Asymptotic behavior of solutions of a linear second-order difference equation

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Abstract


It is known that if \( \sum j | p_j | < \infty \), then the difference equation

\[ \Delta^2 y_{n-1} = p_n y_n, \quad n = 1, 2, \ldots, \]

has solutions \( \{ y_{1n} \} \) and \( \{ y_{2n} \} \) such that \( \lim_{n \to \infty} y_{1n} = \lim_{n \to \infty} y_{2n} / n = 1 \). Here it is shown that this conclusion holds if the series \( \sum j p_j \) converges (perhaps conditionally) and satisfies a second condition which is weaker than absolute convergence. Estimates of \( \{ y_{1n} \} \) and \( \{ \Delta y_{1n} \}, i = 1, 2 \), as \( n \to \infty \) are also given.

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We consider the linear difference equation

\[ \Delta^2 y_{n-1} = p_n y_n, \quad n = 1, 2, \ldots, \tag{1} \]

where \( \Delta \) is the forward difference operator with unit spacing, i.e.,

\[ \Delta u_j = u_{j+1} - u_j. \]

Equations of the form (1) arise in discretizing a second-order linear differential equation to solve it numerically. We will consider the asymptotic behavior of solutions of (1).

The following theorem is well known (see, e.g., [1,2]).

Theorem 1. If

\[ \sum_{j=1}^{\infty} j | p_j | < \infty, \tag{2} \]

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then (1) has solutions \( y_1 = \{ y_{1n} \} \) and \( y_2 = \{ y_{2n} \} \) such that

\[
\lim_{n \to \infty} y_{1n} = 1
\]

and

\[
\lim_{n \to \infty} \frac{y_{2n}}{n} = 1.
\]

Here we will show that (1) has solutions satisfying (3) and (4) if the series \( \sum p_j \) converges conditionally, provided that the rate of convergence is sufficiently rapid. Moreover, our results improve on Theorem 1 in the case where (2) holds, in that they provide estimates of \( y_{in} \) and \( \Delta y_{in} \), \( i = 1, 2 \), as \( n \to \infty \).

Throughout this paper we will write \( z_n = O(q_n) \) when we mean that

\[
\lim_{n \to \infty} \left| \frac{z_n}{q_n} \right| < \infty.
\]

The following theorem is our main result.

**Theorem 2.** Suppose that the series \( \sum p_j \) converges (perhaps conditionally) and let

\[
P_n = \sum_{j=n}^{\infty} p_j.
\]

Suppose also that there is a nonincreasing sequence \( \{ p_n \} \) such that

\[
\lim_{n \to \infty} p_n = 0,
\]

\[
\rho_n \geq \sup_{m \geq n} \left\{ \left| \sum_{j=m}^{\infty} p_j \right| \right\}
\]

and

\[
\lim_{n \to \infty} \frac{1}{\rho_n} \sum_{j=n}^{\infty} |p_j| \mu_j = \theta < \frac{1}{2}.
\]

Then (1) has a solution \( y_1 \) such that

\[
y_{1n} = 1 + O(\rho_{n+1})
\]

and

\[
\Delta y_{1n} = O\left( \frac{\rho_{n+1}}{n+1} \right).
\]

By using summation by parts it can be shown that the series (5) defining \( P_n \) converges and that

\[
|P_n| \leq \frac{2 \rho_n}{n}.
\]
Moreover, our summability assumption on \( \{p_j\} \) is weaker than the standard assumption (2). To see this, assume (2) and let

\[
\hat{p}_n = \sum_{j=n}^{\infty} |p_j|.
\]

Then summation by parts yields

\[
\sum_{j=n}^{M} |p_j| = n\hat{p}_n - M\hat{p}_{M+1} + \sum_{j=n+1}^{M} \hat{p}_j. \tag{12}
\]

Since the left-hand side of (12) converges as \( M \to \infty \), it follows that \( \lim_{M \to \infty} M\hat{p}_{M+1} = 0 \), and therefore

\[
\sum_{j=n+1}^{\infty} \hat{p}_j < \infty.
\]

Hence,

\[
\sum_{j=n+1}^{\infty} |p_j| < \infty,
\]

and consequently (8) holds (with \( \theta = 0 \)) for any nonincreasing sequence \( \{p_n\} \). Therefore, (2) implies the assumptions of Theorem 1. However, the assumptions of Theorem 1 do not imply (2) (see Examples 8 and 9).

It is straightforward to verify that if \( y_1 \) is a sequence such that

\[
y_{1n} = 1 + \sum_{j=n+1}^{\infty} (j-n)p_jy_{1j}, \quad n \geq 0, \tag{13}
\]

then \( v_1 \) satisfies (1) and (3). To put this more conveniently for our purposes, let

\[
f_n = \sum_{j=n+1}^{\infty} (j-n)p_j \tag{14}
\]

and suppose that

\[
h_{1n} = f_n + \sum_{j=n+1}^{\infty} (j-n)p_jh_{1j}, \quad n \geq 1. \tag{15}
\]

Then the sequence \( y_{1n} = 1 + h_{1n} \) satisfies (13) and therefore (1) and (3). Motivated by this, we will obtain a sequence \( h_1 \) which satisfies (15) as a fixed point of the transformation \( v = Th \), where

\[
v_n = f_n + \sum_{j=n+1}^{\infty} (j-n)p_jh_j. \tag{16}
\]

We will show that \( T \) is a contraction mapping of a certain Banach space into itself, and \( h_1 \) will essentially be the unique sequence left fixed by \( T \).

**Lemma 3.** With \( \{f_n\} \) as defined by (14),

\[
|f_n| \leq p_{n+1} \tag{17}
\]
and
\[ |\Delta f_n| \leq \frac{2\rho_{n+1}}{n+1}. \]  
\hfill (18)

**Proof.** Let \( Q_n = \sum_{j=n}^{\infty} p_j \). From (7),
\[ |Q_n| < \rho_n, \]  
and we can rewrite (14) as
\[ f_n = \sum_{j=n+1}^{\infty} \left(1 - \frac{n}{j}\right)(Q_j - Q_{j+1}) = n \sum_{j=n+1}^{\infty} \left(\frac{1}{j-1} - \frac{1}{j}\right)Q_j. \]
This and (19) imply (17). To verify (18) we have only to note that \( \Delta f_n = -P_{n+1} \) and recall (11). \( \Box \)

Since the domain of \( T \) must contain \( f \), Lemma 3 motivates the following definition.

**Definition 4.** For each \( N \geq 1 \) let \( \mathcal{H}_N \) be the Banach space of sequences \( h = \{h_n\}_{n=N}^{\infty} \) such that
\[ h_n = O(\rho_{n+1}) \quad \text{and} \quad \Delta h_n = O\left(\frac{\rho_{n+1}}{n+1}\right), \]
with norm
\[ \|h\| = \sup_{n \geq N} \left\{ \max\left(\frac{|h_n|}{\rho_{n+1}}, \frac{(n+1)|\Delta h_n|}{\rho_{n+1}}\right) \right\}. \]  
\hfill (20)

Now let \( L \) be the transformation defined by \( z = Lh \), where
\[ z_n = \sum_{j=n+1}^{\infty} (j-n)p_j h_j. \]  
\hfill (21)

**Lemma 5.** If \( h \in \mathcal{H}_N \), then the series (21) defining \( z_n \) converges for \( n \geq N \),
\[ |z_n| \leq 2\|h\| \sum_{j=n+1}^{\infty} |P_j| \rho_j \]  
\hfill (22)
and
\[ |\Delta z_n| \leq \left(2\rho_{n+2}\rho_{n+1} + \sum_{j=n+1}^{\infty} |P_{j+1}| \rho_{j+1}\right)\frac{\|h\|}{n+1}. \]  
\hfill (23)

**Proof.** Let \( N < n < M \) and consider the finite sum
\[ z(n; M) = \sum_{j=n+1}^{M} (j-n)p_j h_j, \quad M > n. \]
By using (5) and summation by parts this can be rewritten as
\[ z(n; M) = -(M-n)h_M P_{M+1} + \sum_{j=n+2}^{M} (j-n-1)P_j \Delta h_{j-1} + \sum_{j=n+1}^{M} P_j h_j. \]  
\hfill (24)
From (11) and (20),
\[(M - n) | h_M P_{M+1} | \leq 2 \rho_{M+1}^2 \| h \|,\]
and therefore \( \lim_{M \to \infty} (M - n) h(M) P_{M+1} = 0 \). Moreover, from (20),
\[(j - n - 1) | \Delta h_{j-1} | \leq \rho_j \| h \| \quad \text{and} \quad | h_j | \leq \rho_{j+1} \| h \|\]
for \( j \geq n + 1 \). Therefore, the convergence of the series in (8) implies that we can let \( M \to \infty \) in (24) and conclude that \( z_n \) exists and satisfies (22).

To obtain (23) we note from (21) that
\[\Delta z_n = - \sum_{j=n+1}^{\infty} p_j h_j,\]
which can be rewritten by means of summation by parts as
\[\Delta z_n = -P_{n+1} h_{n+1} - \sum_{j=n+1}^{\infty} P_{j+1} \Delta h_j\] (25)
(see (5)). From (11) and (20),
\[| P_{n+1} h_{n+1} | \leq \frac{2 \rho_{n+1} \rho_{n+2}}{n+1} \| h \| \quad \text{and} \quad | \Delta h_j | \leq \frac{\rho_{j+1}}{j+1} \| h \|.\]

These inequalities and (25) imply (23). \( \square \)

We can now complete the proof of Theorem 2. Because of (6) and (8) we can choose \( N \) so that
\[\max \left( \frac{2}{\rho_{n+1}} \sum_{j=n+1}^{\infty} | p_j \rho_j, 2 \rho_{n+2} + \frac{1}{\rho_{n+1}} \sum_{j=n+1}^{\infty} | P_{j+1} \rho_{j+1} | < \gamma < 1, \quad n \geq N.\]

This, (20) and the inequalities (22) and (23) imply that \( L \) maps \( \mathcal{H}_N \) into itself, and that
\[\| Lh \| < \gamma \| h \|\] (26)
for each \( h \in \mathcal{H}_N \). Now consider the mapping \( v = Th \), where \( h \in \mathcal{H}_N \). From (16) and (21), \( v = f + Lh \); therefore, since \( f \in \mathcal{H}_N \), \( T \) also maps \( \mathcal{H}_N \) into itself. Moreover, if \( h_1 \) and \( h_2 \) are in \( \mathcal{H}_N \), then we can see from (26) that
\[\| Th_1 - Th_2 \| = \| Lh_1 - Lh_2 \| \leq \gamma \| h_1 - h_2 \|.\]

Hence, \( T \) is a contraction of \( \mathcal{H}_N \) and therefore there is a unique sequence \( h_1 \) in \( \mathcal{H}_N \) such that \( Th_1 = h_1 \). Now \( y_1 = 1 + h_1 \) satisfies (1) for \( n \geq N + 1 \); moreover, since \( h_1 \in \mathcal{H}_N \), \( y_1 \) also satisfies (9) and (10). Although \( y_{1n} \) is so far defined only for \( n \geq N \), it can obviously be continued backward to \( n = 0 \) by either (1) or (13). This completes the proof of Theorem 2.

**Theorem 6.** Suppose that the assumptions of Theorem 2 hold, and define
\[\sigma_n = \frac{1}{n} \sum_{j=1}^{n} \rho_j.\]
Then (1) has a solution $y_2$ such that
\[ y_{2n} = n(1 + O(\sigma_n)) \]  
and
\[ \Delta y_{2n} = 1 + O(\rho_{n+1}). \]

(Note that $\lim_{n \to \infty} \sigma_n = 0$, because of (6).)

**Proof.** Choose $M$ so that the solution $y_1$ of (1) found in Theorem 2 satisfies
\[ y_{1j} \neq 0, \quad j \geq M - 1. \]
Then reduction of order yields a second solution
\[ y_{2n} = y_{1n} \sum_{j=M}^{n} \frac{1}{y_{1j}y_{1,j-1}}, \quad n \geq M, \]
of (1). It is straightforward to verify that $y_2$ as defined here satisfies (1) for $n \geq M + 1$, and it can be continued back to $n = 0$ by computing recursively from (1). To prove (27), we first investigate the asymptotic behavior of
\[ \alpha_n = \frac{1}{n} \sum_{j=M}^{n} \frac{1}{y_{1j}y_{1,j-1}} \]
as $n \to \infty$. From (9) there is a constant $A$ such that
\[ |y_{1j} - 1| \leq A\rho_{j+1}, \quad j \geq M - 1. \]
Choose $M_1 > M$ such that $A\rho_{j} < \frac{1}{2}$, $j \geq M_1$. Then (31) and the mean value theorem imply that
\[ \left| \frac{1}{y_{1j}} - 1 \right| \leq 4A\rho_{j+1}, \quad j \geq M_1; \]
consequently,
\[ \frac{1}{y_{1j}y_{1,j-1}} = 1 + \epsilon_j, \]
where
\[ |\epsilon_j| \leq 4A(\rho_j + \rho_{j+1} + 4A\rho_j \rho_{j+1}). \]
From (30) and (33),
\[ \alpha_n = \frac{1}{n} \sum_{j=M}^{M_1-1} \frac{1}{y_{1j}y_{1,j-1}} + \frac{n - M_1 + 1}{n} + \frac{1}{n} \sum_{j=M_1}^{n} \epsilon_j, \quad n \geq M_1. \]
Now (34) and (35) imply that
\[ \alpha_n = 1 + O(\sigma_n). \]
From (9), (29) and (36),
\[ y_{2n} = n y_{1n} \alpha_n = n \left( 1 + O(P_{n+1}) \right) \left( 1 + O(\sigma_n) \right) = n \left( 1 + O(\sigma_n) \right), \]
which proves (27).

To prove (28) we invoke the easily established identity
\[ \Delta y_{2n} = \frac{1}{y_{1n}} + n \alpha_n \Delta y_{1n}, \quad n \geq M. \]

Now (10), (32) and (36) imply (28). \qed

**Theorem 7.** Let \( \{a_n\} \) and \( \{\gamma_n\} \) be such that
\[ |a_1 + a_2 + \cdots + a_n| \leq A < \infty, \quad n = 1, 2, \ldots, \tag{37} \]
\[ 0 < \gamma_{n+1} < \gamma_n, \quad n = 1, 2, \ldots, \quad \lim_{n \to \infty} \gamma_n = 0 \]
and
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{\infty} \gamma_j^2 = \theta_1 < \frac{1}{4A}. \tag{38} \]

Define
\[ \zeta_n = \frac{1}{n} \sum_{j=1}^{n} \gamma_j. \]

Then the difference equation
\[ \Delta^2 y_{n-1} = \frac{a_n \gamma_n}{n} y_n, \quad n = 1, 2, \ldots, \]
has solutions \( y_1 \) and \( y_2 \) such that
\[ y_1 = 1 + O(\gamma_{n+1}), \quad \Delta y_1 = O\left( \frac{\gamma_{n+1}}{n+1} \right), \]
\[ y_2 = n \left( 1 + O(\zeta_n) \right), \quad \Delta y_2 = 1 + O(\gamma_{n+1}). \]

**Proof.** Here \( p_n = a_n \gamma_n / n \) and \( \sum_{j=n}^{\infty} a_j \gamma_j \) converges, by Dirichlet's test. Moreover, summation by parts shows that
\[ \left| \sum_{j=n}^{\infty} a_j \gamma_j \right| \leq 2A \gamma_n \quad \text{and} \quad \left| P_n \right| = \left| \sum_{j=n}^{\infty} p_j \right| = \left| \sum_{j=n}^{\infty} \frac{a_j \gamma_j}{j} \right| \leq \frac{2A \gamma_n}{n}. \]

Therefore, (38) implies (8) with \( \rho_n = 2A \gamma_n \), and the conclusion follow from Theorems 2 and 6. \qed

**Example 8.** Consider the difference equation
\[ \Delta^2 y_{n-1} = \frac{a_n}{n^{a+1}} y_n, \quad n = 1, 2, \ldots, \tag{39} \]
where $\alpha > 0$ and \{$a_n$\} satisfies (37). Then (38) holds with $\gamma_n = n^{-\alpha}$ and $\theta_1 = 0$. Hence, Theorem 7 implies that (39) has solutions $y_1$ and $y_2$ such that

$$
y_{1n} = 1 + O(n^{-\alpha}), \quad \Delta y_{1n} = O(n^{-\alpha - 1}),$$
$$y_{2n} = n(1 + O(\beta_n)), \quad \Delta y_{2n} = 1 + O(n^{-\alpha}),$$

where

$$\beta_n = \begin{cases} 1/n^\alpha, & \text{if } 0 < \alpha < 1, \\
\log n/n, & \text{if } \alpha = 1, \\
1/n, & \text{if } \alpha > 1. \end{cases}$$

Theorem 1 does not apply to (39) unless $\alpha > 1$; for example, in the special case

$$\Delta^2 y_{n-1} = \frac{(-1)^n}{n^{\alpha + 1}} y_n, \quad n = 1, 2, \ldots,$$

we have

$$\sum_{j=1}^n j |p_j| = \sum_{j=1}^n \frac{1}{j^\alpha} = \infty, \quad 0 < \alpha < 1.$$ 

**Example 9.** Consider the difference equation

$$\Delta^2 y_{n-1} = \frac{a_n}{(n + 1)(\log(n + 1)))} y_n, \quad n = 1, 2, \ldots, \quad (40)$$

where $\alpha \geq 1$ and \{$a_n$\} satisfies (37). Here

$$\gamma_n = \frac{n}{(n + 1)(\log(n + 1)))^\alpha}.$$

By elementary arguments similar to those used to prove the integral test for convergence of improper integrals, it can be shown that

$$\sum_{j=n+1}^\infty \frac{1}{(j + 1)(\log(j + 1)))^{2\alpha}} < \frac{1}{(2\alpha - 1)(\log(n + 2)))^{2\alpha - 1}} + \frac{1}{(n + 2)(\log(n + 2)))^{2\alpha}}.$$ 

Hence, (38) holds with $\theta_1 = 0$ if $\alpha > 1$ or with $\theta_1 = 1$ if $\alpha = 1$. Therefore, Theorem 7 implies that (40) has solutions $y_1$ and $y_2$ such that

$$y_{1n} = 1 + O\left(\frac{1}{(\log n)^\alpha}\right), \quad \Delta y_{1n} = O\left(\frac{1}{n(\log n)^\alpha}\right),$$
$$y_{2n} = n(1 + O(\xi_n)), \quad \Delta y_{2n} = 1 + O\left(\frac{1}{(\log n)^\alpha}\right),$$
where
\[ \zeta_n = \frac{1}{n} \sum_{j=1}^{n} \frac{j}{(j+1)(\log(j+1))^{\alpha}}, \]
if either \( \alpha > 1 \) or \( \alpha = 1 \) and \( A < \frac{1}{a} \) in (37).

Theorems 1 and 2 do not apply to (40): for example, in the special case
\[ \Delta^2 y_{n-1} = \frac{(-1)^n}{(n+1)(\log(n+1))^{\alpha}} y_n, \quad n = 1, 2, \ldots, \]
we have
\[ \sum_{j=1}^{\infty} |p_j| = \sum_{j=1}^{\infty} \frac{j}{(j+1)(\log(j+1))^{\alpha}} = \infty. \]

References