 443 exceptional star complements on eight vertices are denoted by \( H_{001}, H_{002}, \ldots, H_{443} \);

(b) the 473 maximal graphs generated from the star complements in (a) are denoted by \( G_{001}, G_{002}, \ldots, G_{473} \).

In case (a) graphs are ordered lexicographically by their spectral moments. In case (b) the graphs are ordered by the number of vertices, the largest eigenvalue (index), and then by vertex degrees, with all these invariants in non-decreasing order. Other names and other ordering of graphs from these two graph sequences have been used in [23,30,31]. The above names and orders appeared in [24] where also mappings between “old” and “new” orders are given in the form of tables.

The distribution over number of vertices of the graphs in (b) is as follows:

<table>
<thead>
<tr>
<th>Number of vertices</th>
<th>22</th>
<th>28</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of graphs</td>
<td>1</td>
<td>1</td>
<td>432</td>
<td>25</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

In all cases graphs with the same index are cospectral.

The maximal exceptional graphs with 29 vertices are all cones except for two, namely \( G_{425} \) and \( G_{430} \). Graphs \( G_{001} \) and \( G_{002} \) are not cones while those on more than 29 vertices cannot be since the maximal vertex degree of graphs representable in \( E_8 \) is 28. The graph \( G_{006} \) is the cone over \( L(K_8) \) while \( G_{003}, G_{004} \) and \( G_{005} \) are cones over the Chang graphs (cf. Example 1.1). The graph \( G_{434} \) is the double cone \( K_2 \vee G \) where \( G \) is the Schlafli graph.

The graph \( G_{001} \) was found by Bridges and Mena [3] as an example of a non-regular graph with just three distinct eigenvalues. The exceptional graphs with this property were found by van Dam [32], and the maximal ones are as follows, with the spectra shown:
The graph $M_{473}$ was the only maximal exceptional graph previously known (cf. [9]).

The graphs $G_{471}$ and $G_{472}$ share the spectrum $20, 5^4, 4^3, (-2)^{26}$.

Their complements are cospectral too. However, they have different vertex degrees and hence different angles [27, Section 4.2].

All of the maximal exceptional graphs are non-regular. Data on their vertex degrees are given in [23,24].

The graphs $G_{001}, G_{002}, G_{425}, G_{430}, G_{455}$ and $G_{463}$ have all vertex degrees smaller than 28. They are constructed without references to a computer search in [30,31]. All other maximal graphs have at least one vertex of degree 28.

The results of the computer search now reveal how the maximal exceptional graphs may be found by theoretical means (in principle without recourse to a computer). First we partition the maximal exceptional graphs into three types: (a) those which are 29-vertex cones, (b) those which have a vertex of degree 28 but have more than 29 vertices, (c) those in which each vertex has degree less than 28. There are 430 graphs of type (a), 37 of type (b) and 6 of type (c).

Since the graph $O^- (8, 2)$ is transitive, we may assume that the vertex of degree 28 is represented by the vector $e$. Then its neighbours are represented by vectors of type $a$ or of type $b$. Hence, we have the following proposition.

**Proposition 3.1.** If a graph, represented in $E_8$, contains a vertex of degree 28, then its neighbours induce a subgraph switching equivalent to $L(K_8)$.

It follows that a maximal exceptional graph of type (a) is a cone over a graph switching equivalent to $L(K_8)$. In order to describe the maximal graphs of type (b), we first establish a criterion for the maximality of such a cone. Let $G(P)$ denote the cone over the graph obtained from $L(K_8)$ by switching with respect to the edge-set $E(P)$, where $P$ is a spanning subgraph of $K_8$. (Thus for each edge $ij$ of $P$ the $a$ type vector $2e_i + 2e_j$ is replaced by the corresponding $b$ type vector $e_i - 2e_j$.) We define properties (I), (II) of $P$ as follows:

(I) $P$ has a four-clique and a four-coclique on disjoint sets of vertices,

(II) $P$ has six vertices adjacent to a seventh and non-adjacent to the eighth.

These configurations are called *dissections* of $P$ of type I or II. A dissection of type I yields a partition of the vertex set into two subsets of cardinality 4, while a dissection of type II yields a partition into subsets of cardinalities 6 and 2.
Now we can characterize the cones which are not maximal:

**Theorem 3.1.** The graph $G(P)$ is maximal if and only if $P$ cannot be dissected (that is, $P$ has neither property (I) nor property (II)).

More generally, we have the following theorem.

**Theorem 3.2.** Let $G$ be an exceptional graph with $29 + k$ vertices ($k \geq 0$), and suppose that $G$ has a vertex $u$ of degree 28. Let $Y$ be the set of vertices not adjacent to $u$. Then $G$ is a maximal exceptional graph if and only if $G - Y$ is isomorphic to a cone $G(P)$ in which $P$ has exactly $k$ dissections.

Thus we can determine the maximal exceptional graphs of type (b) by first finding the graphs $P$ which have a dissection. Note that (i) $P$ and $\overline{P}$ have the same dissections, and (ii) $G(P) = G(\overline{P})$. Therefore we shall say that maximal exceptional graphs of the type described in Theorem 3.2 are generated by sets $\{P, \overline{P}\}$.

**Example 3.1.** (i) If $P = K_{1,7}$ then $P$ has exactly seven dissections, all of type II, and the graph $G473$ is obtained by adding seven vectors of type $c$ to $G(P)$. The graph $G473$ can also be obtained from $G(P)$ when $P = K_1 \cup (K_4 \cup 3K_1)$. No other set $\{P, \overline{P}\}$ generates $G473$.

(ii) If $P = K_1 \cup (K_4 \cup (K_1 \cup 2K_1))$ then $P$ has exactly five dissections and we obtain $G472$ by adding the five corresponding vectors to $G(P)$. Three other sets $\{P, \overline{P}\}$ yield $G472$.

(iii) If $P = K_5 \cup 3K_1$ then $P$ has exactly five dissections, all of type I, and $G471$ is obtained from $G(P)$ by adding five vectors of type $d$. Two other sets $\{P, \overline{P}\}$ yield $G471$.

Complete data on the sets $\{P, \overline{P}\}$ which generate maximal exceptional graphs of type (b) are given in [24].

All the graphs $P$ with at least one dissection have been identified. Since $P$ and $\overline{P}$ yield the same maximal graphs, it suffices to take just one representative from $\{P, \overline{P}\}$. (Of course, at least one of the graphs $P$, $\overline{P}$ is connected and so we can always select a connected representative.)

A computer catalogue of the connected graphs on eight vertices was used to establish that 350 of them have a dissection, and of these, 207 have a disconnected complement. Among the remaining 143 graphs three are self-complementary, and so we obtained $207 + 3 + 140/2 = 280$ representative graphs.

We also note the following result.

**Proposition 3.2.** In an exceptional graph, the vertices of degree 28, if any, induce a clique.
With this proposition we finish our presentation of results from [23]. Based on these results and the notation used, we go on to describe some new results.

The following two theorems have been formulated on the basis of a computer search. Here we prove them theoretically.

**Theorem 3.3.** The spectrum of a maximal exceptional graph of type (a) has the form

\[ x, 4^6, y, (-2)^{21}, \]

where \( x + y = 18 \). The number of edges is given by \( 252 - xy \).

**Proof.** The Seidel spectrum of \( L(K_8) \) is \( 3^{21}, (-9)^7 \), 3 being a main eigenvalue. After switching this graph to a non-regular graph \( H \) the eigenvalue \(-9\) becomes main as well. The 20 eigenvalues 3 and six eigenvalues \(-9\) can be considered as non-main and when passing to the ordinary spectrum they are transformed into 20 eigenvalues \(-2\) and six eigenvalues 4, all non-main. The Seidel main eigenvalues 3 and \(-9\) are transformed into ordinary main eigenvalues \( x_1 \) and \( y_1 \) (cf. [27, pp. 102–103], for details of how main and non-main eigenvalues are transformed between the Seidel and the ordinary spectrum). A maximal exceptional graph of type (a) has the form \( H \vartriangle K_1 \) and we can apply formula (4.5.9) from [27, p. 103], for the characteristic polynomial of \( H \vartriangle K_1 \). Non-main eigenvalues 4 and \((-2)^{21}\) remain in the new graph while three others \( x, y \) and \( z \) appear. We know that one (say \( z \)) is \(-2\). Since the sum of eigenvalues of any graph is equal to 0 we have \( x + y = 18 \). The number of edges is half the sum of squares of eigenvalues and the theorem is proved. \( \Box \)

**Theorem 3.4.** The maximal exceptional graphs of type (b) on 30 vertices have spectra of the form

\[ x, y, 4^5, z, (-2)^{22}, \]

where \( x + y + z = 24 \).

**Proof.** One vertex-deleted subgraph of \( G \) is of the form \( H \vartriangle K_1 \), with spectrum as in the previous theorem. Since the multiplicity of \(-2\) in \( G \) must be 22, we immediately obtain from the interlacing theorem the form of the spectrum as asserted in the theorem. \( \Box \)

As the computer search shows, the following nine graphs, in the role of a star complement, give rise to all maximal graphs except for \( G006 = L(K_8) \vartriangle K_1 \): \( H424, H425, H431, H433, H435, H436, H437, H439, H440 \). Other star complements generate a smaller number of maximal graphs. The graph \( G006 \) has 39 non-isomorphic exceptional star complements. Each of the graphs \( H001, H009 \) and \( H023 \) generates 14 maximal graphs. Other exceptional star complements generate a larger
number of maximal graphs. Each of the maximal graphs which is not a 29-vertex cone has $H_{443}$ as a star complement.

For convenience, complements of some of the mentioned graphs are given in Fig. 4.

We shall now formulate a theorem which asserts that all exceptional graphs can essentially be obtained starting from a single graph, namely $H_{440}$, in the role of star complement for $-2$. Similar theorems can be formulated for each of the other eight of the nine graphs mentioned above.

**Theorem 3.5.** Let $G$ be a connected graph with least eigenvalue $-2$. $G$ is an exceptional graph if and only if both of the following statements hold.

(a) $G$ contains one of the graphs $F_1, F_2, \ldots, F_{20}$ as an induced subgraph;
(b) There exists a graph $H$ such that $G$ is an induced subgraph of $H$ where $H$ has the star complement $E_{440}$ for the least eigenvalue $-2$ or $H$ is the cone over $L(K_8)$.

**Remark.** Condition (a) is already necessary and sufficient for $G$ to be exceptional and is the content of Theorem 2.11. The essence of the theorem is condition (b), which on the other hand is not sufficient. However, it suggests that all exceptional graphs with only minor exceptions can be obtained by the star complement technique starting from a single star complement for $-2$, namely $H_{440}$.
The theorem follows from the above mentioned computer search. It can be proved without recourse to a computer but the proof is very technical and requires a tremendous case analysis. We offer here a very rough outline.

**Sketch of the proof.** Let $x$ be a vertex of degree 7 in $H_{440}$. Using brute force we can check that the switching class of $H_{440} - x$ contains line graphs of certain 11 graphs (on eight vertices). Let us call them $Q_1, Q_2, \ldots, Q_{11}$. In each of these graphs one can specify switching sets which yield $H_{440} - x$.

We have to prove that $H_{440}$ is an induced subgraph of all maximal exceptional graphs except for $H_{006}$.

The six maximal exceptional graphs of type (c) are constructed explicitly in [30,31] and in each of them one can directly identify an induced subgraph isomorphic to $H_{440}$.  

As proved in [23], the maximal exceptional graphs of type (b) contain an induced subgraph isomorphic to $S(K_1, 7)$, where $S(Q)$ denotes the subdivision of the graph $Q$. Since $H_{440}$ is an induced subgraph of $S(K_1, 7)$ we are done.

Consider a maximal exceptional graph $G$ of type (a). It is a 29-vertex cone. By Theorem 3.1, $G$ can be represented as $G(P)$ for some eight vertex graph $P$ without dissections. Suppose (what we have to prove) that $G$ contains an induced subgraph $E$ isomorphic to $H_{440}$. Let $x$ be a vertex of $G$ and of $E$ whose degree in $E$ is equal to 7. Next suppose that subgraph $E$ and its vertex $x$ can be selected in such a way that degree of $x$ in $G$ is equal to 28. We can assume that the vertex $x$ is represented by the all-1 vector. The obtained representation of $G$ is of the form $G(P)$ where $P$ is fully determined. Then $E - x = H_{440} - x$ is represented by vectors of types $a$ and $b$. This representation determines a line graph $L(F)$ of a graph $F$ on eight vertices to which $E - x$ is switching equivalent. The graph $F$ and the switching set $F_1$ which yields $E - x$ should coincide with one of the 11 cases mentioned above. Consider the partition $F_1 \cup F_2$ of the edge set of $F$. We must have either $F_1 \subset E(P)$ and $F_2 \subset E(\overline{P})$ or $F_1 \subset E(\overline{P})$ and $F_2 \subset E(P)$. Now, if we prove that for any graph $P$ without dissections there exist one of the 11 graphs $Q_i$ with the corresponding switching set $F_1$ such that the previous statement holds, we are done.

To prove the last statement we have to consider several cases. The proof of Theorem 3.5 can be completed in this way. \[\square\]

In the following proposition we characterize explicitly the exceptional graphs that are induced subgraphs of the cone over $L(K_8)$ which appear in (b) of Theorem 3.5.

**Proposition 3.3.** Let $H$ be a graph on eight vertices. The graph $G = L(H) \nabla K_1$ is exceptional if and only if $H$ contains one of the graphs of Fig. 5 as a subgraph.

**Proof.** If $G$ is exceptional it contains an induced subgraph $F$ isomorphic to $G_i$ of Fig. 2 for some $i = 12, 13, \ldots, 31$. Let $x$ be the vertex of $G$ of degree 28. The subgraph $F$ contains the vertex $x$. The vertex $x$ is of degree 5 in $F$. Among graphs
Fig. 5.

$G_{12}, G_{13}, \ldots, G_{31}$ the graphs $G_i$ for $i = 14, 17, 23, 24, 25, 26, 27, 29, 30, 31$ con-
tain a vertex $x$ of degree 5. The graphs $G_i - x (i = 14, 17, 23, 24, 25, 26, 27, 29, 30, 31)$ are line graphs whose root graphs are given in Fig. 5.

This completes the proof. □

We can further improve our results on the common star complement for excep-
tional graphs. To that end we adopt some new terminology.

A graph $G$ is said to be star covered by a graph $K$ for the eigenvalue $\mu$ if $G$ is an induced subgraph of a graph which has the graph $K$ as a star complement for eigenvalue $\mu$.

**Theorem 3.6.** An exceptional graph is either star covered by the graph $H_{440}$ for the eigenvalue $-2$ or is isomorphic to the cone over $L(K_8)$.

**Proof.** The graph $G_{006} = L(K_8) \Join K_1$ has a vertex of degree 28 and 28 vertices of degree 13. Let $x$ be a vertex of degree 13. All exceptional graphs which are proper induced subgraphs of $G_{006}$ are induced subgraphs of $G_{006} - x$. Consider the graph $H$ obtained from $L(K_8)$ by switching w.r.t. the single vertex $x$. Since the graph $K_2 \cup 6K_1$ has no dissections, the cone $H \Join K_1$ is a maximal exceptional graph by Theorem 3.1. (It can be identified in the tables of [24] as the graph $G_{013}$ with vertex degrees $12^{12}, 14^{15}, 16, 28$.) Since $G_{006} - x$ is an induced subgraph of $G_{013}$, all exceptional graphs contained in $G_{006}$ as induced subgraphs are contained also in $G_{013}$, except for $G_{006}$ itself. Hence, all exceptional graphs are star covered by $H_{440}$ for the eigenvalue $-2$, except for $G_{006}$.

This completes the proof. □
Our results can be reformulated also in the following way.

**Theorem 3.7.** A connected graph $G$ is exceptional if and only if either $G = L(K_8) \cong K_1$ or the following two conditions hold:

(a) $G$ contains one of the graphs $F_1, F_2, \ldots, F_{20}$ as an induced subgraph;

(b) $G$ is star covered by the graph $H_{440}$ for the eigenvalue $-2$.

It is interesting to note that $H_{440} = G_{29} \cong K_2$ and $G_{29} = L(Z) \cong K_1$, where $Z$ is represented in Fig. 5. Having in view Theorem 3.5 and Proposition 3.2 one can say that all exceptional graphs, other than the cone over $L(K_8)$, can be obtained by starting from a line graph $L(Z)$ (on five vertices), extending it to $H_{440}$ (by a threefold cone forming operation according to the above relations), and by applying the star complement technique to $H_{440}$.

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**References**


