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A Generalization of the Duality and Sum Formulas on the Multiple Zeta Values

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In this paper we present a relation among the multiple zeta values which generalizes simultaneously the "sum formula" and the "duality" theorem. As an application, we give a formula for the special values at positive integral points of a certain zeta function of Arakawa and Kaneko in terms of multiple harmonic series. © 1999 Academic Press

The multiple zeta values (or Euler–Zagier sums) seem to be related to many kind of mathematical subjects. Recently A. Granville, D. Zagier, and others proved a conjecture known as the "sum formula" (or "sum conjecture") (cf. [2, 4]) which gives a remarkable relation between the multiple zeta values and special values of Riemann zeta function. In this note we prove a generalization of the "sum formula" which is at the same time a generalization of another remarkable identity referred to as the "duality" theorem (cf. [9]).

The multiple zeta values are defined for integers $k_1, ..., k_{n-1} \geqslant 1$ and $k_n \geqslant 2$ by

$$\zeta(k_1, k_2, ..., k_n) = \sum_{0 < m_1 < m_2 < \cdots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

For any index set $(k_1, k_2, ..., k_n)$ satisfying the condition above and any integer $l \geqslant 0$, we define

$$Z(k_1, k_2, ..., k_n; l) = \sum_{\substack{c_1 + c_2 + \cdots + c_n = l \\ \forall c_j \geqslant 0}} \zeta(k_1 + c_1, k_2 + c_2, ..., k_n + c_n).$$

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For any integer $s \geqslant 1$ and $a_1, b_1, a_2, b_2, ..., a_s, b_s \geqslant 1$, we define two index sets which are "dual" to each other by

$$\mathbf{k} = (\underbrace{1, ..., 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, ..., 1}_{a_2 - 1}, b_2 + 1, ..., \underbrace{1, ..., 1}_{a_s - 1}, b_s + 1)$$

and

$$\mathbf{k}' = (\underbrace{1, ..., 1}_{b_s - 1}, a_s + 1, \underbrace{1, ..., 1}_{b_{s-1} - 1}, a_{s-1} + 1, ..., \underbrace{1, ..., 1}_{b_1 - 1}, a_1 + 1).$$

Our main theorem is then the following.

THEOREM 1. $Z(\mathbf{k}'; l) = Z(\mathbf{k}; l)$.

Remark. For $l = 0$, this is just the duality theorem $\zeta(\mathbf{k}') = \zeta(\mathbf{k})$ (cf. [9]), while in the case

$$\mathbf{k} = (n+1), \qquad \mathbf{k}' = (\underbrace{1, ..., 1}_{n-1}, 2), \qquad l = k - n - 1$$

(i.e., $s = a_1 = 1$, $b_1 = n$) Theorem 1 specializes to the "sum formula" (cf. [2, 4])

$$\sum_{\substack{k_1, k_2, ..., k_{n-1} > 0,\ k_n > 1, \\ k_1 + k_2 + \cdots + k_n = k}} \zeta(k_1, k_2, ..., k_n) = \zeta(k).$$

Theorem 1 also contains Theorem 5.1 in M. Hoffman [4] as a special case when $l = 1$.

As an application of Theorem 1, we get following Theorem 2.

Recently T. Arakawa and M. Kaneko [1] defined for $k \geqslant 1$ the function $\xi_k(s)$ by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t})\, dt,$$

where $Li_k(z)$ denotes the kth polylogarithm $\sum_{m=0}^{\infty} (z^m/m^k)$. The integral converges for $Re(s) > 0$ and the function $\xi_k(s)$ continues to an entire function of s. They proved that the special values of $\xi_k(s)$ at non-positive integers are given by poly-Bernoulli numbers and established a connection between $\xi_k(s)$ and the multiple zeta values. Our theorem gives an expression of special values of $\xi_k(s)$ at *positive* integers.

THEOREM 2. *For integers $k \geqslant 1$ and $n \geqslant 1$, we have*

$$\xi_k(n) = \sum_{0<m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n} \frac{1}{m_1 m_2 \cdots m_{n-1} m_n^{k+1}}.$$

Now, we sketch our proofs.

Proof of Theorem 1. First, we review the definition of "Drinfel'd integral" following Zagier [9]. For $\varepsilon_1 = 1$, $\varepsilon_k = 0$ and $\varepsilon_2, ..., \varepsilon_{k-1} \in \{0, 1\}$, we define

$$I(\varepsilon_1, ..., \varepsilon_k) = \int \cdots \int_{0<t_1<\cdots<t_k<1} \frac{dt_1}{A_{\varepsilon_1}(t_1)} \cdots \frac{dt_k}{A_{\varepsilon_k}(t_k)},$$

where we denote $A_0(t) = t$ and $A_1(t) = 1 - t$. It is known that there is an identity between the multiple zeta values and "Drinfel'd integral," namely

$$\zeta(\mathbf{k}) = I(\underbrace{1, ..., 1}_{a_1}, \underbrace{0, ..., 0}_{b_1}, \underbrace{1, ..., 1}_{a_2}, \underbrace{0, ..., 0}_{b_2}, ..., \underbrace{1, ..., 1}_{a_s}, \underbrace{0, ..., 0}_{b_s},).$$

For integers $l_i \geqslant 0$ $(i = 1, ..., s)$ satisfying $l_1 + \cdots + l_s = l$ and for integers d_i satisfying $1 \leqslant d_i \leqslant a_i + l_i$ $(i = 1, ..., s)$, we put $S_{\mathbf{k}}$ as

$$\begin{aligned} &S_{\mathbf{k}}(d_1, ..., d_s; l_1, ..., l_s) \\ &\quad = \sum_{\substack{\varepsilon_{i,2} + \cdots + \varepsilon_{i,a_i+l_i} = d_i - 1 \\ \varepsilon_{1,2}, ..., \varepsilon_{i,a_i+l_i} \in \{0,1\} \text{ for } \forall i}} I(1, \varepsilon_{1,2}, ..., \varepsilon_{1,a_1+l_1}, \underbrace{0, ..., 0}_{b_1}, \\ &\qquad 1, \varepsilon_{2,2}, ..., \varepsilon_{2,a_2+l_2}, \underbrace{0, ..., 0}_{b_2}, ..., 1, \varepsilon_{s,2}, ..., \varepsilon_{s,a_s+l_s}, \underbrace{0, ..., 0}_{b_s}). \end{aligned}$$

Then we have

$$Z(\mathbf{k}; l) = \sum_{\substack{l_1 + l_2 + \cdots + l_s = l \\ l_i \geqslant 0 \text{ for } \forall i}} S_{\mathbf{k}}(a_1, ..., a_s; l_1, ..., l_s).$$

We make a generating function of $S_{\mathbf{k}}$ as

$$\sum_{1 \leqslant d_i \leqslant a_i + l_i \text{ for } \forall i} \left(S_{\mathbf{k}}(d_1, ..., d_s; l_1, ..., l_s) \prod_{j=1}^{s} X_j^{d_j - 1} \right).$$

Following the manner of Zagier's proof of the "sum formula," we calculate the generating function. We put $t_{2s+1} = 1$. Then we get the following expression of $S_{\mathbf{k}}$:

$$S_{\mathbf{k}}(a_1, ..., a_s; l_1, ..., l_s)$$
$$= \left(\prod_{i=1}^{s} (l_i!(a_i-1)!\,(b_i-1)!) \right)^{-1} \int_{0<t_1<t_2<\cdots<t_{2s}<1} \cdots \int \prod_{i=1}^{s} \left(\left(\log \frac{t_{2i}}{t_{2i-1}} \right)^{l_i} \right.$$
$$\times \left. \left(\log \frac{1-t_{2i-1}}{1-t_{2i}} \right)^{a_i-1} \left(\log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right) \frac{dt_1\, dt_2\, dt_3 \cdots dt_{2s}}{(1-t_1)\, t_2(1-t_3) \cdots t_{2s}}.$$

We can write $Z(\mathbf{k}; l)$ as

$$Z(\mathbf{k}; l) = \left(l! \prod_{i=1}^{s} ((a_i-1)!\,(b_i-1)!) \right)^{-1}$$
$$\times \int_{0<t_1<t_2<\cdots<t_{2s}<1} \cdots \int \left(\log \left(\prod_{i=1}^{s} \frac{t_{2i}}{t_{2i-1}} \right) \right)^{l}$$
$$\times \prod_{i=1}^{s} \left(\left(\log \frac{1-t_{2i-1}}{1-t_{2i}} \right)^{a_i-1} \left(\log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right)$$
$$\times \frac{dt_1\, dt_2\, dt_3 \cdots dt_{2s}}{(1-t_1)\, t_2(1-t_3) \cdots t_{2s}}.$$

We change the variables for $i = 1, 2, ..., s$ by

$$x_{2i-1} = \log \frac{1-t_{2i-1}}{1-t_{2i}}, \qquad x_{2i} = \log \frac{t_{2i+1}}{t_{2i}}, \qquad \text{and}$$
$$\frac{dt_1\, dt_2\, dt_3 \cdots dt_{2s}}{(1-t_1)\, t_2(1-t_3) \cdots t_{2s}} = dx_1\, dx_2 \cdots dx_{2s}$$

and put $f(x_1, ..., x_{2s}) = \sum_{j=0}^{2s} ((-1)^j \exp(\sum_{r=1,\, r:\, \text{odd}}^{j} x_r + \sum_{r=j+1,\, r:\, \text{even}}^{2s} x_r)) = (\prod_{i=1}^{s} (t_{2i}/t_{2i-1}))^{-1}$. Then we get

$$Z(\mathbf{k}; l) = \left(l! \prod_{i=1}^{s} ((a_i-1)!\,(b_i-1)!) \right)^{-1}$$
$$\times \int_{\substack{x_i>0,\ 1 \leqslant i \leqslant 2s, \\ f(x, x_2, ..., x_{2s})>0}} \cdots \int (\log(f(x_1, x_2, ..., x_{2s})^{-1}))^{l}$$
$$\times \prod_{i=1}^{s} (x_{2i-1}^{a_i-1} x_{2i}^{b_i-1})\, dx_1\, dx_2 \cdots dx_{2s}.$$

Since we have $f(x_{2s}, x_{2s-1}, ..., x_1) = f(x_1, x_2, ..., x_{2s})$, the change of variables $(x_{2s}, x_{2s-1}, ..., x_1) \leftrightarrow (x_1, x_2, ..., x_{2s})$ leads us to Theorem 1. Q.E.D

Proof of Theorem 2. For integers $k \geqslant 1$ and $n \geqslant 1$, the expression given by Arakawa and Kaneto [1] means

$$\begin{aligned}\xi_k(n) &= \sum_{\substack{a_1+a_2+\cdots+a_k=n-1\\ \forall a_j \geqslant 0}} (a_k+1)\,\zeta(a_1+1, a_2+1, ..., a_{k-1}+1, a_k+2)\\ &= \sum_{t=1}^{n} \sum_{\substack{a_1+a_2+\cdots+a_k=n-t\\ \forall a_j \geqslant 0}} \zeta(a_1+1, a_2+1, ..., a_{k-1}+1, a_k+t+1).\end{aligned}$$

So we use Theorem 1 in case of

$$\mathbf{k} = (\underbrace{1, ..., 1}_{k-1}, t+1), \qquad \mathbf{k}' = (\underbrace{1, ..., 1}_{t-1}, k+1), \qquad l = n-t$$

(i.e., $s=1$, $a_1=k$, $b_1=t$), and we get Theorem 2. Q.E.D

Remark. As Zagier pointed out, Theorem 2 can also be proved directly without using the result of Arakawa and Kaneko and Theorem 1.

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