Long-time asymptotic expansion for the damped semilinear wave equation

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Abstract

The first initial-boundary value problem is considered for the damped semilinear wave equation with the quadratic nonlinearity. For small initial data and homogeneous boundary conditions its solution is constructed in the form of a series in the eigenfunctions of the Laplace operator. The long-time asymptotic expansion is obtained which shows the nonlinear effects of amplitude and frequency multiplication. The same results hold for the admissible initial data that are not small.

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1. Introduction

We are concerned with studying the long-time behavior of solutions of the semilinear damped wave equation

\[ u_{tt} + 2bu_t - \Delta u = f(u) \]  

(1.1)

with \( b = \text{const} > 0 \). More precisely, we propose a certain method of constructing solutions of dissipative semilinear evolution equations in bounded domains and obtaining their long-time asymptotic expansions. We give its application to (1.1) in a three-dimensional ball with \( f(u) = u^2 \). In principle, power nonlinearities of the type of \( u^p \) with integer \( p \) can be considered as well. We shall limit ourselves...
to a second-order expansion, although our technique permits one to calculate the higher-order terms.

Since the literature on the nonlinear wave equations is extensive, we shall mention below only some results pertaining to the problem in question. Cauchy problems for (1.1) have been studied in [7–9,16,18,19] (see also references therein). Feireisl [8] considered (1.1) with \( f(u) = -|u|^{p-1}u \) and proved that for long times strong global-in-time solutions of the initial value problem split into a finite number of travelling waves. For a more general nonlinearity he established in [7] that solutions of the Cauchy problem, for which the energy remains bounded and sufficiently small, are global and converge to a spatial shift of a ground state as \( t \to \infty \). Keller [19] examined the question of stability for the Cauchy problem for (1.1) in \( \mathbb{R}^N \). On the other hand, Levine [20, Part IV] showed that solutions corresponding to certain large initial data with negative energy blow up in finite time. Using energy methods Feireisl [9] studied the long-time behavior of the strong solutions of the equation
\[
 u_{tt} + 2bu_t - \Delta u + \mu u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t > 0,
\]
with \( b, \mu > 0, \quad N \geq 3, \) and suitable initial data.

Todorova and Yordanov [28] examined the Cauchy problem for the equation
\[
 u_{tt} + u_t - \Delta u = |u|^p, \quad x \in \mathbb{R}^N, \quad t > 0,
\]
with small initial data. Their main goal was to study the critical exponent \( p_c(N) \) defined by the following property. If \( p > p_c(N) \), then all the small data solutions are global, while if \( 1 < p < p_c(N) \), then all solutions with the positive average of the initial data blow up in finite time regardless of the smallness of the data. For the damped wave equation \( p_c(N) = 1 + 2/N \), the same as for the nonlinear heat equation \( u_t - \Delta u = |u|^p \).

Yang [16] considered the initial value problem for the singularly perturbed equation
\[
 \delta u_{tt} + u_t - \Delta u = f(u, \nabla u), \quad x \in \mathbb{R}^N, \quad t > 0,
\]
\[
 u(x, 0) = \epsilon u_0(x), \quad u_t(x, 0) = \epsilon u_1(x), \quad x \in \mathbb{R}^N,
\]
with small initial data and established the existence and the life span of the classical solutions of this problem. He also proved the weak convergence of the solution to that of the degenerate problem (with \( \delta = 0 \)). Milani [22] examined the Cauchy problem for the equation
\[
 \delta u_{tt} + u_t - \sum_{i,j} a_{ij}(\nabla u)\partial_i\partial_j u = 0
\]
in \( \mathbb{R}^N \). By means of the energy methods he proved the existence of such \( \delta_0 > 0 \) that for each \( \delta \leq \delta_0 \) the problem admits a unique classical solution \( u_\delta \) in the space \( X(T) = \bigcap_{j=0}^{s+2} C^j([0, T], H^{s+2-j}) \), \( s > N/2 + 1 \). He proved that \( u_\delta \) converges weakly to the solution of the degenerate problem.
Introducing scaling variables and using energy estimates Gallay and Raugel [12] computed the long-time asymptotic expansion of small solutions of the nonlinear damped equation
\[ \varepsilon u_{tt} + u_t = \left( a(x)u_x \right)_x + N(u, u_x, u_t), \quad x \in \mathbb{R}, \ t > 0, \]
under the assumption that the diffusion coefficient \( a(x) \) has positive limits \( a_\pm \) as \( x \to \pm \infty \). The authors showed that this expansion is determined, up to the second order, by a linear parabolic equation which depends on the limiting values \( a_\pm \).

In the paper [13] they considered (1.1) with \( f(u) = u - u^2 \). Such an equation appears in McKean’s model for particles that undergo a binary branching process and perform the Brownian motion.

Karch [18] studied the long-time behavior of solutions to the generalized damped wave equation
\[ u_{tt} + Au_t + \nu Bu + F(x, t, u, u_t \nabla u) = 0 \]
and obtained sufficient conditions for the asymptotics to be self-similar.

Now we pass to initial-boundary value problems. Long and Pham [24] studied the spatially 1D mixed problem
\[
\begin{align*}
&u_{tt} - u_{xx} + f(u, u_t) = 0, \quad x \in (0, 1), \ t \in (0, T), \\
&u_x(0, t) - hu(0, t) = g(t), \quad u(1, t) = 0, \\
&u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),
\end{align*}
\]
where \( h = \text{const} > 0 \). Having imposed some restrictions on \( f(u, u_t) \), they provided sufficient conditions for the local existence and uniqueness of the solutions and generalized the earlier results of Ang and Pham [5] valid for \( h = 0 \) and \( f(u, u_t) = |u_t|^\alpha \text{sgn}(u_t) \), \( 0 < \alpha < 1 \). Long and Pham used Galerkin’s method and energy inequalities in their considerations.

Georgiev and Todorova [14] examined the first mixed problem for the wave equation with nonlinear damping
\[ u_{tt} - \Delta u + u_t |u_t|^{m-1} = u|u|^{p-1}, \quad x \in \Omega, \ t > 0, \quad (1.2) \]
where \( p, m < 1 \), and \( \Omega \in \mathbb{R}^N, \ N \geq 1 \), is a bounded domain with a smooth boundary. For \( 1 < p \leq m \) the authors proved a global existence theorem for large initial data. For \( 1 < m < p \) they established a blow up result for sufficiently large initial data.

Biazuti [2] studied the global existence and long-time behavior of weak solutions for a class of abstract Cauchy problems related to the equation
\[ u_{tt} - \sum_i \partial_i \left( \sigma \left( \partial_i u \right) \right) - \Delta u_t + |u_t|^{\alpha} \text{sgn}(u_t) = F(x, t), \quad x \in \Omega, \ t > 0, \]
where \( \sigma, \alpha < 1 \).
where $\Omega \in \mathbb{R}^N$ is a bounded domain with a smooth boundary. Her result generalizes the earlier ones of [5,33,34]. A more specific problem with a different nonlinear term $g(u)$ was considered by Ma and Soriano [21], namely

$$u_{tt} - \text{div}(|\nabla u|^{n-2}\nabla u) - \Delta u_t + g(u) = F(x,t), \quad x \in \Omega, \ t > 0,$$

with the boundary and initial conditions (1.2), where $\Omega \in \mathbb{R}^N$, $N \geq 2$, is a bounded domain, $g(u)$ grows like $\exp(|u|^{n/(n-1)})$ and satisfies the condition $g(u)u \geq 0$. Existence, uniqueness, and time decay estimates were obtained in this paper.

Returning from nonlinear dissipation and strong damping to the case of weak dissipation of (1.1) we must note that the existence and uniqueness of solutions of the first mixed problem for this equation (with some restrictions on $f(u)$) were proved in the monograph of Babine and Vishik [1] by means of the Galerkin method. The properties of the linear semigroups associated with this problem (uniform continuity, boundedness, absorbing sets) were also established there. A detailed analysis of inertial manifolds and global attractors can be found in [4,6,15]. The abstract Cauchy problem generalizing the first mixed problem for (1.1) with $f(u) = -u|u|^{p-1}$, $p > 1$, was studied by de Brito [3]. Existence and uniqueness of weak solutions were proved by Galerkin’s method, and the exponential decay was established.

As regards general methods of obtaining the long-time asymptotics of solutions of mixed problems for dissipative evolution equations, we must mention the method of Foias and Saut [10,11] introduced in their studies of the Navier–Stokes equation in a bounded domain. They reduce the initial-boundary value problem to the “normal form” by means of a nonlinear functional transformation. Recently, Shi [26] has applied this approach for constructing an asymptotic expansion of solutions of the boundary-value problem for the semilinear wave equation

$$u_{tt} - \Delta u + au_t + bu = f(u_t, u)$$

with the boundary condition $u|_{\partial \Omega} = 0$ in a bounded domain $\Omega$ with a smooth boundary $\partial \Omega$. Here $a, b = \text{const}$ and $a > 0$. He proved that the small solutions admit a Foias–Saut type of expansion $\sum_{\mu \in \Pi(L)} \exp(-\mu t)w_{\mu}(t)$, where $\Pi(L)$ is the additive semigroup generated by the spectrum of the infinitesimal generator $L$ of the linear operator semigroup associated to the problem in question and $w_{\mu}(t)$ is a polynomial in $t$ whose coefficients are functions of $x$ in a Sobolev space. However, the normalization mapping and the functions $w_{\mu}(t)$ can not be expressed explicitly (like, e.g., the Hopf–Cole transform). The question of the convergence of the series also remains open.

In the present note we propose a method of constructing solutions of some nonlinear evolution equations in bounded domains in the form of an eigenfunction expansion. The convergence of the series is proved in a Sobolev space $H^s$. Then the long-time asymptotic expansion of the constructed solution is obtained.
by means of rather transparent calculations. The next approximation is calculated on the basis of the previous one. Naturally, we need a concrete problem with a specified geometry for getting the subtle estimates of the eigenfunction expansion coefficients. These estimates secure the convergence of the series in the appropriate function space.

In the analysis to follow we shall consider the first initial-boundary value problem for (1.1) with homogeneous boundary conditions and \( f(u) = u^2 \) in a ball and construct its global-in-time solution for small initial data. Existence and uniqueness simply follow from the construction. Note that the exponential decay of solutions can be established by the general methods of [1–3,7,15], but our goal is first to construct solutions that exist globally in time and then to compute the higher-order asymptotics as \( t \to \infty \). In contrast to Galerkin’s method, this approach is based on projecting the nonlinearity onto the infinite-dimensional space of eigenfunctions of the Laplace operator in the corresponding domain. The same idea can work for other nonlinear evolution equations that possess large initial data solutions which do not blow up in finite time (see, e.g., [30,33,34]). If such solutions have any time decay, then at some sufficiently large time \( T \) they will become small. Then the analogous problem can be posed with the initial data prescribed at \( t = T \), and the same asymptotic expansion will be valid that was obtained before for the small initial data solution.

The basic ideas of this method were developed in the papers [30–34], where several initial-boundary value problems were considered for dissipative parabolic semilinear equations (the Kuramoto–Sivashinsky equation in [30], the damped Boussinesq equation in [31,32], and the fractional Laplacian heat equation in [33,34]). We shall provide below the general functional description of this approach as applied to the hyperbolic equation (1.1).

Let \( H \) be a Hilbert space with the scalar product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \), and let the operator \( A \) be defined on \( D(A) \) dense in \( H \). Assume that \( A \) is closed and possesses a complete orthogonal system of eigenvectors \( \{ e_j \}_{j=0}^\infty \), \( e_j \in D(A) \),

\[
A e_j = \Lambda_j e_j, \quad \text{Re} \, \Lambda_j > 0, \quad j \in \mathbb{N} \cup \{ 0 \},
\]

where the eigenvalues \( \Lambda_j \) are numbered in increasing order of \( \text{Re} \, \Lambda_j \). Moreover, assume that \( |\Lambda_j| \to +\infty \) as \( j \to +\infty \). We do not suppose that the vectors \( e_j \) are normalized. It is not a matter of principle, but will be convenient for deducing the estimates of the eigenfunction coefficients.

Introduce the space

\[
H^s = \left\{ u = \sum_{i=0}^{\infty} \hat{u}_i e_i : \sum_{i=0}^{\infty} |\hat{u}_i|^2 |\Lambda_i|^s \| e_i \|^2 < +\infty \right\}
\]

endowed with the norm (see [1, Chapter 4])
\[ \|u\|^2_s = \sum_{i=0}^{\infty} |\hat{u}_i|^2 |A_i|^s \|e_i\|^2, \quad \hat{u}_i = \frac{\langle u, e_i \rangle}{\|e_i\|^2}. \]

Consider the following abstract Cauchy problem:

\[ u''(t) + 2bu'(t) + Au(t) = B(u(t), u(t)), \quad t > 0, \]
\[ u(0) = \varepsilon^2 \phi, \quad u'(0) = \varepsilon^2 \psi, \quad (1.3) \]

where \( b, \varepsilon = \text{const} > 0; u(t) : [0, \infty) \to H \) is a continuous function, and \( B(\cdot, \cdot) \) is a bilinear form in \( H \).

Denote by \( I \) the identity operator and assume that \( A - b^2 I > 0 \). This condition guarantees the existence of damped oscillations and the absence of aperiodic processes. If \( A - b^2 I \) is not definite, the linear stability criteria are also satisfied, but aperiodic processes appear, which leads to the presence of the exponential operator functions \( \exp\{-t[bI \pm (b^2 I - A)^{1/2}]\} \) in the representation of the solution of the linear problem. These cases do not add anything new to the picture of the wave propagation.

Setting \( \sigma(A) = (A - b^2 I)^{1/2} \) and integrating (1.3) with respect to \( t \) we reduce it to the integral equation

\[ u(t) = \varepsilon^2 \exp(-bt) \left[ C(t)\phi + S(t)(b\phi + \psi) \right] + \int_0^t \exp([-b(t - \tau)]S(t - \tau)B(u(\tau), u(\tau)) \, d\tau, \quad (1.4) \]

where

\[ C(t) = \cos(\sigma(A)t), \quad S(t) = \sigma^{-1}(A) \sin(\sigma(A)t) \]

are the strongly continuous cosine and sine operator functions in \( H \).

**Definition.** The function \( u(t) \) is called a strong solution of (1.3) if each term of this equation is a continuous \( H \)-valued function of \( t \). The function \( u(t) \) is called a mild solution of (1.3) if it satisfies the integral equation (1.4) in the Banach space \( C^0([0, \infty), H^s) \).

We seek mild solutions of (1.3) in the form

\[ u(t) = \sum_{m=0}^{\infty} \hat{u}_m(t)e_m, \quad \hat{u}_m(t) = \frac{\langle u(t), e_m \rangle}{\|e_m\|^2}. \quad (1.5) \]

Expand the nonlinear term into the series

\[ B(u(t), u(t)) = \sum_{m=0}^{\infty} \hat{B}_m(t)e_m, \]
\[ \hat{B}_m(t) = \frac{1}{\|e_m\|^2} \left( B \left( \sum_{m=0}^{\infty} \hat{u}_p(t) e_p \sum_{m=0}^{\infty} \hat{u}_k(t) e_k \right), e_m \right) \]
\[ = \sum_{p, k \geq 0} b(m, p, k) \hat{u}_p(t) \hat{u}_k(t), \]
\[ b(m, p, k) = \frac{(B(e_p, e_k), e_m)}{\|e_m\|^2}. \quad (1.6) \]

The absolute and uniform in \( m \geq 0, t \geq 0 \) convergence of the series (1.6) is the essential condition of the application of the method in question.

Expanding the initial data into the series
\[ \phi = \sum_{m=0}^{\infty} \hat{\phi}_m e_m, \quad \psi = \sum_{m=0}^{\infty} \hat{\psi}_m e_m, \]
we substitute them and (1.5), (1.6) into (1.4), and obtain for all integers \( m \geq 0 \)
\[ \hat{u}_m(t) = \varepsilon^2 \exp(-bt) \left[ \cos(\sigma_m t) + b \frac{\sin(\sigma_m t)}{\sigma_m} \right] \hat{\phi}_m + \frac{\sin(\sigma_m t)}{\sigma_m} \hat{\psi}_m \]
\[ + \int_0^t \exp[-b(t-\tau)] \frac{\sin(\sigma_m (t-\tau))}{\sigma_m} \]
\[ \times \sum_{p, k \geq 0} b(m, p, k) \hat{u}_p(\tau) \hat{u}_k(\tau) d\tau, \quad (1.7) \]

where \( \sigma_m = \sqrt{\Lambda_m - b^2} \).

Next, we seek the coefficients \( \hat{u}_m(t) \) in the form of the series in \( \varepsilon \)
\[ \hat{u}_m(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{v}_m^{(N)}(t). \quad (1.8) \]

Setting \( \hat{\Phi}_m = \varepsilon \hat{\phi}_m, \hat{\Psi}_m = \varepsilon \hat{\psi}_m \) (it is convenient to keep \( \varepsilon \) in the coefficients in order to simplify some estimates) we substitute (1.8) into (1.7) and get the recurrence formulas
\[ \hat{v}_m^{(0)}(t) = \exp(-bt) \left[ \cos(\sigma_m t) + b \frac{\sin(\sigma_m t)}{\sigma_m} \right] \hat{\phi}_m + \frac{\sin(\sigma_m t)}{\sigma_m} \hat{\psi}_m, \]
\[ \hat{v}_m^{(N)}(t) = \int_0^t \exp[-b(t-\tau)] \frac{\sin(\sigma_m (t-\tau))}{\sigma_m} Q_m^{(N)}(\hat{v}(\tau)) d\tau, \quad N \geq 1, \]
\[ Q_m^{(N)}(\hat{v}(t)) = \sum_{p, k \geq 0} b(m, p, k) \sum_{j=1}^{N} \hat{v}_p^{(j-1)}(t) \hat{v}_k^{(N-j)}(t). \quad (1.9) \]
In order to secure the absolute and uniform in 
\( m \geq 0, t \geq 0 \) convergence of the series \( Q_m^{(N)}(\hat{v}(t)) \) (which in its own turn guarantees the absolute and uniform convergence of the series (1.6)) we should establish the following estimates for integers \( N \geq 0, m \geq 0, \) and real \( t \geq 0 \):

\[
|\hat{v}_m^{(N)}(t)| \leq c^N f_1(N) f_2(m) \exp(-bt),
\]

where the constant \( c > 0 \) is independent of \( N, m, b, t; f_1(N) \to 0 \) as \( N \to \infty \) sufficiently fast to guarantee the absolute and uniform convergence of

\[
\sum_{j=1}^N \hat{v}_p^{(j-1)}(t) \hat{v}_k^{(N-j)}(t) \quad \text{as} \quad N \to \infty,
\]

and \( f_2(m) \to 0 \) as \( m \to \infty \) sufficiently fast to secure the absolute and uniform convergence of the series \( Q_m^{(N)}(\hat{v}(t)) \). The inequalities (1.10) are established by induction. Then we choose \( \epsilon \in [0, \epsilon_0] \) with \( \epsilon_0 < 1/c \), where \( c \) is the constant from (1.10), so that the series (1.8) converges absolutely and uniformly with respect to \( t \geq 0, \epsilon \in [0, \epsilon_0] \).

Thus, we construct mild solutions of (1.3) in the form (1.5), (1.8), (1.9), and the estimates (1.10) allow us to establish the index \( s \) such that

\[ u(t) \in C_0^0([0, \infty), H^s). \]

Moreover, (1.10) also serves for calculating the long-time asymptotics. To this end we add and subtract integrals from \( t \) to \( \infty \) in the integral representations (1.9) and rewrite them as

\[
\hat{v}_m^{(0)}(t) = \exp(-bt) \left[ A_m^{(0)} \cos(\sigma_m t) + B_m^{(0)} \sin(\sigma_m t) \right],
\]

\[
\hat{v}_m^{(N)}(t) = \exp(-bt) \left[ \left( -A_m^{(N)} + R_m^{(N)}(t) \right) \cos(\sigma_m t) \right. \\
\left. + \left( B_m^{(N)} + R_m^{(N)}(t) \right) \sin(\sigma_m t) \right], \quad N \geq 1,
\]

where

\[
A_m^{(0)} = \hat{\Phi}_m, \quad B_m^{(0)} = \frac{b \hat{\Phi}_m + \hat{\Psi}_m}{\sigma_m},
\]

\[
A_m^{(N)} = \int_0^\infty e^{bt} \frac{\sin(\sigma_m \tau)}{\sigma_m} Q_m^{(N)}(\hat{v}(t)) \, d\tau,
\]

\[
B_m^{(N)} = \int_0^\infty e^{bt} \frac{\cos(\sigma_m \tau)}{\sigma_m} Q_m^{(N)}(\hat{v}(t)) \, d\tau,
\]

\[
R_m^{(N)}(t) = \int_t^\infty e^{bt} \frac{\sin(\sigma_m \tau)}{\sigma_m} Q_m^{(N)}(\hat{v}(t)) \, d\tau,
\]

\[
R_m^{(N)}(t) = -\int_t^\infty e^{bt} \frac{\cos(\sigma_m \tau)}{\sigma_m} Q_m^{(N)}(\hat{v}(t)) \, d\tau.
\]
Next, we prove that the residual terms $R^{(N)}_{A,m}(t)$ and $R^{(N)}_{B,m}(t)$ satisfy (1.10) which gives us the asymptotic formula for $\hat{v}_m^{(N)}(t)$ and consequently for $\hat{u}_m(t)$ (see (1.8)). By means of (1.5) we can obtain the following first-order long-time asymptotics:

$$\|u(t) - \tilde{u}_0(t)\|_s \leq C \exp(-2bt),$$

$$\tilde{u}_0(t) = \exp(-bt) \sum_{m=0}^{\infty} \left[-A_m \cos(\sigma mt) + B_m \sin(\sigma mt)\right] e_m,$$

$$A_m = \sum_{N=0}^{\infty} \varepsilon^{N+1} A^{(N)}_m, \quad B_m = \sum_{N=0}^{\infty} \varepsilon^{N+1} B^{(N)}_m. \quad (1.12)$$

This process can be continued, and the next terms of the asymptotics can be obtained. However, we shall postpone the calculation of the higher-order terms until Section 6 in order not to go into details.

The paper is organized as follows. In Section 2 we collect the information on the radial and angular eigenfunctions used for constructing solutions of (1.1) in a ball. In Section 2 we state the problem and formulate the main results, i.e., Theorems 1 and 2. Theorem 1 is dedicated to the construction of small global-in-time mild solutions of (1.1) in a unit ball. The existence and uniqueness results follow from the construction. Theorem 2 is devoted to obtaining the long-time asymptotic expansion of the solutions in question. In Section 4 several propositions are presented which allow to estimate the eigenfunction expansion coefficients of the initial data and the solutions. In Section 5 Theorem 2 is proved. The proof of Theorem 2 is provided in Section 6. Some final remarks are given in Section 7.

2. Preliminaries

Denote by $B$ a ball of a unit radius and put the origin of the coordinate system in its centre, so that in the spherical coordinates $B = \{(r, \theta, \varphi): |r| < 1, \ 0 \leq \theta \leq \pi, \ 0 \leq \varphi < 2\pi\}$. For studying the first initial-boundary value problem for the damped semilinear wave equation in $B$ we shall employ the expansion in the eigenfunctions of the Laplace operator in this ball.

We set $H = L_2(B)$, the space of real functions square integrable over $B$ endowed with the inner product

$$\langle f, g \rangle = \int_0^1 \int_0^{2\pi} \int_0^\pi f(r, \theta, \varphi)g(r, \theta, \varphi)r^2 \sin \theta \, d\theta \, d\varphi \, dr,$$
and denote by \(\| \cdot \|\) the corresponding norm. A function \(f(r, \theta, \varphi) \in L_2(B)\) can be expanded into the series
\[
f(r, \theta, \varphi) = \sum_{m \geq 0, n \geq 1} \widehat{f}_{mn} \chi_{mn}(r, \theta, \varphi), \quad \widehat{f}_{mn} = \langle f, \chi_{mn} \rangle.
\]
where \(\chi_{mn}\) are the eigenfunctions of the Laplace operator in \(B\), i.e., they satisfy
\[
\Delta \chi = -\Lambda \chi, \quad (r, \theta, \varphi) \in B, \quad \chi|_{S} = 0,
\]
\[
|\chi(0, \theta, \varphi)| < +\infty, \quad \chi(r, \theta, \varphi + 2\pi) = \chi(r, \theta, \varphi), \quad (2.1)
\]
where
\[
\Delta = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} \Delta_{\theta, \varphi} \quad \text{and} \quad \Delta_{\theta, \varphi} = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial^2_\varphi.
\]

The angular eigenfunctions \(Y(\theta, \varphi)\) are the nontrivial solutions of the following problem on the unit sphere \(S\):
\[
\Delta_{\theta, \varphi} Y + \mu Y = 0, \quad (\theta, \varphi) \in S,
\]
\[
|Y|_{\theta=0,\pi} < +\infty, \quad Y(\theta, \varphi + 2\pi) = Y(\theta, \varphi).
\]
The corresponding eigenvalues are
\[
\mu_m = m(m + 1), \quad m \in \mathbb{N} \cup \{0\}.
\]
The radial eigenfunctions are the nontrivial solutions of the problem
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left( \Lambda - \frac{m(m + 1)}{r^2} R \right) = 0,
\]
\[
R(1) = 0, \quad |R(0)| < +\infty.
\]
They are represented by the formula [17]
\[
R_{mn}(r) = j_m(\lambda_{mn} r) = \sqrt{\frac{\pi}{2r}} J_{m+1/2}(\lambda_{mn} r)
\]
and are called the spherical Bessel functions. The eigenvalues of the Laplace operator in the ball \(B\) are
\[
\Lambda_{mn} = \lambda_{mn}^2, \quad m \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N},
\]
where \(\lambda_{mn}\) are the positive zeros of the Bessel function \(J_{m+1/2}(x)\) numbered in increasing order, \(n\) is the number of the zero. Note that \(\lambda_{0n} = \pi n\).

Thus, we have for integers \(m \geq 0, n \geq 1\)
\[
\chi_{mn}(r, \theta, \varphi) = j_m(\lambda_{mn} r) Y_m(\theta, \varphi). \quad (2.2)
\]
Denote by $L_{2,r^2}(0, 1)$ the real space $L_2(0, 1)$ with the weight $r^2$ and the inner product $(f, g) = \int_0^1 f(r)g(r)r^2\,dr$. For the norm of the spherical Bessel functions in this space we have the formula [29]

$$
\|j_m\|_{(n)}^2 = \int_0^1 j_m^2(\lambda_{mn}r)r^2\,dr = \frac{\pi}{4} [J_{m+3/2}(\lambda_{mn})]^2.
$$

It follows from the estimates of the Bessel functions that for sufficiently large $\lambda_{mn} > 0$ [29, p. 219]

$$
\frac{c_1}{\lambda_{mn}} \leq \|j_m\|_{(n)}^2 \leq \frac{c_2}{\lambda_{mn}}.
$$

For large positive zeros of $J_m(x)$ the following asymptotics holds (called McMahon’s expansion, see [17]):

$$
\lambda_{mn} = \varpi_{mn} + O\left(\frac{1}{\varpi_{mn}}\right), \quad \varpi_{mn} = \left(m + 2n - \frac{1}{2}\right)\frac{\pi}{2},
$$

$$
m \ll n, \ n \to \infty.
$$

Denote by $P_m(x)$ the Legendre polynomial of order $m$ and consider two points, $P$ and $Q$, on the unit sphere $S$. Let $O$ be the centre of the sphere and let $\gamma(P, Q)$ be the angle (between 0 and $\pi$) formed by two vector radii $OP$ and $OQ$. For fixed $P$ and $Q$ varying over $S$ the function $P_m[\cos(\gamma(P, Q))]$ is a spherical harmonic of the $m$th order of the spherical coordinates of $Q$. For fixed $Q$ and variable $P$ this function is also a spherical harmonic with respect to $P$ (see [25]).

Introduce the real space $L_2(S)$ endowed with the inner product $(f, g)_S = \int_S fg\,dS$ and denote by $\|\cdot\|_S$ the corresponding norm. The following formulas hold [25]:

$$
(Y_m, Y_k)_S = \int_S Y_m(Q)P_k[\cos(\gamma(P, Q))]\,dS_Q = 0, \quad m \neq k,
$$

$$
\|Y_m\|_S^2 = \frac{4\pi}{2m + 1},
$$

$$
\frac{2m + 1}{4\pi} \int_S Y_m(Q)P_m[\cos(\gamma(P, Q))]\,dS_Q = Y_m(P),
$$

$$
\frac{2m + 1}{4\pi} \int_S P_m[\cos(\gamma(P, Q'))]P_m[\cos(\gamma(Q, Q'))]\,dS_{Q'} = P_m[\cos(\gamma(P, Q))].
$$

Introduce the Sobolev space $H^s(B)$ endowed with the norm

$$
\|f\|_S^2 = \sum_{m \geq 0, \ n \geq 1} \lambda_{mn}^{2s} \|\hat{f}_{mn}\|^2 \|\chi_{mn}\|^2
$$

and set $H_0^s(B) = H^s(B) \cap \{u|_S = 0\}$. 

3. Main results

Consider the first initial-boundary value problem for the semilinear damped wave equation in the unit ball $B$

$$u_{tt} + 2bu_t - \Delta u = u^2, \quad x \in B, \ t > 0,$$

$$u(x, 0) = \varepsilon^2 \phi(x), \quad u_t(x, 0) = \varepsilon^2 \psi(x), \quad x \in B,$$

periodicity conditions in $\varphi$ with the period $2\pi$,

$$u|_S = 0, \quad \varphi|_S = 0, \quad t > 0, \quad (3.1)$$

where $x = (r, \theta, \varphi); \ b, \varepsilon = \text{const} > 0; \ \phi(x) \text{ and } \psi(x)$ are real valued functions. Set $A_0 = -\Delta$ defined on sufficiently smooth functions satisfying the conditions (2.1).

**Definition.** The function $u(t)$ is called a *mild solution* of the problem (3.1) if it satisfies the integral equation (1.4) with $A = A_0, B(u(t), u(t)) = u^2(t)$ in the Banach space $C([0, \infty), H^0_0(B)).$

We examine below only the most interesting and complicated case of small dissipation, when $\lambda^2_{mn} > b^2$ for all $m \geq 0, n \geq 1.$ Then all the eigenvalues of the linear operator of the equation (3.1) lie on the line $\text{Re} \ z = -b.$ It corresponds to the existence of an infinite number of damped oscillations. If $\lambda^2_{mn} \leq b^2$ for some $m, n,$ then several negative eigenvalues of the linear operator are located in the region $-b \leq \text{Re} \ z < 0.$ This case is sufficiently easier from the point of view of obtaining long-time asymptotics and corresponds to the presence of both aperiodic processes and damped oscillations. Note that $\lambda^2_{01} = \pi^2.$

We shall use the notation $D_\theta = (-1/\sin \theta) \partial_\theta$ and denote by $V^1_0(f(r, Q))$ the total variation of the function $f(r, Q), \ Q \in S,$ in $r \in [0, 1].$ We formulate some assumptions on a sufficiently smooth function $f(r, Q), \ r \in (0, 1), \ Q \in S.$

**Assumption A.**

1. $\int_S ds_Q \int^1_0 (|f(r, Q)| + |r \partial_r f(r, Q)|) \, dr < +\infty;$
2. $\int_S ds_Q \int^1_0 (|D_\theta f(r, Q)| + |r \partial_r D_\theta f(r, Q)|) \, dr < +\infty.$

**Assumption B.**

1. $f(1, Q) = 0, \ \int_S ds_Q \int^1_0 (|f(r, Q)| + |\partial_r f(r, Q)|) \, dr < +\infty;$
2. $D^2_\theta f(1, Q) = 0, \ \int_S ds_Q \int^1_0 (|D^2_\theta f(r, Q)| + |r \partial_r D^2_\theta f(r, Q)| + |r \partial_r^2 D^2_\theta f(r, Q)|) \, dr < +\infty.$
Theorem 1. If $b^2 < \pi^2$ and the initial functions $\psi$ and $\varphi$ satisfy Assumptions A and B respectively, then there exists a mild solution of (3.1) from the space $C([0, \infty), H^s_0(B)), s < 3/2$, which can be represented as

$$u(r, \theta, \varphi, t) = \sum_{m \geq 0, n \geq 1} \hat{u}_{mn}(t) j_m(\lambda_{mn} r) Y_m(\theta, \varphi),$$

where the coefficients $\hat{u}_{mn}(t)$ are defined below (see (5.5), (4.4)). If $-3/2 + \epsilon < s < 3/2$ where $\epsilon > 0$ is small, the solution is unique.

Remark 3.1. Assumptions A and B consist of two parts for the following reasons. The series representing the solution in question contains two series, $\sum_{m=0, n \geq 1}$ and $\sum_{m,n \geq 1}$. Assumptions A(1) and B(1) secure the convergence of the series with $m = 0, n \geq 1$. The eigenfunction expansion coefficients may grow with $m$, therefore some restrictions are needed to compensate this growth. They form Assumptions A(2) and B(2) and guarantee the convergence of the second series.

Theorem 2. Under the assumptions of Theorem 1, there exists such a constant $C$ independent of $t, \epsilon, b$, such that the following estimate holds for the mild solution of the problem (1.1) as $-3/2 + \epsilon < s < 3/2, t \geq 0$:

$$\|u(t) - \tilde{u}_0(t) - \tilde{u}_1(t)\|_s \leq C \exp(-3bt), \quad (3.2)$$

where

$$\tilde{u}_0(x, t) = \exp(-bt) \sum_{m \geq 0, n \geq 1} \left[ -A_{mn} \cos(\sigma_{mn}t) + B_{mn} \sin(\sigma_{mn}t) \right] \chi_{mn}(x),$$

$$A_{mn} = \sum_{N=0}^{\infty} \epsilon^{N+1} A^{(N)}_{mn}, \quad B_{mn} = \sum_{N=0}^{\infty} \epsilon^{N+1} B^{(N)}_{mn},$$

$$\sigma_{mn} = \sqrt{\lambda_{mn}^2 - b^2} > 0.$$

The coefficients $A^{(N)}_{mn}$ and $B^{(N)}_{mn}$ are defined by (6.1), the functions $\chi_{mn}(x)$ by (2.2), and

$$\tilde{u}_1(x, t) = \exp(-2bt) \sum_{m \geq 0, n \geq 1} a_{mn}(t) \chi_{mn}(x),$$

$$a_{mn}(t) = \sum_{p, k \geq 0; q, s \geq 1} \left\{ \frac{A_{pqks} - B_{pqks}}{4\sigma_{mn}} \right\} \frac{b \sin(\sigma_{mn}^+ t) + (\sigma_{mn}^+ - \sigma_{mn}) \cos(\sigma_{mn}^+ t)}{\Delta^+(m, n, p, q, k, s)} \times \frac{D_{pqks} b \cos(\sigma_{mn}^+ t) - (\sigma_{mn}^+ - \sigma_{mn}) \sin(\sigma_{mn}^+ t)}{2\sigma_{mn}} \frac{1}{\Delta^+(m, n, p, q, k, s)}.$$
\[
\begin{align*}
+ \frac{A_{pqks} - B_{pqks}}{4\sigma_{mn}} b \sin(\sigma_{mnpqks}^+ t) + (\sigma_{mnpqks}^- - \sigma_{mn}) \cos(\sigma_{mnpqks}^- t) \\
+ \frac{D_{pqks}}{2\sigma_{mn}} \frac{b \cos(\sigma_{mnpqks}^- t) - (\sigma_{mnpqks}^- - \sigma_{mn}) \sin(\sigma_{mnpqks}^- t)}{\Delta^- (m, n, p, q, k, s)} \\
- \frac{D_{pqks}}{2\sigma_{mn}} \frac{b \sin((\sigma_{pq} + \sigma_{qs})_t) + (\sigma_{mnpqks}^- - \sigma_{mn}) \cos((\sigma_{pq} + \sigma_{qs})_t)}{\Delta^- (m, n, p, q, k, s)} \\
+ \frac{D_{pqks}}{2\sigma_{mn}} \frac{b \cos((\sigma_{pq} - \sigma_{qs})_t) + (\sigma_{mnpqks}^+ - \sigma_{mn}) \sin((\sigma_{pq} - \sigma_{qs})_t)}{\Delta^+ (m, n, p, q, k, s)} \\
+ \frac{A_{pqks} + B_{pqks}}{4\sigma_{mn}} \times \frac{b \sin((\sigma_{pq} - \sigma_{qs})_t) - (\sigma_{mnpqks}^+ - \sigma_{mn}) \cos((\sigma_{pq} - \sigma_{qs})_t)}{\Delta^+ (m, n, p, q, k, s)} \\
- \frac{A_{pqks} + B_{pqks}}{4\sigma_{mn}} \times \frac{b \sin((\sigma_{pq} - \sigma_{qs})_t) - (\sigma_{mnpqks}^+ - \sigma_{mn}) \cos((\sigma_{pq} - \sigma_{qs})_t)}{\Delta^+ (m, n, p, q, k, s)} \\
- \frac{D_{pqks}}{2\sigma_{mn}} \frac{b \cos((\sigma_{pq} - \sigma_{qs})_t) + (\sigma_{mnpqks}^+ - \sigma_{mn}) \sin((\sigma_{pq} - \sigma_{qs})_t)}{\Delta^+ (m, n, p, q, k, s)} \\
\times \chi_{mn}(x),
\end{align*}
\]

\[
A_{pqks} = \sum_{N=0}^{\infty} \varepsilon^{N+1} \sum_{j=1}^{N} A_{pq}^{(j-1)} A_{ks}^{(N-j)},
\]

\[
B_{pqks} = \sum_{N=0}^{\infty} \varepsilon^{N+1} \sum_{j=1}^{N} B_{pq}^{(j-1)} B_{ks}^{(N-j)},
\]

\[
D_{pqks} = \sum_{N=0}^{\infty} \varepsilon^{N+1} \sum_{j=1}^{N} A_{pq}^{(j-1)} B_{ks}^{(N-j)},
\]

\[
\sigma_{mnpqks}^+ = \sigma_{mn} + \sigma_{pq} + \sigma_{ks}, \quad \sigma_{mnpqks}^- = \sigma_{mn} - \sigma_{pq} - \sigma_{ks},
\]

\[
\sigma_{mnpqks}^\pm = \sigma_{mn} + \sigma_{pq} - \sigma_{ks}, \quad \sigma_{mnpqks}^\mp = \sigma_{mn} - \sigma_{pq} + \sigma_{ks},
\]

\[
\Delta^+ (m, n, p, q, k, s) = b^2 + (\sigma_{mnpqks}^+ - \sigma_{mn})^2,
\]

\[
\Delta^- (m, n, p, q, k, s) = b^2 + (\sigma_{mnpqks}^- - \sigma_{mn})^2,
\]

\[
\Delta^\pm (m, n, p, q, k, s) = b^2 + (\sigma_{mnpqks}^\pm - \sigma_{mn})^2,
\]

\[
\Delta^\mp (m, n, p, q, k, s) = b^2 + (\sigma_{mnpqks}^\mp - \sigma_{mn})^2.
\]
and the series in \( p, q, k, s \) converges absolutely and uniformly with respect to \( m, n, t \).

4. Auxiliary results

In this section we shall present several results concerning the estimates of the eigenfunction expansion coefficients. Let the function \( f(r, Q) \) be defined in the unit ball \( B \), \( Q \) being the point on the unit sphere \( S \). Consider for integer \( m \geq 0 \), real \( \lambda > 0 \), and \( Q \in S \)

\[
\Im_m(\lambda, Q) = \int_0^1 r^{3/2} f(r, Q) J_{m+1/2}(\lambda r) \, dr.
\]

Lemma 1. Suppose that a function \( f(r, Q) \) satisfies Assumption A(1). Then there exists such \( C(Q) \in L^1(S) \) that for \( m \geq 0 \), \( \lambda > 0 \), and a.e. in \( Q \in S \)

\[
|\Im_m(\lambda, Q)| \leq \frac{C(Q)}{\lambda^{3/2}}.
\]

Proof. Denote \( F(r, Q) = rf(r, Q) \). Assumption A(1) implies that the function \( F(r, Q) \) belongs to the Sobolev space \( W^1_1((0,1)) \) with respect to \( r \) and to \( L^1(S) \) with respect to \( Q \). Therefore, it has a bounded total variation in \( r \in [0,1] \) absolutely integrable in \( Q \) over \( S \), i.e., \( V^1_0(F(r, Q)) = V(Q) \in L^1(S) \) and \( \lim_{r \to 0^+} F(r, Q) = F(0, Q) \in L^1(S) \). Then \( F(r, Q) \) can be represented as a difference of two monotone in \( r \) functions \( F(r, Q) = F_1(r, Q) - F_2(r, Q) \). Therefore,

\[
\Im_m(\lambda, Q) = \int_0^1 \left[ F_1(r, Q) - F_2(r, Q) \right] \sqrt{r} J_{m+1/2}(\lambda r) \, dr.
\]

By the second mean value theorem for integrals (Bonnet’s theorem),

\[
\left| \int_0^1 F_1(r, Q) \sqrt{r} J_{m+3/2}(\lambda r) \, dr \right|
\leq \left| F_1(0, Q) \right| \int_0^\xi \sqrt{r} J_{m+3/2}(\lambda r) \, dr + \left| F_1(1, Q) \right| \int_\xi^1 \sqrt{r} J_{m+3/2}(\lambda r) \, dr
\leq C_1(Q) \lambda^{-3/2}, \quad \text{where } C_1(Q) \in L^1(S).
\]

The integral containing \( F_2(r, Q) \) can be estimated in an analogous way. \( \Box \)
Lemma 2. Let \( f(r, Q) \) satisfy Assumptions B(1). Then there exists such \( C(Q) \in L_1(S) \) that for \( m \geq 0, \lambda > 0, \) and a.e. in \( Q \in S \)
\[
| \Im_m(\lambda, Q) | \leq \frac{C(Q)(m+1)}{\lambda^{5/2}}.
\]

Proof. Changing the variable \( \xi = \lambda r \), integrating by parts and using the boundary condition \( f(1, Q) = 0 \) leads to
\[
\Im_m(\lambda, Q) = \frac{1}{\lambda^2} \int_0^1 \xi^{3/2} \partial_\xi f(\xi/\lambda, Q) J_{m+1/2}(\xi) \, d\xi
\]
\[
= -\frac{1}{\lambda^2} \int_0^1 \left[ \frac{\xi}{\lambda} \partial_\xi f(\xi/\lambda, Q) - mf_Q(\xi/\lambda) \right] J_{m+3/2}(\xi) \, d\xi
\]
\[
= -\frac{1}{\lambda} \int_0^1 \left[ r \partial_r f(r, Q) - mf_Q(r) \right] \sqrt{r} J_{m+3/2}(\lambda r) \, dr.
\]
Assumptions B(1) imply that the function \( \tilde{F}(r, Q) = r \partial_r f(r, Q) - mf(r, Q) \) belongs to the Sobolev space \( W^1_1((0, 1)) \) in \( r \) and to \( L_1(S) \) with respect to \( Q \). Therefore it has a bounded total variation in \( r \in [0, 1] \) which is absolutely integrable over \( S \), and there exist the limits \( \tilde{F}(0, Q), \tilde{F}(1, Q) \in L_1(S) \). Repeating the same arguments as in the proof of Lemma 1 yields the required estimate. \( \square \)

Next, consider the integral
\[
H(m, n, p, k, \lambda_{mn}, \lambda_j) = \int_0^1 J_{m+1/2}(\lambda_{mn} r) J_{p+1/2}(\lambda_1 r) J_{k+1/2}(\lambda_2 r) \sqrt{r} \, dr,
\]
\[
j = 1, 2,
\]
where \( m, p, k \geq 0, n \geq 1 \) are integers, \( \lambda_1, \lambda_2 > 0, \) and \( \lambda_{mn} \) is one of the positive zeros of the function \( J_{m+1/2}(x) \).

Lemma 3. For any fixed \( n \geq 1, \) any \( m, p, k \geq 0, \) and real positive \( \lambda_1, \lambda_2 \to +\infty \) there exists such a constant \( C \) independent of \( m, n, p, k, j \) that the following estimates hold:
\[
\left| H(m, n, p, k, \lambda_{mn}, \lambda_j) \right| \leq C \frac{\lambda_{mn}^{-3/2}}{\lambda_j^{1/2}} \begin{cases} 
\lambda_1^{-1/2}, & \lambda_1 > \lambda_2; \\
\lambda_2^{-1/2}, & \lambda_1 < \lambda_2; \\
\lambda_1 = \lambda_2 = \lambda. & \lambda_1 = \lambda_2 = \lambda.
\end{cases}
\]

Proof. It is a slight modification of that of [34, Lemma 2]. \( \square \)
Next, we study the integral
\[ I_{pkm} = \langle Y_p Y_k, Y_m \rangle_S = \int_S Y_p(Q) Y_k(Q) Y_m(Q) \, dS_Q \]
from the point of view of obtaining the decay estimate in \( m \) and tracing the dependence on \( p \) and \( k \) at the same time.

**Lemma 4.** For all integers \( p, k, m \geq 0 \) there exists such a constant \( C \) independent of \( p, k, m \) that
\[ |I_{pkm}| \leq \frac{C}{\sqrt{(m+1)(p+1)(k+1)}}. \]

**Proof.** See [34, Lemma 3]. \( \square \)

**Lemma 5.** For all integers \( p, k, m \geq 1 \)
\[ |I_{pkm}| \leq \frac{4\sqrt{\pi}}{(2m+1)\sqrt{m+1}} \left[ \sqrt{p+1 \ln(p+1)} + \sqrt{k+1 \ln(k+1)} + \ln 2 \frac{1}{2} \left( \sqrt{p+1} + \sqrt{k+1} \right) + 4 \right]. \]

**Proof.** See [34, Lemma 4]. \( \square \)

**Lemma 6.** (i) If the function \( f(r, Q) \) satisfies Assumptions A, then there exists such a constant \( C \) independent of \( m \) and \( n \) that for all integers \( m \geq 0, n \geq 1 \)
\[ \left| \hat{f}_{mn} \right| \leq \frac{C}{\lambda_{mn}^{1/2} \sqrt{m+1}}. \quad (4.1) \]
(ii) If \( f(r, Q) \) satisfies Assumptions B, then for integers \( m \geq 0, n \geq 1 \)
\[ \left| \hat{f}_{mn} \right| \leq \frac{C}{\lambda_{mn}^{3/2} \sqrt{m+1}}. \quad (4.2) \]
where \( C \) is independent of \( m \) and \( n \).

**Proof.** Consider first \( m = 0, 1 \). In the chosen coordinate system with the pole at the point \( P \)
\[ \hat{f}_{mn} = \frac{1}{\|f_m\|_{\alpha}^2 \|Y_m\|_S^2} \int_S P_m \left[ \cos \gamma(P, Q) \right] dS_Q \int_0^1 r^2 j_m(\lambda_{mn} r) f(r, Q) \, dr. \]
Then in the case (i) we have, by Lemma 1 and (2.3),
\[ |\hat{f}_{mn}| \leq C_{\lambda_{mn}} \int_{0}^{2\pi} d\varphi \int_{0}^{\pi} \left| P_m(\cos \theta) \right| \sin \theta d\theta \left| \int_{0}^{1} r^2 f_m(\lambda_{mn} r) f(r, \theta, \varphi) dr \right| \leq C_{\lambda_{mn}}^{-1/2}. \]

In the case (ii) we deduce (4.2) with \( m = 0, 1 \) by means of applying Lemma 2.

Assume now that \( m \geq 2 \). We set \( \cos \theta = z \) and introduce primitive Legendre polynomials

\[ \varphi_m^{(1)}(z) = \int_{-1}^{1} P_m(\xi) d\xi, \]
\[ \varphi_m^{(2)}(z) = \int_{-1}^{1} d\xi \int_{-1}^{\xi} P_m(\eta) d\eta = \int_{-1}^{1} (z - \xi) P_m(\xi) d\xi. \]

Note that [25]

\[ \left| \varphi_m^{(1)}(z) \right| \leq \frac{4}{\sqrt{\pi}} \frac{1}{\sqrt{m + (2m + 1)}}, \]
\[ \left| \varphi_m^{(2)}(z) \right| \leq \frac{4}{\sqrt{\pi}} \frac{1}{(m + 1)^{3/2}(2m + 1)} \]

and \( \varphi_m^{(1)}(1) = \varphi_m^{(2)}(1) = 0 \).

We introduce the integral

\[ \langle f \rangle(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r, \theta, \varphi) d\varphi = F_3(r, \cos \theta) = F_3(r, z) \]

which represents the mean value of the function \( f(r, \theta, \varphi) \) along the parallel, all of whose points have colatitude \( \theta \). Each plane characterized by the condition \( \theta = \text{const} \) has a distance \( z = \cos \theta \) from the centre of the unit sphere \( S \).

Next, we study the function

\[ G_m(r) = \int_{0}^{\pi} F_3(r, \cos \theta) P_m(\cos \theta) \sin \theta d\theta = \int_{-1}^{1} F_3(r, z) P_m(z) dz. \]

Integrating by parts we get

\[ G_m(r) = -\int_{-1}^{1} \varphi_m^{(1)}(z) \partial_z F_3(r, z) dz. \]
We can write that
\[ \hat{f}_{mn} = \frac{1}{\| j_m \|^2} \int_0^\pi r^2 j_m(\lambda_{mn} r) \, dr \int_0^{\pi/2} \phi_m^{(1)}(\cos \theta) \, d\theta \langle f \rangle (r, \theta) \, d\theta. \]

Changing the order of integration and using (4.3) and Lemma 1 we obtain
\[
| \hat{f}_{mn} | \leq C \lambda_{mn} (2m + 1) \\
\times \int_0^{2\pi} d\phi \int_0^\pi |\phi_m^{(1)}(\cos \theta)| \left| \int_0^1 r^2 j_m(\lambda_{mn} r) \, f(r, \theta, \varphi) \, dr \right| \leq \frac{C}{\lambda_{mn}^{1/2} \sqrt{m + 1}}.
\]

In the case (ii) we integrate two times by parts in $G_m(r)$ and find that
\[
G_m(r) = \int_{-1}^1 \phi_m^{(2)}(z) \partial_z^2 F_3(r, z) \, dz = \pi \int_0^\pi \phi_m^{(2)}(\cos \theta) D^2_\theta F_3(r, \cos \theta) \sin \theta \, d\theta.
\]

Applying Lemma 2 and (4.3) we get
\[
| \hat{f}_{mn} | \leq C \lambda_{mn} (2m + 1) \int_0^{2\pi} d\phi \int_0^\pi |\phi_m^{(2)}(\cos \theta)| \, d\theta,
\]
\[
\left| \int_0^1 r^2 j_m(\lambda_{mn} r) \, \partial_\theta \left( \frac{1}{\sin \theta} \partial_\theta \right) f(r, \theta, \varphi) \, dr \right| \leq \frac{C}{\lambda_{mn}^{3/2} \sqrt{m + 1}},
\]
which completes the proof. \(\square\)

In the sequel we should estimate the functions
\[
\hat{v}_{mn}^{(0)}(t) = \exp(-bt) \left\{ \cos(\sigma_{mn} t) + b \frac{\sin(\sigma_{mn} t)}{\sigma_{mn}} \hat{\Phi}_{mn} + \frac{\sin(\sigma_{mn} t)}{\sigma_{mn}} \hat{\Psi}_{mn} \right\},
\]
\[
\hat{v}_{mn}^{(N)}(t) = \int_0^t \exp[-b(t - \tau)] \frac{\sin(\sigma_{mn}(t - \tau))}{\sigma_{mn}} Q_{mn}^{(N)}(\hat{v}(\tau)) \, d\tau, \quad N \geq 1,
\]
\[
Q_{mn}^{(N)}(\hat{v}(t)) = \sum_{p, k \geq 0; \ q, s \geq 1} b(m, n, p, q, k, s) \sum_{j=1}^N \hat{v}_{pq}^{(j-1)}(t) \hat{v}_{ks}^{(N-j)}(t) \quad (4.4)
\]

obtained from (1.9) by changing $m$ to $m, n$; $p$ to $p, q$; and $k$ to $k, s$. 
Lemma 7. The following estimates hold for the functions $\hat{v}_{mn}^{(N)}(t)$ and their derivatives for integers $m \geq 0$, $n \geq 1$, $N \geq 0$, and real $t \geq 0$:

$$
\begin{align*}
|\hat{v}_{mn}^{(N)}(t)| &\leq cN(N + 1)^{-2} \lambda_{mn}^{-3/2}(m + 1)^{-1/2}\exp(-bt), \\
\left|\frac{d}{dt}\hat{v}_{mn}^{(N)}(t)\right| &\leq cN(N + 1)^{-2} \lambda_{mn}^{-1/2}(m + 1)^{-1/2}\exp(-bt),
\end{align*}
$$

(4.5)

where the constant $c$ is independent of $m$, $n$, $N$, $b$, and $t$.

Proof. We use induction on the number $N$. For $N = 0$ the inequalities (4.1) follow from Lemma 6. Assume that they hold for $\hat{v}_{mn}^{(l)}(t)$ and $(d/dt)\hat{v}_{mn}^{(N)}(t)$ with $0 \leq l \leq N - 1$. We can write that

$$
\begin{align*}
\hat{v}_{mn}^{(N)}(t) &= \exp(-bt)\left[ \sin(\sigma_{mn}t)\overline{F}_{mn}^{(N)}(t) - \cos(\sigma_{mn}t)\overline{\tilde{F}}_{mn}^{(N)}(t) \right], \\
\frac{d}{dt}\hat{v}_{mn}^{(N)}(t) &= \exp(-bt)\left[ -b\left[ \sin(\sigma_{mn}t)\overline{F}_{mn}^{(N)}(t) - \cos(\sigma_{mn}t)\overline{\tilde{F}}_{mn}^{(N)}(t) \right] \\
&\quad + \left[ \sigma_{mn}\cos(\sigma_{mn}t)\overline{F}_{mn}(t) + \sigma_{mn}\sin(\sigma_{mn}t)\overline{\tilde{F}}_{mn}(t) \right] \right],
\end{align*}
$$

(4.6, 4.7)

Integrating by parts in $\tau$ in the integral representation of $\overline{F}_{mn}^{(N)}(t)$ we get

$$
\overline{F}_{mn}^{(N)}(t) = \int_0^t \exp(b\tau) \frac{\cos(\sigma_{mn}\tau)}{\sigma_{mn}} Q_{mn}^{(N)}(\hat{v}(\tau)) d\tau,
$$

$$
\overline{\tilde{F}}_{mn}^{(N)}(t) = \int_0^t \exp(b\tau) \frac{\sin(\sigma_{mn}\tau)}{\sigma_{mn}} Q_{mn}^{(N)}(\hat{v}(\tau)) d\tau.
$$

An analogous expression can be obtained for $\overline{\tilde{F}}_{mn}^{(N)}(t)$.

Using (2.3), (2.4) and Lemmas 3 and 5 we deduce that for all $m$, $n$, $p$, $k$ and sufficiently large $q$, $s$

$$
|b(m,n,p,q,k,s)| \leq C \frac{\lambda_{mn}^{1/2}}{\sqrt{m + 1}}\left[ \sqrt{p + 1}\ln(p + 1) + \sqrt{k + 1}\ln(k + 1) \right]
$$

$$
\times \begin{cases} 
\lambda_{pq}^{-3/2}\lambda_{ks}^{-1/2}, & \lambda_{pq} > \lambda_{ks}; \\
\lambda_{pq}^{-1/2}\lambda_{ks}^{-3/2}, & \lambda_{pq} < \lambda_{ks}; \\
\lambda_{pq}^{-1}, & \lambda_{pq} = \lambda_{ks}.
\end{cases}
$$

(4.8)
As an example we estimate the term in the expression for \( \hat{v}_{mn}^{(N)}(t) \) that contains the sum \( \sum_{j=1}^{N} \hat{v}_{pq}^{(j)}(\tau)(d/d\tau)\hat{v}_{ks}^{(N-j)}(\tau) \) in the integrand. Denoting this term by \( \bar{v}_{mn}^{(N)}(t) \) and applying (4.5) for \( \hat{v}_{pq}^{(j)}(\tau) \) and \( (d/dt)\hat{v}_{mn}(t) \) with \( 0 \leq j \leq N - 1 \) we get

\[
|\bar{v}_{mn}^{(N)}(t)| \leq \frac{C^{1/2}}{\sigma_{mn}^{2}} \exp(-bt) \Gamma^{(N)}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5),
\]

\[
\Sigma_1 = \sum_{p,q,k,s: \lambda_{pq} > \lambda_{ks}} \frac{\ln(p + 1)}{\lambda_{pq}^3 \lambda_{ks} \sqrt{k + 1}},
\]

\[
\Sigma_2 = \sum_{p,q,k,s: \lambda_{pq} > \lambda_{ks}} \frac{\ln(k + 1)}{\lambda_{pq}^3 \lambda_{ks} \sqrt{p + 1}},
\]

\[
\Sigma_3 = \sum_{p,q,k,s: \lambda_{pq} < \lambda_{ks}} \frac{\ln(p + 1)}{\lambda_{pq}^2 \lambda_{ks}^2 \sqrt{k + 1}},
\]

\[
\Sigma_4 = \sum_{p,q,k,s: \lambda_{pq} < \lambda_{ks}} \frac{\ln(k + 1)}{\lambda_{pq}^2 \lambda_{ks}^2 \sqrt{p + 1}},
\]

\[
\Sigma_5 = 2 \sum_{p,q} \frac{\ln(p + 1)}{\lambda_{pq}^3 \sqrt{p + 1}},
\]

where

\[
\Gamma^{(N)} = \sum_{j=1}^{N} c^{j-1} c^{N-j-2}(N + 1 - j)^{-2} \leq c^{N-1}(N + 1)^{-2}.
\]

Here we have used the inequality [23, p. 181]

\[
j^{-2}(N + 1 - j)^{-2} \leq 2^2(N + 1)^{-2}[j^{-2} + (N + 1 - j)^{-2}].
\]

We take the sum \( \Sigma_1 \) as an example and prove its convergence. Taking some small \( \delta > 0 \) and using the asymptotics (2.4) we can write that

\[
\Sigma_1 \leq \sum_{p,q} \frac{\ln(p + 1)}{\lambda_{pq}^{2+\delta}} \sum_{k,s} \frac{1}{\lambda_{ks}^{2-\delta} \sqrt{k + 1}} < +\infty.
\]

In a similar way the convergence of the other sums \( \Sigma_i \) is established. The first of the inequalities (4.5) is proved. The second of these inequalities follows from (4.7) by analogous considerations. \( \square \)

5. Proof of Theorem 1

We seek mild solutions of (3.1) in the form

\[
u(r,\theta,\varphi, t) = \sum_{m \geq 0, n \geq 1} \hat{u}_{mn}(t)\chi_{mn}(r, \theta, \varphi), \quad (5.1)
\]

where

\[
\hat{u}_{mn}(t) = \frac{\langle u, \chi_{mn} \rangle}{\|\chi_{mn}\|^2}
\]
and \( \chi_{mn}(r, \theta, \varphi) \) are defined by (2.2). Expanding the initial data into the analogous series we get

\[
\phi(r, \theta, \varphi) = \sum_{m \geq 0, n \geq 1} \hat{\phi}_{mn} \chi_{mn}(r, \theta, \varphi), \quad \hat{\phi}_{mn} = \frac{\langle \phi, \chi_{mn} \rangle}{\| \chi_{mn} \|^2},
\]

\[
\psi(r, \theta, \varphi) = \sum_{m \geq 0, n \geq 1} \hat{\psi}_{mn} \chi_{mn}(r, \theta, \varphi), \quad \hat{\psi}_{mn} = \frac{\langle \psi, \chi_{mn} \rangle}{\| \chi_{mn} \|^2}. \quad (5.2)
\]

According to (1.6), the eigenfunction expansion of the nonlinearity takes the form

\[
(u^2)_{mn}^{\wedge}(t) = \frac{1}{\| \chi_{mn} \|^2} \left( \sum_{p, q} \hat{u}_{pq}(t) \chi_{pq} \sum_{k, s} \hat{u}_{ks}(t) \chi_{ks} \chi_{mn} \right)
\]

\[
= \sum_{p, q, k, s} b(m, n, p, q, k, s) \hat{u}_{pq}(t) \hat{u}_{ks}(t),
\]

\[
b(m, n, p, q, k, s) = \frac{(j_p j_k, j_{m(n)}) (q, s, n)}{\| f_m \|^2 (n) \| Y_m \|^2 S}.
\]

(5.3)

Substituting (5.1)–(5.3) into (3.1) we obtain the following Cauchy problem:

\[
\hat{u}_{mn}''(t) + 2b \hat{u}_{mn}'(t) + \lambda_{mn}^2 \hat{u}_{mn}(t) = (u^2)_{mn}^{\wedge}(t), \quad t > 0,
\]

\[
\hat{u}_{mn}(0) = \varepsilon^2 \hat{\phi}_{mn}, \quad \hat{u}_{mn}'(0) = \varepsilon^2 \hat{\psi}_{mn}. \quad (5.4)
\]

Setting \( \hat{\Phi}_{mn} = \varepsilon^2 \hat{\phi}_{mn}, \hat{\Psi}_{mn} = \varepsilon^2 \hat{\psi}_{mn} \) we integrate (5.4) in \( t \) and get the integral equation (1.7) with the index \( m \) replaced by \( m, n; p \) by \( p, q; k \) by \( k, s \); and \( \sigma_{mn} = \sqrt{\lambda_{mn}^2 - b^2} \). For solving this equation we represent \( \hat{u}_{mn}(t) \) as a formal series in \( \varepsilon \)

\[
\hat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{v}_{mn}^{(N)}(t), \quad (5.5)
\]

substitute this expansion into the integral equation and obtain the formulas (4.4). By Lemma 7, the estimates (4.5) hold for \( \hat{v}_{mn}^{(N)}(t) \) with \( m \geq 0, n \geq 1, N \geq 0 \).

Now we should prove that the formally constructed function (5.1), (5.5), (4.4) is really a mild solution of (3.1) in the space \( C_0([0, \infty), H_0^s(B)) \), \( s < 3/2 \). Choosing \( \varepsilon \) so that \( 0 < \varepsilon \leq \varepsilon_0 < 1/c \), where \( c \) is the constant in the estimates (4.5), we obtain for \( m \geq 0, n \geq 1, t > 0 \)

\[
|\hat{u}_{mn}(t)| \leq c\lambda_{mn}^{-3/2}(m + 1)^{-1/2} \exp(-bt). \quad (5.6)
\]

By means of (2.3)–(2.5) we deduce that for \( s < 3/2 \) the series

\[
\| u(t) \|_S^2 = \sum_{m \geq 0, n \geq 1} \lambda_{mn}^{2s} |\hat{u}_{mn}(t)|^2 \| Y_m \|^2 S \| j_m \|_{(n)}^2
\]
converges absolutely and uniformly with respect to \( t \geq 0 \). To this end we apply the Fubini–Tonelli theorem and prove the convergence of the iterated series \( \Sigma_m \Sigma_n \) via the comparison with the iterated integral

\[
\int_{A}^{\infty} \frac{dm}{(m+1)(2m+1)} \int_{B}^{\infty} (m+2n-1/2)^{2s-4} \, dn
\]

with sufficiently large \( A, B > 0 \). The condition \( s < 3/2 \) guarantees the convergence of the internal integral.

The uniqueness of solutions for \(-3/2 + \epsilon < s < 3/2\) can be established in the same way as in [30]. \( \square \)

6. Proof of Theorem 2: long-time asymptotics

The series representation (5.1) is well adapted for obtaining the long-time asymptotic expansion. Following the scheme described in the Introduction we rewrite the formulas (4.4) as

\[
\hat{v}^{(0)}_{mn}(t) = \exp(-bt) \left[ A^{(0)}_{mn} \cos(\sigma_{mn}t) + B^{(0)}_{mn} \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \right], \\
\hat{v}^{(N)}_{mn}(t) = \exp(-bt) \left[ (-A^{(N)}_{mn} + R^{(N)}_{A,mn}(t)) \cos(\sigma_{mn}t) \right. \\
\left. + (B^{(N)}_{mn} + R^{(N)}_{B,mn}(t)) \sin(\sigma_{mn}t) \right], \quad N \geq 1,
\]

\[
A^{(0)}_{mn} = \hat{\Phi}_{mn}, \quad B^{(0)}_{mn} = \frac{b \hat{\Phi}_{mn} + \hat{\Psi}_{mn}}{\sigma_{mn}},
\]

\[
A^{(N)}_{mn} = \int_{0}^{\infty} \exp(b\tau) \frac{\sin(\sigma_{mn}\tau)}{\sigma_{mn}} Q^{(N)}_{mn}(\hat{v}(\tau)) \, d\tau,
\]

\[
B^{(N)}_{mn} = \int_{0}^{\infty} \exp(b\tau) \frac{\cos(\sigma_{mn}\tau)}{\sigma_{mn}} Q^{(N)}_{mn}(\hat{v}(\tau)) \, d\tau,
\]

(6.1)

\[
R^{(N)}_{A,mn}(t) = \frac{1}{\sigma_{mn}} \int_{t}^{\infty} e^{b\tau} \frac{\sin(\sigma_{mn}\tau)}{\sigma_{mn}} Q^{(N)}_{mn}(\hat{v}(\tau)) \, d\tau,
\]

\[
R^{(N)}_{B,mn}(t) = -\frac{1}{\sigma_{mn}} \int_{t}^{\infty} e^{b\tau} \frac{\cos(\sigma_{mn}\tau)}{\sigma_{mn}} Q^{(N)}_{mn}(\hat{v}(\tau)) \, d\tau,
\]

(6.2)

and \( Q^{(N)}_{mn}(\hat{v}(t)) \) is defined by (4.4). After establishing the estimates of the type of (4.5) for the residual terms \( R^{(N)}_{A,mn}(t) \) and \( R^{(N)}_{B,mn}(t) \) we will be able to use (6.1) as an asymptotic formula for \( \hat{v}^{(N)}_{mn}(t), \quad N \geq 1. \)
Integrating by parts we get
\[
R^{(N)}_{A,mn}(t) = \frac{1}{\sigma^2_{mn}} e^{bt} Q^{(N)}_{mn}(\hat{v}(t))
\]
\[
+ \frac{1}{\sigma^2_{mn}} \int_{t}^{\infty} e^{b\tau} \cos(\sigma_{mn} \tau) \left[ b Q^{(N)}_{mn}(\hat{v}(\tau)) + \frac{d}{d\tau} Q^{(N)}_{mn}(\hat{v}(\tau)) \right] d\tau.
\]

Using the estimates (4.5) and (4.8) we obtain
\[
|R^{(N)}_{A,mn}(t)| \leq cN(N + 1)^{-2} \lambda_{mn}^{-3/2} (m + 1)^{-1/2} \exp(-bt). \tag{6.3}
\]

Analogous estimate holds for \( R^{(N)}_{B,mn}(t) \). For convenience we shall write \( f^{(N)}_{mn}(t) = O(e^{-bt}) \) if this function satisfies the inequality (6.3).

We have proved that for \( t > 0 \) and integers \( m \geq 0, n \geq 1, N \geq 1 \)
\[
\hat{v}^{(N)}_{mn}(t) = \exp(-bt) \left\{ \left[ -A^{(N)}_{mn} + O(e^{-bt}) \right] \cos(\sigma_{mn}t) + \left[ B^{(N)}_{mn} + O(e^{-bt}) \right] \sin(\sigma_{mn}t) \right\}, \tag{6.4}
\]
where
\[
\left| A^{(N)}_{mn} \right|, \left| B^{(N)}_{mn} \right| \leq cN(N + 1)^{-2} \lambda_{mn}^{-3/2} (m + 1)^{-1/2}.
\]

Substituting the asymptotics (6.4) into the integral representations (6.2) we obtain for \( R^{(N)}_{A,mn}(t) \) and \( R^{(N)}_{B,mn}(t) \) the asymptotic formulas (6.2) with
\[
Q^{(N)}_{mn}(\hat{v}(\tau)) = \sum_{p,q,k,s} b(m,n,p,q,k,s)
\]
\[
\times \sum_{j=1}^{\infty} \left[ -A^{(j-1)}_{pq} \cos(\sigma_{pq} \tau) + B^{(j-1)}_{pq} \sin(\sigma_{pq} \tau) + O(e^{-b\tau}) \right]
\]
\[
\times \left[ -A^{(N-j)}_{ks} \cos(\sigma_{ks} \tau) + B^{(N-j)}_{ks} \sin(\sigma_{ks} \tau) + O(e^{-b\tau}) \right].
\]

Calculating the integrals in \( \tau \) by means of the formulas
\[
\int_{t}^{\infty} e^{-b\tau} \cos(\sigma \tau) d\tau = e^{-bt} \frac{b \cos(\sigma t) - \sigma \sin(\sigma t)}{b^2 + \sigma^2},
\]
\[
\int_{t}^{\infty} e^{-b\tau} \sin(\sigma \tau) d\tau = e^{-bt} \frac{b \sin(\sigma t) + \sigma \cos(\sigma t)}{b^2 + \sigma^2},
\]
substituting the results into the representations (6.1) for \( \hat{v}^{(N)}_{mn}(t) \), and recalling (5.1), (5.5) we obtain the second-order long-time asymptotics of the solution in question. The proof is complete. \( \square \)
7. Conclusion

We have concentrated our attention on the most interesting case, when the spectrum of the linear operator of the equation lies on the line Re $z = -b$. It is also the most complicated case from the point of view of obtaining the long-time asymptotics. It would be interesting to compare it with that of the damped Boussinesq equation [32]

$$u_{tt} - 2b \Delta u_t = -\alpha \Delta^2 u + \Delta u + \Delta (u^2),$$  

(7.1)

where $\alpha, b = \text{const} > 0$, $\alpha$ is the dispersion parameter and $b$ is the dissipation coefficient. Choosing the time dependence in the form $\exp(zt)$ we obtain for the 3D ball that

$$z_{1,2} = -b \sqrt{\lambda_{mn}^2 - \alpha - b^2} i \sqrt{\lambda_{mn}^2} + 1,$$  

(7.2)

where $\lambda_{mn}^2 > b^2$. Since the long-time behavior of solutions is primarily determined by the real parts of the eigenvalues and according to (7.2) these real parts form a negative decreasing sequence, the terms of the long-time asymptotic expansion of solutions for (7.1) are “more ordered” with respect to $\lambda_{mn}$, than those corresponding to (7.3). The latter ones all make contribution even to the major term of the asymptotics (to say nothing of the subsequent terms). As a result the asymptotic expansion for (7.1) as $t \to \infty$ is sufficiently simpler than that of the present study.

We have calculated two terms of the long-time asymptotic expansion of the solution. The coefficient in the first term is computed on the basis of nonlinear iterations. In the second term one can easily trace the nonlinear effects of amplitude and frequency multiplication. Following the same scheme we can obtain the next terms of the long-time asymptotics.

If we consider a two-dimensional ball (a disc), then the long-time asymptotic expansion analogous to (3.2) can be obtained by replacing the eigenfunctions of the Laplace operator in a 3D ball by those of a disc, namely $\chi_{mn}(x) = J_m(\lambda_{mn} r) e^{im\theta}$ (see the analogous estimates in [31]). In this case it is sufficiently easier to prove the convergence of the series.

For the problem in the $N$-dimensional ball $B_N = \{0 < r < 1; 0 < \theta < \pi; 0 < \varphi_i < \pi, i = 1, 2, \ldots, N - 3; 0 < \varphi_{N-2} < 2\pi\}$ the eigenfunction expansion of the solution should be

$$u(x, t) = \sum_{m \geq 0, n \geq 1} \hat{u}_{mn}(t) \chi_{mn}(x),$$  

(7.4)
where
\[ \chi_{mn}(x) = \frac{1}{r^\nu} J_{m+\nu}(\lambda_{m+\nu,n} r) C^\nu_m(\cos \theta), \]

\(\nu = (N - 2)/2\) is the space dimension index, \(\lambda_{m+\nu,n}\) are the positive zeros of the Bessel function \(J_{m+\nu}(z)\) numbered in increasing order, \(n\) is the number of the zero, \(C^\nu_m(z)\) are the ultraspherical (Gegenbauer) polynomials. The formula (7.4) is written in the convenient “zonal representation” (see [27, p. 231]), when there is no dependence on the angles \(\varphi_i\). The boundedness of the solution at \(r = 0\) is already satisfied here. Other initial-boundary-value problems with concrete geometry can be considered for the equation in question, for example, (1.1) in an ellipsoid or in a torus.

We must point out that the key issue in the application of our method is the convergence of the series (1.6) representing the eigenfunction expansion of the nonlinearity. We have restricted our consideration to the quadratic nonlinearity. However, we can also treat the power nonlinearity in (1.2), i.e., \((u(t))^s\) with integer \(s \geq 3\). Then instead of (1.6) we will get

\[ (u(t))^s = \sum_{m=0}^{\infty} \hat{U}_{sm}(t) e_m, \]

where
\[ \hat{U}_{sm}(t) = \frac{\langle \sum_{p_1=1}^{\infty} \hat{u}_{p_1}(t)e_{p_1} \sum_{p_2=1}^{\infty} \hat{u}_{p_2}(t)e_{p_2} \ldots \sum_{p_s=1}^{\infty} \hat{u}_{p_s}(t)e_{p_s}, e_m \rangle}{\|e_m\|^2}, \]

and apply the same procedure as above. Our main concern would be the convergence of the series in (7.5) and much will depend on obtaining the subtle estimates of the coefficients \(b(m, p_1, p_2, \ldots, p_s)\). If the appropriate estimates can be obtained, then other power nonlinearities can be considered in a similar way.

In conclusion, we would like to return to the issue of the smallness of initial data. Although this assumption is needed for the global-in-time existence for (1.1), some equations admit global-in-time solutions for large initial data. Such solutions may have time decay due to dissipation or dispersion (or both). Then our approach based on constructing the small data solutions can be applied and the long-time asymptotic expansion can be obtained. Due to the time decay the solutions will become small beginning from some time \(T\) and the asymptotics will be valid for the large initial data solutions as \(t \to \infty\).
References