ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 226–238
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Global existence of solutions for compressible Navier–Stokes equations with vacuum ☆

Xulong Qin [a,*], Zheng-an Yao [a], Wenshu Zhou [b]

[a] Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, PR China
[b] Department of Mathematics, Jilin University, Changchun 130012, PR China

Received 13 September 2006
Available online 25 August 2007
Submitted by M.D. Gunzburger

Abstract

In this paper, we will investigate the global existence of solutions for the one-dimensional compressible Navier–Stokes equations when the density is in contact with vacuum continuously. More precisely, the viscosity coefficient is assumed to be a power function of density, i.e., $\mu(\rho) = A\rho^\theta$, where A and θ are positive constants. New global existence result is established for $0 < \theta < 1$ when the initial density appears vacuum in the interior of the gas, which is the novelty of the presentation.

Keywords: Navier–Stokes equations; Free boundary; Vacuum; Existence

1. Introduction

In this paper, we will study the free boundary problem of the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity in the Eulerian coordinates

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0, \\ (\rho u)_\tau + \big(\rho u^2 + P(\rho)\big)_\xi = \big(\mu(\rho) u_\xi\big)_\xi, \quad a(\tau) < \xi < b(\tau),\ \tau > 0, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u)(\xi, 0) = \big(\rho_0(\xi), u_0(\xi)\big), \quad a = a(0) \leqslant \xi \leqslant b(0) = b, \tag{1.2}$$

where $\rho(\xi,\tau)$, $u(\xi,\tau)$ and $P(\rho) = B\rho^\gamma$ denote the density, velocity, and pressure of the flows, respectively; $\mu(\rho) = A\rho^\theta$ is the viscosity coefficient. Without loss of generality we assume that $A = B = 1$. $a(\tau)$ ($b(\tau)$) is the free boundary defined by

$$a'(\tau) = u\big(a(\tau), \tau\big), \qquad b'(\tau) = u\big(b(\tau), \tau\big),$$

☆ This work is supported in part by NNSFC (10531040, 10471156), Tianyuan Youth Foundation (10626056) and 985 Program of Jilin University.
* Corresponding author.
E-mail addresses: qin_xulong@163.com (X. Qin), mcsyao@mail.sysu.edu.cn (Z. Yao), wolfzws@163.com (W. Zhou).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.08.033

and satisfied

$$\big(-P(\rho)+\mu(\rho)u_\xi\big)(\xi,\tau)=0, \quad \xi=a(\tau),b(\tau). \tag{1.3}$$

As it is known, a real gas is well approximated by an ideal gas within moderate temperature and density when the conductivity k and viscosity μ are constants. However, the conductivity k and viscosity coefficient μ vary with the temperature and the density at high temperature. There are extensive discussions and experimental evidence in [16] and [17]. In mathematics, Hoff and Serre [12] showed the failure of the continuous dependence on the initial data of solutions to the Navier–Stokes equations with vacuum and constant viscosity coefficient. T.P. Liu, Z. Xin and T. Yang [11] pointed out the main reason came from the independence of the kinematic viscosity coefficient on the density and proved that the system was local well-posedness when the viscosity coefficient depends on the density. On the other hand, one can find the viscosity depends on the temperature if the Navier–Stokes equations can be derived from the Boltzman equation by exploiting Chapaman–Buskog expansion up to the second-order. For isentropic flows, however, the viscosity depends on the density.

In order to solve the free boundary problem (1.1)–(1.3) more conveniently, we introduce the Lagrangian coordinates by transformation as in [1–10]

$$x=\int_{a(\tau)}^{\xi}\rho(z,\tau)\,dz, \quad t=\tau. \tag{1.4}$$

Then the free boundaries $\xi=a(\tau)$ and $\xi=b(\tau)$ become $x=0$ and $x=\int_{a(\tau)}^{b(\tau)}\rho(z,\tau)\,dz=\int_a^b\rho_0(z)\,dz$ by the conservation of mass, respectively. Without loss of generality we assume $\int_a^b\rho_0(z)\,dz=1$. Therefore, the system (1.1) becomes in the Lagrangian coordinates

$$\begin{cases}\rho_t+\rho^2u_x=0,\\ u_t+P(\rho)_x=\big(\mu(\rho)\rho u_x\big)_x, \quad 0<x<1,\ t>0,\end{cases} \tag{1.5}$$

with boundary conditions instead of (1.3)

$$\big(-P(\rho)+\rho^{1+\theta}u_x\big)(d,t)=0, \quad d=0,1, \tag{1.6}$$

and the initial data

$$(\rho,u)(x,0)=\big(\rho_0(x),u_0(x)\big), \quad 0\leqslant x\leqslant 1. \tag{1.7}$$

Thus the free boundary problem (1.1)–(1.3) becomes the mixed problem (1.5)–(1.7).

The existence and uniqueness of a generalized global-in-time solution for the initial boundary problem (1.5)–(1.7) were studied by many authors. Among them, T.P. Liu, Z. Xin and T. Yang [11] obtained the local existence of weak solutions to the Navier–Stokes equations when the initial density connects to vacuum with discontinuities, i.e., $\inf_{[0,1]}\rho_0(x)>0$. M. Okada, Š. Matušů-Nečasová and T. Makino [6] proved the global existence of weak solutions in the case of isentropic flow for $0<\theta<\frac{1}{3}$. Following [6], T. Yang, Z. Yao and C.J. Zhu [9] obtained the global existence result for $0<\theta<\frac{1}{2}$. Recently, S. Jiang, Z. Xin and P. Zhang [5] improved the results in [6,9] to the case $0<\theta<1$. D.Y. Fang and T. Zhang [3] investigated the discontinuous solutions with the same scope of θ as in [5].

When the density connects to vacuum continuously, the boundary condition should be replaced by

$$\rho(0,t)=\rho(1,t)=0. \tag{1.8}$$

There are also many investigations on the problem (1.5), (1.7)–(1.8). For example, the authors in [1] and [2] gave the global existence of weak solutions for $0<\theta<\frac{2}{9}$ and $0<\theta<\frac{1}{3}$, respectively. D.Y. Fang and T. Zhang [4] generalized the result to $0<\theta<\frac{1}{2}$ under some additional restrictions on the initial data. In addition, M. Okada [7] obtains the global existence if $0<\theta<\frac{1}{4}$ and if one boundary is fixed and the other is free. The common of these papers is that the initial density's vacuum only appears on the boundary.

However, the initial density may vanish in the interior of the gas for more general realistic model. Obviously, this generality is more delicate than the previous works [1,2,4,6,7,9] in which the initial density $\rho_0(x)$ has a positive lower bound or only degenerates on the boundary. To the best of our knowledge, there is no result on (local or global) existence of weak solutions when the viscosity depends on density with vacuum in the interior of the gas. In addition, there are also some relevant results when the viscosity μ is constant, see for example [13–15,18].

The main feature of this paper is the initial density is degenerate in the interior of the gas when the viscosity depends on density, which is an important case from the physical point of view. Motivated by [1,2], we obtain the lower bound of density dominated by the initial density. On the other hand, the density persists vacuum on the boundary for ever. To achieve the aim, we establish some new and novel a priori estimates in Section 2. Moreover, the interval of θ requested here is larger than that of previous works.

Throughout paper, we make the following assumptions on the initial data:

(H1) $0<\theta<1<\gamma$.
(H2) $\rho_0(0)=\rho_0(x_0)=\rho_0(1)=0$, $\rho_0(x)>0$, $\forall x\in(0,1)\setminus(x_0)$ for some $x_0\in(0,1)$, $(\rho_0^\theta)_x(x)\in C[0,1]$.
(H3) $u_0(x)\in L^\infty[0,1]$.

Now we give the main result of this paper.

Theorem 1.1. *Suppose the hypotheses of* (H1)–(H3) *are satisfied with the system* (1.5), (1.7)–(1.8), *then there exists a global weak solution in the sense*

$$C_1\rho_0^k(x)\leqslant\rho(x,t)\leqslant C_2,\quad (x,t)\in[0,1]\times[0,T], \tag{1.9}$$

$$\rho(0,t)=\rho(1,t)=0, \tag{1.10}$$

$$\rho\in L^\infty\big([0,1]\times[0,T]\big)\cap C^{\frac12}\big([0,T];L^2[0,1]\big), \tag{1.11}$$

$$u\in L^2\big([0,T];L^2[0,1]\big), \tag{1.12}$$

$$u_x\in L^2\big([0,T];L^2\big([\delta,x_0-\delta]\cup[x_0+\delta,1-\delta]\big)\big), \tag{1.13}$$

$$\rho^{1+\theta}u_x\in L^2\big([0,T];L^2[0,1]\big), \tag{1.14}$$

for any $T>0$, $k>\max\{\frac{2(1+\theta)}{1-\theta},1+\frac{2}{1-\theta}\}$, $\delta\in(0,1/2)$ *and for some positive constants* C_i $(i=1,2)$ *depending on* k, *and the following equations hold*:

$$\rho_t+\rho^2u_x=0,\quad \rho(x,0)=\rho_0(x),\quad \text{for a.e. } x\in(0,1) \text{ and } t\geqslant 0, \tag{1.15}$$

and

$$\int_0^T\int_0^1\big(u\phi_t+\big(P(\rho)-\mu\rho u_x\big)\phi_x\big)\,dx\,dt+\int_0^1u_0(x)\phi(x,0)\,dx=0 \tag{1.16}$$

for any $\phi(x,t)\in C_0^\infty(\Omega)$ *with* $\Omega=\{(x,t):\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T\}$.

Remark 1.1. We note that the set of initial data $\rho_0(x)$ satisfying all the assumptions in Theorem 1.1 contains a quite general family of functions. For example, we can choose $\rho_0(x)=[x(x-x_0)^2(1-x)]^{\frac1\theta}$ for some $x_0\in(0,1)$.

In the next section, we will give the proof of Theorem 1.1. To get rid of the degeneracy of the initial density in the interior of the gas, we will mollify the initial data and consider the corresponding approximation problem (2.9)–(2.11) (see below). We establish a series of new a priori estimates for the approximation solution, a key one of among which shows that the approximation density can be dominated by the approximation initial density. Then we will take limit to the approximation solution and prove the limits satisfy (1.9)–(1.16).

2. Proof of Theorem 1.1

2.1. The approximation of the Navier–Stokes equations

For simplicity, we still denote by $(\rho_0(x),u_0(x))$ the extension of $(\rho_0(x),u_0(x))$ in R, i.e.,

$$\rho_0(x) := \begin{cases} \rho_0(1), & x > 1, \\ \rho_0(x), & x \in [0,1], \\ \rho_0(0), & x < 0, \end{cases} \qquad u_0(x) := \begin{cases} u_0(x), & x \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

Similar to [5], we define the approximate initial data

$$\begin{aligned} \rho_{0\varepsilon}(x) &= \left(j_\varepsilon * \rho_0^\theta\right)^{\frac{1}{\theta}}(x) + \varepsilon, \\ u_{0\varepsilon}(x) &= (u_0 * j_\varepsilon)(x)\left[1 - \varphi_\varepsilon(x) - \varphi_\varepsilon(1-x)\right] + (u_0 * j_\varepsilon)(0)\varphi_\varepsilon(x) + (u_0 * j_\varepsilon)(1)\varphi_\varepsilon(1-x) \\ &\quad + \left(\rho_{0\varepsilon}(0)\right)^{\gamma-\theta-1}\int_0^x \varphi_\varepsilon(y)\,dy - \left(\rho_{0\varepsilon}(1)\right)^{\gamma-\theta-1}\int_x^1 \varphi_\varepsilon(1-y)\,dy, \end{aligned} \tag{2.1}$$

where $0 < \varepsilon < 1$, $j_\varepsilon(x)$ denotes the Friedrichs mollifier, and $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$ defined by $\varphi(x) \in C_0^\infty(R)$ satisfying $\varphi(x) = 1$ when $|x| < 1$, and $\varphi(x) = 0$ when $|x| > 2$. Clearly, $\rho_{0\varepsilon}(x) \in C^{1+\alpha}[0,1]$, $u_{0\varepsilon}(x) \in C^{2+\alpha}[0,1]$ for any $\alpha \in (0,1)$. Moreover $\rho_{0\varepsilon}(x)$ and $u_{0\varepsilon}(x)$ are compatible with the boundary conditions (2.10) and satisfy

$$\left|u_{0\varepsilon}(x)\right|_\infty \leqslant C, \tag{2.2}$$

$$\left|\left(\rho_{0\varepsilon}^\theta(x)\right)_x\right|_\infty \leqslant C, \tag{2.3}$$

for some positive constants C independent of ε. In addition, we have

$$\rho_{0\varepsilon}(x) \to \rho_0(x), \quad \text{uniformly in } [0,1], \text{ as } \varepsilon \to 0,$$

and

$$u_{0\varepsilon}(x) \to u_0(x) \quad \text{in } L^{2n}(0,1). \tag{2.4}$$

Indeed, by the definition of Friedrichs mollifier and noticing $\theta \in (0,1)$, we get

$$\begin{aligned} \left|\left(\rho_{0\varepsilon}(0)\right)^{\gamma-\theta-1}\int_0^x \varphi_\varepsilon(y)\,dy\right| &= \left(\rho_{0\varepsilon}(0)\right)^{\gamma-\theta-1}\varepsilon\left|\int_0^{\frac{x}{\varepsilon}} \varphi(z)\,dz\right| \\ &= \left(\rho_{0\varepsilon}(0)\right)^{\gamma-1}\varepsilon^{1-\theta}\left(\frac{\varepsilon}{\rho_{0\varepsilon}(0)}\right)^\theta\left|\int_0^{\frac{x}{\varepsilon}} \varphi(z)\,dz\right| \\ &\leqslant C\varepsilon^{1-\theta} \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{2.5}$$

Similarly, we find

$$\left(\rho_{0\varepsilon}(1)\right)^{\gamma-\theta-1}\int_x^1 \varphi_\varepsilon(1-y)\,dy \to 0 \quad (\varepsilon \to 0). \tag{2.6}$$

On the other hand, we obtain

$$\left|(u_0 * j_\varepsilon)(0)\right|^{2n}\int_0^1 \varphi_\varepsilon^{2n}(x)\,dx \leqslant C\varepsilon \to 0 \quad (\varepsilon \to 0), \quad \text{for all } n \in \mathcal{N}. \tag{2.7}$$

Similarly, we discover

$$\left|(u_0 * j_\varepsilon)(1)\right|^{2n}\int_0^1 \varphi_\varepsilon^{2n}(1-x)\,dx \to 0 \quad (\varepsilon \to 0), \quad \text{for all } n \in \mathcal{N}. \tag{2.8}$$

Collecting (2.5)–(2.8) implies (2.4).

We consider the following approximation Navier–Stokes equations:

$$\begin{cases} \rho_{\varepsilon t} + \rho_\varepsilon^2 u_{\varepsilon x} = 0, \\ u_{\varepsilon t} + P(\rho_\varepsilon)_x = \big(\mu(\rho_\varepsilon)\rho_\varepsilon u_{\varepsilon x}\big)_x, \quad 0 < x < 1,\ t > 0, \end{cases} \tag{2.9}$$

with the boundary conditions

$$\rho_\varepsilon^\gamma(d,t) = \rho_\varepsilon^{1+\theta} u_{\varepsilon x}(d,t), \quad d = 0, 1, \tag{2.10}$$

and the initial data

$$(\rho_\varepsilon, u_\varepsilon)(x,0) = \big(\rho_{0\varepsilon}(x), u_{0\varepsilon}(x)\big), \quad 0 < x < 1. \tag{2.11}$$

For any $\theta \in (0,1)$, by the same argument as in [4], we can obtain a unique solution $(\rho_\varepsilon, u_\varepsilon)$ with $\rho_\varepsilon, \rho_{\varepsilon x}, \rho_{\varepsilon t}, \rho_{\varepsilon tx}, u_\varepsilon, u_{\varepsilon x}, u_{\varepsilon t}, u_{\varepsilon xx} \in C^{\beta,\beta/2}(\Omega_t)$ for some $\beta \in (0,1)$ and with $\rho_\varepsilon > 0$ in $\Omega_t := (0,1) \times (0,t)$.

In the sequel, the letter $C(C(T))$ will denote a generic positive constant independent of ε.

Lemma 2.1. *One has the following equalities*:

$$\rho_\varepsilon^\theta(x,t) + \theta \int_0^t \rho_\varepsilon^\gamma(x,s)\,ds = \big(\rho_{0\varepsilon}^\theta\big)(x) - \theta \int_0^x u_\varepsilon(y,t)\,dy + \theta \int_0^x u_{0\varepsilon}(y)\,dy, \tag{2.12}$$

$$\big[\rho_\varepsilon^\theta(x,t)\big]_x + \theta \int_0^t \big(\rho_\varepsilon^\gamma\big)_x(x,s)\,ds = \big(\rho_{0\varepsilon}^\theta\big)_x(x) - \theta u_\varepsilon(x,t) + \theta u_{0\varepsilon}(x), \tag{2.13}$$

$$\rho_\varepsilon(d,t) = \rho_{0\varepsilon}(d)\left(\frac{1}{(\gamma-\theta)t + \rho_{0\varepsilon}^{\theta-\gamma}(d)}\right)^{\frac{1}{\gamma-\theta}}, \quad d = 0, 1, \tag{2.14}$$

for $0 \leqslant x \leqslant 1$ *and* $0 \leqslant t \leqslant T$.

Proof. The proofs of (2.12)–(2.14) can be found in [8]. □

In the following, we will give some a priori estimates for $(\rho_\varepsilon(x,t), u_\varepsilon(x,t))$.

Lemma 2.2. *There hold*

$$\int_0^1 \big(u_\varepsilon^2 + \rho_\varepsilon^\gamma\big)\,dx + \int_0^t \int_0^1 \rho_\varepsilon^{1+\theta} u_{\varepsilon x}^2\,dx\,ds \leqslant C, \tag{2.15}$$

$$\int_0^1 \big[\big(\rho_\varepsilon^\theta\big)_x\big]^{2n}(x,t)\,dx \leqslant C, \tag{2.16}$$

$$\int_0^1 u_\varepsilon^{2n}\,dx + n(2n-1)\int_0^t \int_0^1 u_\varepsilon^{2n-2}\rho_\varepsilon^{1+\theta} u_{\varepsilon x}^2\,dx\,ds \leqslant C, \tag{2.17}$$

for any $\varepsilon \in (0,1)$, $t \in [0,T]$ *and* $n \in \mathcal{N}$,

Proof. There are detailed proofs of (2.15)–(2.17) in [5] and [8]. □

The following lemma gives the uniform upper bound of approximation density ρ_ε.

Lemma 2.3. *We have*

$$\rho_\varepsilon(x,t) \leqslant C, \quad (x,t) \in [0,1] \times [0,T]. \tag{2.18}$$

Proof. From (2.12), we deduce that

$$\rho_\varepsilon^\theta \leqslant \rho_{0\varepsilon}^\theta - \theta \int_0^x u_\varepsilon(y,t)\,dy + \theta \int_0^x u_{0\varepsilon}(y)\,dy.$$

Using Hölder's inequality and noticing (2.2) and (2.15), we obtain

$$\begin{aligned}\rho_\varepsilon^\theta &\leqslant \rho_{0\varepsilon}^\theta - \theta \int_0^x u_\varepsilon(y,t)\,dy + \theta \int_0^x |u_{0\varepsilon}|\,dy \\ &\leqslant \rho_{0\varepsilon}^\theta + C\left(\int_0^1 u_\varepsilon^2(x,t)\,dx\right)^{\frac{1}{2}} + C \leqslant C.\end{aligned}$$

The lemma follows. $\square$

The next lemma embodies some relation between the approximate density and velocity.

Lemma 2.4. *For any $\varepsilon_0 > 0$ and for any $0 < t \leqslant T$, we have*

$$\int_0^t \left\|\rho_\varepsilon^{\varepsilon_0} u_\varepsilon\right\|_\infty ds \leqslant C. \tag{2.19}$$

Proof. We observe that there holds for any $n \in \mathcal{N}$,

$$\int_0^t \left\|\rho_\varepsilon^{\varepsilon_0} u_\varepsilon\right\|_\infty ds = \int_0^t \left\|\rho_\varepsilon^{n\varepsilon_0} u_\varepsilon^n\right\|_\infty^{\frac{1}{n}} ds. \tag{2.20}$$

From the embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$, we obtain

$$\begin{aligned}\left\|\rho_\varepsilon^{n\varepsilon_0} u_\varepsilon^n\right\|_\infty &\leqslant \int_0^1 \left|\rho_\varepsilon^{n\varepsilon_0} u_\varepsilon^n\right| dx + \int_0^1 \left|\left(\rho_\varepsilon^{n\varepsilon_0} u_\varepsilon^n\right)_x\right| dx \\ &\leqslant \int_0^1 \rho_\varepsilon^{n\varepsilon_0} |u_\varepsilon|^n\,dx + n\varepsilon_0 \int_0^1 \rho_\varepsilon^{n\varepsilon_0 - 1} |\rho_{\varepsilon x}||u_\varepsilon|^n\,dx + n\int_0^1 \rho_\varepsilon^{n\varepsilon_0} |u_\varepsilon|^{n-1} |u_{\varepsilon x}|\,dx.\end{aligned}$$

Using (2.17)–(2.18) and Cauchy–Schwartz inequality, we have

$$\begin{aligned}\left\|\rho_\varepsilon^{n\varepsilon_0} u_\varepsilon^n\right\|_\infty &\leqslant \frac{1}{2}\int_0^1 \rho_\varepsilon^{2n\varepsilon_0}\,dx + \int_0^1 u_\varepsilon^{2n}\,dx + \frac{n\varepsilon_0}{2}\int_0^1 \rho_\varepsilon^{2(n\varepsilon_0-1)} \rho_{\varepsilon x}^2\,dx + \frac{n}{2}\int_0^1 \rho_\varepsilon^{2n\varepsilon_0} u_\varepsilon^{2(n-1)} u_{\varepsilon x}^2\,dx + C \\ &\leqslant C + \frac{n\varepsilon_0}{2}\int_0^1 \rho_\varepsilon^{2(n\varepsilon_0-1)} \rho_{\varepsilon x}^2\,dx + \frac{n}{2}\int_0^1 \rho_\varepsilon^{2n\varepsilon_0} u_\varepsilon^{2(n-1)} u_{\varepsilon x}^2\,dx.\end{aligned} \tag{2.21}$$

This leads to

$$\begin{aligned}\int_0^t \left\|\rho_\varepsilon^{\varepsilon_0} u_\varepsilon\right\|_\infty ds &\leqslant \int_0^t \left(C + \frac{n\varepsilon_0}{2}\int_0^1 \rho_\varepsilon^{2(n\varepsilon_0-1)} \rho_{\varepsilon x}^2\,dx + \frac{n}{2}\int_0^1 \rho_\varepsilon^{2n\varepsilon_0} u_\varepsilon^{2(n-1)} u_{\varepsilon x}^2\,dx\right)^{\frac{1}{n}} ds \\ &\leqslant C + C\int_0^t\int_0^1 \rho_\varepsilon^{2(n\varepsilon_0-1)} \rho_{\varepsilon x}^2\,dx\,ds + C\int_0^t\int_0^1 \rho_\varepsilon^{2n\varepsilon_0} u_\varepsilon^{2(n-1)} u_{\varepsilon x}^2\,dx\,ds,\end{aligned} \tag{2.22}$$

where we used (2.20) and Young's inequality. From Lemmas 2.2 and 2.3, we find

$$\int_0^1 \rho_\varepsilon^{2(n\varepsilon_0-1)}\rho_{\varepsilon x}^2\,dx \leqslant C,$$
$$\int_0^t\int_0^1 \rho_\varepsilon^{2n\varepsilon_0}u_\varepsilon^{2(n-1)}u_{\varepsilon x}^2\,dx\,ds \leqslant C, \tag{2.23}$$

since there exists $n \in \mathcal{N}$, such that $2n\varepsilon_0 \geqslant 1+\theta$. Combing (2.22) and (2.23), we complete the proof of the lemma. □

Lemma 2.5. *Let $\varepsilon_0 > 0$ and $n \in \mathcal{N}$ such that $\theta + \frac{\varepsilon_0}{2n} \leqslant \gamma$. Then for any $0 < t \leqslant T$, there holds*

$$\int_0^t \left[\left(\rho_\varepsilon^{\theta+\frac{\varepsilon_0}{2n}}\right)_x\right]^{2n} ds \leqslant C(T). \tag{2.24}$$

Proof. By simple calculations, we arrive at

$$\int_0^t \left[\left(\rho_\varepsilon^{\theta+\frac{\varepsilon_0}{2n}}\right)_x\right]^{2n} ds = \frac{(\theta+\frac{\varepsilon_0}{2n})^{2n}}{\theta^{2n}}\int_0^t \rho_\varepsilon^{\varepsilon_0}\left[\left(\rho_\varepsilon^\theta\right)_x\right]^{2n} ds. \tag{2.25}$$

Noticing (2.13), we obtain

$$\int_0^t \rho_\varepsilon^{\varepsilon_0}\left[\left(\rho_\varepsilon^\theta\right)_x\right]^{2n} ds = \int_0^t \rho_\varepsilon^{\varepsilon_0}\left[\left(\rho_{0\varepsilon}^\theta\right)_x - \theta u_\varepsilon + \theta u_{0\varepsilon} - \theta\int_0^s \left(\rho_\varepsilon^\gamma\right)_x d\tau\right]^{2n} ds.$$

Using Cauchy–Schwartz inequality, we discover

$$\begin{aligned}
\int_0^t \rho_\varepsilon^{\varepsilon_0}\left[\left(\rho_\varepsilon^\theta\right)_x\right]^{2n} ds &\leqslant C\int_0^t \rho_\varepsilon^{\varepsilon_0}\left(\left(\rho_{0\varepsilon}^\theta\right)_x^{2n} + u_\varepsilon^{2n} + u_{0\varepsilon}^{2n} + \int_0^s \left[\left(\rho_\varepsilon^\gamma\right)_x\right]^{2n} d\tau\right) ds \\
&\leqslant C\int_0^t \rho_\varepsilon^{2\varepsilon_0} ds + C\int_0^t \left(\rho_{0\varepsilon}^\theta\right)_x^{4n} ds + C\int_0^t u_{0\varepsilon}^{2n}\,ds \\
&\quad + C\int_0^t \left\|\rho_\varepsilon^{\varepsilon_0}u_\varepsilon^{2n}\right\|_\infty ds + C\int_0^t \rho_\varepsilon^{\varepsilon_0}\int_0^s \left[\left(\rho_\varepsilon^\gamma\right)_x\right]^{2n} d\tau\,ds \\
&\leqslant C\int_0^t \rho_\varepsilon^{2\varepsilon_0}\,ds + \int_0^t \left(\rho_{0\varepsilon}^\theta\right)_x^{4n} ds + \int_0^t u_{0\varepsilon}^{2n}\,ds \\
&\quad + C\int_0^t \left\|\rho_\varepsilon^{\frac{\varepsilon_0}{2n}}u_\varepsilon\right\|_\infty^{2n} ds + C\int_0^t \rho_\varepsilon^{\varepsilon_0}\int_0^s \left[\left(\rho_\varepsilon^\gamma\right)_x\right]^{2n} d\tau\,ds.
\end{aligned}$$

Following (2.2)–(2.3) and (2.18)–(2.19), we get

$$\int_0^t \rho_\varepsilon^{\varepsilon_0}\left[\left(\rho_\varepsilon^\theta\right)_x\right]^{2n} ds \leqslant C + C\int_0^t\int_0^s \left[\left(\rho_\varepsilon^\gamma\right)_x\right]^{2n} d\tau\,ds. \tag{2.26}$$

By (2.25) and (2.26), we have

$$\begin{aligned}\int_0^t \big[\big(\rho_\varepsilon^{\theta+\frac{\varepsilon_0}{2n}}\big)_x\big]^{2n}\,ds &\leqslant C\int_0^t \rho_\varepsilon^{\varepsilon_0}\big[\big(\rho^\theta\big)_x\big]^{2n}\,ds\\
&\leqslant C + C\int_0^t\int_0^s \big[\big(\rho_\varepsilon^\gamma\big)_x\big]^{2n}\,d\tau\,ds\\
&\leqslant C + C\int_0^t\int_0^s \big|\rho_\varepsilon^{2n[\gamma-(\theta+\frac{\varepsilon_0}{2n})]}\big|\big[\big(\rho_\varepsilon^{\theta+\frac{\varepsilon_0}{2n}}\big)_x\big]^{2n}\,d\tau\,ds\\
&\leqslant C + C\int_0^t\int_0^s \big[\big(\rho_\varepsilon^{\theta+\frac{\varepsilon_0}{2n}}\big)_x\big]^{2n}\,d\tau\,ds, \qquad (2.27)\end{aligned}$$

which and Gronwall's inequality imply (2.24). Thus the proof of the lemma is completed. □

Lemma 2.6. *For any $\alpha \geqslant 1$ and $k > \frac{2}{1-\theta}$ $(0<\theta<1)$, there holds*

$$\int_0^1 \left(\frac{\rho_{0\varepsilon}^k(x)}{\rho_\varepsilon(x,t)}\right)^\alpha dx \leqslant C, \quad \forall 0 < t \leqslant T. \qquad (2.28)$$

Proof. We replace ρ_ε by $v_\varepsilon = \frac{1}{\rho_\varepsilon}$ in the first equation of (2.9), and find

$$\big(v_\varepsilon^\alpha\big)_t = \alpha v_\varepsilon^{\alpha-1} u_{\varepsilon x}. \qquad (2.29)$$

Multiplying (2.29) by $\rho_{0\varepsilon}^{k\alpha}(x)$ and integrating it over $[0,1]\times[0,t]$, we obtain

$$\int_0^1 \rho_{0\varepsilon}^{k\alpha} v_\varepsilon^\alpha\,dx = \alpha\int_0^t\int_0^1 v_\varepsilon^{\alpha-1}u_{\varepsilon x}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + \int_0^1 \rho_{0\varepsilon}^{k\alpha} v_{0\varepsilon}^\alpha\,dx.$$

Integrating by parts for the second term of the above integral equality leads to

$$\begin{aligned}&\alpha\int_0^t\int_0^1 v_\varepsilon^{\alpha-1}u_{\varepsilon x}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&\quad= \alpha\int_0^t \big(v_\varepsilon^{\alpha-1}\rho_{0\varepsilon}^{k\alpha}u_\varepsilon\big)(x,s)\Big|_0^1\,ds - \alpha\int_0^t\int_0^1 u_\varepsilon\big(v_\varepsilon^{\alpha-1}\rho_{0\varepsilon}^{k\alpha}\big)_x\,dx\,ds\\
&\quad= \alpha\int_0^t \big(v_\varepsilon^{\alpha-1}\rho_{0\varepsilon}^{k\alpha}u_\varepsilon\big)(x,s)\Big|_0^1\,ds - \alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha-2}v_{\varepsilon x}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds - k\alpha^2\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha-1}\rho_{0\varepsilon}^{k\alpha-1}\rho_{0\varepsilon x}\,dx\,ds\\
&\quad= I_1 + I_2 + I_3.\end{aligned}$$

By (2.14) and (2.19) (taking $\varepsilon_0 = 1$), we get

$$\begin{aligned}I_1 &= \int_0^t v_\varepsilon^\alpha\rho_{0\varepsilon}^{k\alpha}\rho_\varepsilon u_\varepsilon(x,s)\Big|_0^1\,ds\\
&= \int_0^t \big(v_\varepsilon^\alpha\rho_{0\varepsilon}^{k\alpha}\rho_\varepsilon u_\varepsilon\big)(1,s) - \big(v_\varepsilon^\alpha\rho_{0\varepsilon}^{k\alpha}\rho_\varepsilon u_\varepsilon\big)(0,s)\,ds\end{aligned}$$

$$\leqslant C\int_0^t \|\rho_\varepsilon u_\varepsilon\|_\infty\,ds \leqslant C. \tag{2.30}$$

Recalling (2.13), we obtain

$$\begin{aligned}
I_2 &= -\alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha-2} v_{\varepsilon x}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&= \frac{\alpha(\alpha-1)}{\theta}\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha+\theta-1}\big(\rho_\varepsilon^\theta\big)_x\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&= \frac{\alpha(\alpha-1)}{\theta}\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha+\theta-1}\big(\rho_{0\varepsilon}^\theta\big)_x\rho_{0\varepsilon}^{k\alpha}\,dx\,ds - \alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&\quad + \alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon u_{0\varepsilon} v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds - \alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\int_0^s\big(\rho_\varepsilon^\gamma\big)_x\,d\tau\,dx\,ds,
\end{aligned}$$

and by Young's inequality, we find

$$\begin{aligned}
I_2 &\leqslant \frac{3}{4}\alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds - \alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&\quad + \alpha(\alpha-1)\int_0^t\int_0^1 v_\varepsilon^{\alpha+\theta-1}A_\varepsilon\rho_{0\varepsilon}^{k\alpha}\,dx\,ds\\
&\leqslant -\frac{1}{4}\alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + \alpha(\alpha-1)\int_0^t\int_0^1 v_\varepsilon^{\alpha+\theta-1}A_\varepsilon\rho_{0\varepsilon}^{k\alpha}\,dx\,ds
\end{aligned}$$

where $A_\varepsilon = (\frac{(\rho_{0\varepsilon}^\theta(x))_x}{\theta})^2 + u_{0\varepsilon}^2(x) + (\int_0^s(\rho_\varepsilon^\gamma)_x\,d\tau)^2$. From (2.2), (2.3) and (2.24), there holds $A_\varepsilon \leqslant C$ for $(x,t)\in[0,1]\times[0,T]$. Therefore,

$$I_2 \leqslant -\frac{1}{4}\alpha(\alpha-1)\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + C\alpha(\alpha-1)\int_0^t\int_0^1 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds. \tag{2.31}$$

By (H2) and Young's inequality, we arrive at

$$\begin{aligned}
I_3 &= -k\alpha^2\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\theta+\alpha-1}\rho_\varepsilon^\theta\rho_{0\varepsilon}^{k\alpha-1}\rho_{0\varepsilon x}\,dx\,ds\\
&\leqslant C(k,\alpha)\int_0^t\int_0^1 u_\varepsilon v_\varepsilon^{\theta+\alpha-1}\rho_{0\varepsilon}^{k\alpha-1}\,dx\,ds\\
&\leqslant \frac{\alpha(\alpha-1)}{8}\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + C(k,\alpha)\int_0^t\int_0^1 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha-2}\,dx\,ds
\end{aligned}$$

$$\leqslant \frac{\alpha(\alpha-1)}{8}\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + C(k,\alpha)\int_0^t\int_0^1 \left(v_\varepsilon\rho_{0\varepsilon}^k\right)^{\alpha+\theta-1}\rho_{0\varepsilon}^{k(1-\theta)-2}\,dx\,ds.$$

Since $k > \frac{2}{1-\theta}$, we obtain

$$I_3 \leqslant \frac{\alpha(\alpha-1)}{8}\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds + C(k,\alpha)\int_0^t\int_0^1 \left(v_\varepsilon\rho_{0\varepsilon}^k\right)^{\alpha+\theta-1}\,dx\,ds. \tag{2.32}$$

Using the above estimates (2.30)–(2.32), we get

$$\int_0^1 \left(\rho_{0\varepsilon}^k v_\varepsilon\right)^\alpha dx + \frac{\alpha(\alpha-1)}{8}\int_0^t\int_0^1 u_\varepsilon^2 v_\varepsilon^{\alpha+\theta-1}\rho_{0\varepsilon}^{k\alpha}\,dx\,ds \leqslant C + C(k,\alpha)\int_0^t\int_0^1 \left(\rho_{0\varepsilon}^k v_\varepsilon\right)^{\alpha+\theta-1}\,dx\,ds,$$

which implies Lemma 2.6 by Gronwall's inequality. □

Lemma 2.7. *For any $k > \max\{\frac{2(1+\theta)}{1-\theta}, 1 + \frac{2}{1-\theta}\}$, there holds*

$$\rho_\varepsilon(x,t) \geqslant C\rho_{0\varepsilon}^k(x), \quad (x,t)\in[0,1]\times[0,T].$$

Proof. From the imbedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$, we have

$$\begin{aligned}
\frac{\rho_{0\varepsilon}^k}{\rho_\varepsilon} &\leqslant \int_0^1 \frac{\rho_{0\varepsilon}^k}{\rho_\varepsilon}\,dx + \int_0^1 \left|\left(\frac{\rho_{0\varepsilon}^k}{\rho_\varepsilon}\right)_x\right| dx \\
&\leqslant C + \int_0^1 \left|\frac{k\rho_{0\varepsilon}^{k-1}\rho_{0\varepsilon x}}{\rho_\varepsilon}\right| dx + \int_0^1 \left|\frac{\rho_{0\varepsilon}^k\rho_{\varepsilon x}}{\rho_\varepsilon^2}\right| dx \\
&\leqslant C + \int_0^1 \left|\frac{k\rho_{0\varepsilon}^{k-1}\rho_{0\varepsilon x}}{\rho_\varepsilon}\right| dx + \frac{1}{\theta}\int_0^1 \left|\frac{\rho_{0\varepsilon}^k(\rho_\varepsilon^\theta)_x}{\rho_\varepsilon^{1+\theta}}\right| dx.
\end{aligned}$$

Using the inequality $ab \leqslant \frac{1}{2}(a^2+b^2)$ $(a,b\in R)$, we obtain

$$\begin{aligned}
\frac{\rho_{0\varepsilon}^k}{\rho_\varepsilon} &\leqslant C + C\int_0^1 \frac{\rho_{0\varepsilon}^{k-1}}{\rho_\varepsilon}\,dx + \frac{1}{2\theta}\int_0^1 \left[\left(\frac{\rho_{0\varepsilon}^k}{\rho_\varepsilon^{1+\theta}}\right)^2 + \left(\rho_\varepsilon^\theta\right)_x^2\right] dx \\
&\leqslant C, \quad \text{by (2.3), (2.16) and (2.28).}
\end{aligned}$$

This completes the proof of the lemma. □

2.2. *The limit process of the approximation solution*

By virtue of Lemmas 2.1–2.7 and (2.9), we obtain

$$C_1\rho_{0\varepsilon}^k(x) \leqslant \rho_\varepsilon(x,t) \leqslant C_2, \quad (x,t)\in[0,1]\times[0,T],$$

$$\int_0^1 u_\varepsilon^{2n}\,dx + \int_0^1 \left[\left(\rho_\varepsilon^\theta\right)_x\right]^{2n} dx \leqslant C, \quad t\in[0,T], \tag{2.33}$$

$$\int_0^t\int_0^1 \left(\rho_{\varepsilon t}\right)^2 dx\,dt + \int_0^t\int_0^1 \rho_\varepsilon^{1+\theta}u_{\varepsilon x}^2\,dx\,dt \leqslant C, \tag{2.34}$$

$$\int_0^T \int_\delta^{x_0+\delta} u_{\varepsilon x}^2\,dx\,ds + \int_0^T \int_{x_0+\delta}^{1-\delta} (u_{\varepsilon x})^2\,dx\,ds \leqslant C_\delta,$$

where $n \in \mathcal{N}$, $\delta \in (0, \frac{1}{2})$, C and C_δ are positive constants independent of ε. Then we can extract a subsequence of $(\rho_\varepsilon, u_\varepsilon)$, still denoted by $(\rho_\varepsilon, u_\varepsilon)$, such that as $\varepsilon \to 0$,

$$u_\varepsilon \rightharpoonup u \quad \text{weak-}* \text{ in } L^\infty\big([0,T]; L^{2n}[0,1]\big), \tag{2.35}$$

$$u_{\varepsilon x} \rightharpoonup u_x \quad \text{weakly in } L^2\big(0,T; L^2\big([\delta, x_0-\delta] \cup [x_0+\delta, 1-\delta]\big)\big) \tag{2.36}$$

and

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{weak-}* \text{ in } L^\infty\big([0,T]; W^{1,2n}[0,1]\big),$$
$$\rho_{\varepsilon t} \rightharpoonup \rho_t \quad \text{weakly in } L^2\big([0,T]; L^2[0,1]\big).$$

From the embedding theorem $W^{1,2n}(0,1) \hookrightarrow C^{1-\frac{1}{2n}}[0,1]$, we have

$$\big|\rho_\varepsilon(x_1,t) - \rho_\varepsilon(x_2,t)\big| \leqslant C|x_1-x_2|^{1-\frac{1}{2n}}, \quad \forall x_1, x_2 \in [0,1],\ t \in [0,T]. \tag{2.37}$$

From Lions–Aubin's lemma and (2.34), we obtain

$$\begin{aligned}
\big\|\rho_\varepsilon(t_1) - \rho_\varepsilon(t_2)\big\|_{L^\infty} &\leqslant \delta\big\|\rho_\varepsilon(t_1) - \rho_\varepsilon(t_2)\big\|_{W^{1,p}} + C_\delta\big\|\rho_\varepsilon(t_1) - \rho_\varepsilon(t_2)\big\|_{L^2} \\
&\leqslant 2\delta\big\|\rho_\varepsilon(t)\big\|_{W^{1,p}} + C_\delta|t_1-t_2|^{\frac{1}{2}}\|\rho_{\varepsilon t}\|_{L^2([0,T];L^2)} \\
&\leqslant C\delta + C_\delta|t_1-t_2|^{\frac{1}{2}}.
\end{aligned} \tag{2.38}$$

Since (2.37), (2.38) and the triangle inequality, we deduce that $\rho_\varepsilon(x,t)$ is equicontinuous on $[0,1]\times[0,t]$. Thus, we can extract a subsequence by Arzelà–Ascoli's lemma and a diagonal process if necessary, such that

$$\rho_\varepsilon(x,t) \to \rho(x,t) \quad \text{strongly in } C^0\big([0,1]\times[0,T]\big). \tag{2.39}$$

Furthermore, we discover from (2.38)

$$\rho \in C^{\frac{1}{2}}\big([0,T]; L^2(0,1)\big).$$

Recalling (2.33)–(2.34) and $0<\theta<1$, we have

$$\begin{aligned}
\int_0^t \int_0^1 \big(\rho_\varepsilon^{\frac{1+\theta}{2}} u_\varepsilon\big)_x^2\,dx\,ds &= \int_0^t \int_0^1 \big[\rho_\varepsilon^{\frac{1+\theta}{2}} u_{\varepsilon x} + \big(\rho_\varepsilon^{\frac{1+\theta}{2}}\big)_x u_\varepsilon\big]^2\,dx\,ds \\
&\leqslant 2\int_0^t \int_0^1 \big(\rho_\varepsilon^{1+\theta} u_{\varepsilon x}^2 + \big(\rho_\varepsilon^{\frac{1+\theta}{2}}\big)_x^2 u_\varepsilon^2\big)\,dx\,ds \\
&\leqslant C.
\end{aligned} \tag{2.40}$$

This shows that, as $\varepsilon \to 0$,

$$\big(\rho_\varepsilon^{\frac{1+\theta}{2}} u_\varepsilon\big)_x \rightharpoonup \big(\rho^{\frac{1+\theta}{2}} u\big)_x \quad \text{weakly in } L^2\big([0,1]\times[0,T]\big). \tag{2.41}$$

Multiplying (2.9) by $\varphi \in C_0^\infty((0,1)\times(0,T))$, we obtain

$$\int_0^t \int_0^1 \big(u_\varepsilon \varphi_t + \big(\rho_\varepsilon^\gamma - \rho_\varepsilon^{1+\theta} u_{\varepsilon x}\big)\varphi_x\big)\,dx\,ds + \int_0^1 u_{0\varepsilon}(x)\varphi(x,0)\,dx = 0.$$

It is easy to see that, as $\varepsilon \to 0$,

$$\int_0^t\int_0^1 u_\varepsilon\varphi_t\,dx\,ds \to \int_0^t\int_0^1 u\varphi_t\,dx\,ds,$$

$$\int_0^1 u_{0\varepsilon}(x)\varphi(x,0)\,dx \to \int_0^1 u_0(x)\varphi(x,0)\,dx.$$

We claim that, as $\varepsilon \to 0$,

$$\int_0^t\int_0^1 \big(\rho_\varepsilon^\gamma - \rho_\varepsilon^{1+\theta}u_{\varepsilon x}\big)\varphi_x\,dx\,ds \to \int_0^t\int_0^1 \big(\rho^\gamma - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds. \tag{2.42}$$

Firstly it yields

$$\begin{aligned} A_\varepsilon &\triangleq \int_0^t\int_0^1 \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds \\ &= \int_0^T\int_0^\delta \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds + \int_0^T\int_\delta^{x_0-\delta} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds \\ &\quad + \int_0^T\int_{x_0-\delta}^{x_0+\delta} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds + \int_0^T\int_{x_0+\delta}^{1-\delta} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds \\ &\quad + \int_0^T\int_{1-\delta}^{1} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\varphi_x\,dx\,ds \\ &\triangleq A_{\varepsilon 1}^\delta + A_{\varepsilon 2}^\delta + A_{\varepsilon 3}^\delta + A_{\varepsilon 4}^\delta + A_{\varepsilon 5}^\delta. \end{aligned}$$

Using (2.40)–(2.41) and Hölder's inequality, we find

$$\begin{aligned} \big|A_{\varepsilon 1}^\delta\big| &= \left|\int_0^T\int_0^\delta \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\phi_x\,dx\,ds\right| \\ &\leqslant C\left(\int_0^t\int_0^1 \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)^2\,dx\,ds\right)^{\frac12}(T\delta)^{\frac12} \\ &\leqslant C\delta^{\frac12}. \end{aligned}$$

Similarly we have

$$\big|A_{\varepsilon 3}^\delta\big| = \left|\int_0^T\int_{x_0-\delta}^{x_0+\delta} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\phi_x\,dx\,ds\right| \leqslant C\delta^{\frac12},$$

$$\big|A_{\varepsilon 5}^\delta\big| = \left|\int_0^T\int_{1-\delta}^{1} \big(\rho_\varepsilon^{1+\theta}u_{\varepsilon x} - \rho^{1+\theta}u_x\big)\phi_x\,dx\,ds\right| \leqslant C\delta^{\frac12}.$$

Thus we obtain

$$\big|A_{\varepsilon 1}^\delta\big| + \big|A_{\varepsilon 3}^\delta\big| + \big|A_{\varepsilon 5}^\delta\big| \leqslant C\delta^{1/2},$$

and hence, there exists $\delta_0 > 0$ for any $\eta > 0$ such that

$$\left|A_{\varepsilon 1}^{\delta_0}\right| + \left|A_{\varepsilon 3}^{\delta_0}\right| + \left|A_{\varepsilon 5}^{\delta_0}\right| < \eta/2.$$

Furthermore, it follows from (2.34) and (2.38) that there exists $\varepsilon_0 > 0$ for the above fixed $\delta_0 > 0$, such that

$$\left|A_{\varepsilon 2}^{\delta_0}\right| + \left|A_{\varepsilon 4}^{\delta_0}\right| < \eta/2, \quad \forall \varepsilon < \varepsilon_0.$$

Thus the claim (2.42) holds, and we have

$$\int_0^t \int_0^1 \left(u\varphi_t + \left(\rho^\gamma - \rho^{1+\theta} u_x\right)\varphi_x\right) dx\, ds + \int_0^1 u_0(x)\varphi(x,0)\, dx = 0.$$

To complete the proof of Theorem 1.1, it only remains to show that $\rho(1,t) = \rho(0,t) = 0$. From (2.11), we have

$$\rho_\varepsilon^\theta(1,t) \leqslant \rho_{0\varepsilon}^\theta(1) - \int_0^1 \left(u_\varepsilon(x,t) - u_{0\varepsilon}\right)(x)\, dx = \rho_{0\varepsilon}^\theta(1).$$

Then $\rho(1,t) = 0$ by taking limit to the above inequality as $\varepsilon \to 0$. Similarly, $\rho(0,t) = 0$. Thus (ρ, u) is a weak solution, and the proof of Theorem 1.1 is completed.

References

[1] T. Yang, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys. 230 (2002) 329–363.

[2] S.W. Vong, T. Yang, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum (II), J. Differential Equations 192 (2003) 475–501.

[3] D.Y. Fang, T. Zhang, Compressible Navier–Stokes equations with vacuum state in the case of general pressure law, Math. Methods Appl. Sci. 29 (2006) 1081–1106.

[4] D.Y. Fang, T. Zhang, Compressible Navier–Stokes equations with vacuum state in one dimension, Commun. Pure Appl. Anal. 3 (2004) 675–694.

[5] S. Jiang, Z.P. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier–Stokes equations with density-dependent viscosity, Methods Appl. Anal. 12 (3) (2005) 239–251.

[6] M. Okada, Š. Matušů-Nečasová, T. Makino, Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, Ann. Univ. Ferrara Sez. VII (N.S.) 48 (2002) 1–20.

[7] M. Okada, Free boundary problem for one-dimensional motions of compressible gas and vacuum, Japan J. Indust. Appl. Math. 21 (2004) 109–128.

[8] T. Yang, Z.A. Yao, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Partial Differential Equations 26 (5–6) (2001) 965–981.

[9] T. Yang, H.J. Zhao, A vacuum problem for the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity, J. Differential Equations 184 (1) (2002) 163–184.

[10] T. Luo, Z. Xin, T. Yang, Interface of compressible Navier–Stokes equations with vacuum, SIAM J. Math. Anal. 31 (2000) 1175–1191.

[11] T.P. Liu, Z. Xin, T. Yang, Vacuum states of compressible flow, Discrete Contin. Dyn. Syst. 4 (1998) 1–32.

[12] D. Hoff, D. Serre, The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow, SIAM J. Appl. Math. 51 (1991) 887–898.

[13] H. Kim, Global existence of strong solutions of the Navier–Stokes equations for one-dimensional isentropic compressible fluids, unpublished result.

[14] H.J. Choe, H. Kim, Global existence of the radically symmetric solutions of the Navier–Stokes equations for the isentropic compressible fluids, Math. Methods Appl. Sci. 2005 (28) (2005) 1–28.

[15] Z. Xin, Blow up of smooth solutions to the compressible Navier–Stokes equation with compact density, Comm. Pure Appl. Math. 51 (1998) 229–240.

[16] E. Beckere, Gasdynamik, Teubner Verlag, Stuttgart, 1966.

[17] Y.B. Zeldovich, Y.P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, vol. II, Academic Press, New York, 1967.

[18] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2, Oxford Lecture Ser. Math. Appl., vol. 10, 1998.