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Asymptotic First Eigenvalue Estimates for the Biharmonic Operator on a Rectangle

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We find an asymptotic expression for the first eigenvalue of the biharmonic operator on a long thin rectangle. This is done by finding lower and upper bounds which become increasingly accurate with increasing length. The lower bound is found by algebraic manipulation of the operator, and the upper bound is found by [illegible] groundstate is small. © 1997 Academic Press

1. INTRODUCTION

There is a substantial amount of literature on numerical studies of the biharmonic operator acting in $L^2(\Omega)$ for particular regions $\Omega \subseteq \mathbf{R}^2$, especially the square, disk and punctured disk. In this paper we study the operator acting in $L^2(R_h)$, where R_h is the rectangle $[0, h] \times [0, 1]$. Difficulties in studying the biharmonic operator arise because the eigenvalue problem

$$\Delta^2 f = \mu f, \qquad f = \frac{\partial f}{\partial n} = 0 \qquad \text{on} \quad \partial R_h \tag{1}$$

is not exactly soluble. The boundary conditions in this problem are called Dirichlet boundary conditions or clamped plate boundary conditions. Numerical analysts have succeeded in proving a number of interesting results about the groundstate of the biharmonic operator and the corresponding eigenvalue for the square and some rectangles; however, there are very few previous results concerning the h-dependence of spectral quantities. For the unit square the best current enclosure

$$\mu_1 \leqslant 1294.9339^{88}_{40}$$

for the first eigenvalue is due to C. Wieners [13] using the Tempel–Lehmann–Görisch method to obtain the lower bound, and minimisation of

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the quadratic form of the operator on a certain space of test functions for the upper bound. The enclosure is guaranteed by interval arithmetic programming. The accuracy with which this value has been computed has increased with computing power over the last sixty years. In 1937 Weinstein [12] introduced a method which theoretically enabled him to calculate a limiting sequence of lower bounds for μ_1, although his hand calculations

$$1294.956 \leqslant \mu_1 \leqslant 1302.360$$

were slightly inaccurate. Results of intermediate accuracy have been given by Aronszajn (1950) [1] (from a citation in [7, p. 74]), Bazley, Fox and Stadter (1967) [3], De Vito, Fichera, Fusciardi and Schärf (1966) [11] and many others.

The biharmonic operator, which we shall denote by $(\Delta^2)_{\mathrm{DIR}}$, is defined as the non-negative self-adjoint operator associated with the closed quadratic form

$$Q(f) = \begin{cases} \int_{R_h} |\Delta f|^2, & \text{if } f \in W_0^{2,2}(R_h); \\ \infty, & \text{otherwise.} \end{cases} \tag{2}$$

See [6, Theorem 4.4.2] for details. Using the Rayleigh–Ritz formula [6, Section 4.5], the first eigenvalue of the biharmonic operator is given by the expression

$$\mu_1(h) = \inf\{Q(f) \colon f \in L^2(R_h),\ \|f\|_2 = 1\}. \tag{3}$$

We give formulae for lower and upper bounds $\lambda_1(h)$, $\nu_1(h)$ for $\mu_1(h)$ and use their asymptotic expressions to prove that

$$\mu_1(h) = c^4 + 2dc^2\pi^2 h^{-2} + O(h^{-3})$$

as $h \to \infty$, where $c \approx 4.73004$ is the first positive solution of the transcendental equation

$$\cosh c \cos c = 1,$$

and

$$d = \frac{2 \tanh c \tan c - c \tanh c - c \tan c}{c \tanh c - c \tan c} \approx 0.54988. \tag{4}$$

Elementary algebraic manipulation is used to find the lower bound; no benefit is derived for large h by using more involved methods such as Weinstein's truncation of operators. The upper bound is found, as in most other papers, by minimisation of the quadratic form of the operator over

TABLE I

Values of λ_1, ν_1 and a Guaranteed Error Estimate on μ_1

h	λ_1	ν_1	% err on μ_1
1.0	1286.66	1295.93	0.720
1.2	940.070	946.421	0.676
1.4	776.088	780.618	0.584
1.6	687.796	691.129	0.485
1.8	635.529	638.044	0.396
2.0	602.282	604.221	0.322
3.0	537.444	538.111	0.124
4.0	519.496	519.794	0.058
5.0	512.080	512.237	0.031

a certain test space of functions. Observing that eigenfunctions of the biharmonic operator are close to being separable functions, we choose our test space to consist of all such functions. This simplicity of approach allows us to find the asymptotic formulae. The lower and upper bounds we find are also useful for small values of h. They are within 0.72% of the actual eigenvalue for all $h \in [1, \infty)$. See Fig. 3 and Table I.

A fundamental issue in the study of fourth order operators is the fact that the corresponding semigroup is not positivity preserving. This is exhibited by non-positivity of the groundstate of the biharmonic operator for certain domains Ω, a feature first noticed by Bauer and Reiss (1972) [2] for the square. A rigorous proof of this fact has been given by Wieners (1995) [14] who found a function which is slightly negative very near the corners of the square and pointwise extremely close to the actual groundstate. Kozlov, Kondrat'ev, and Maz'ya (1990) [9] had in fact already obtained a more informative result. They managed to show that if the region has an internal angle of less than 146.30° then the groundstate oscillates infinitely often in sign as one approaches the corner, although there are no explicit data concerning where the first oscillation occurs.

Using the bounds λ_1, ν_1 and also a lower bound λ_3 on the third eigenvalue, we prove the bound

$$\frac{\|f_1^-\|_2}{\|f_1\|_2} \leqslant \frac{(\nu_1 - \lambda_1)^{1/2}}{(\lambda_3 - \nu_1)^{1/2}} = O(h^{-1/2}) \tag{5}$$

as $h \to \infty$, on the size of the negative part f_1^- of the groundstate f_1 of $(\Delta^2)_{\mathrm{DIR}}$. See Fig. 4 and Table II for a plot and values of this function for small values of h. In particular we see that for the case of the square,

$$\frac{\|f_1^-\|_2}{\|f_1\|_2} \leqslant 0.0484.$$

TABLE II

h	$(\nu_1-\lambda_1)^{1/2}/(\lambda_3-\nu_1)^{1/2}$
1	0.0484
10	0.0336
20	0.0249
40	0.0179
60	0.0147
80	0.0128
100	0.0114

By "negative part of the groundstate" we mean the negative part when the eigenfunction is positive in the centre of the square.

It is possible to use the L^2 bound and a Sobolev embedding theorem to imply that

$$\frac{\|f_1^-\|_\infty}{\|f_1\|_2}\leqslant\frac{(\nu_1-\lambda_1)^{1/2}\,\lambda_3^{1/4}}{2(\lambda_3-\nu_1)^{1/2}}=O(h^{-1/2})$$

as $h\to\infty$, and so the negative part of the eigenfunction f_1, already small for $h=1$, vanishes asymptotically as $h\to\infty$ in the L^2 and L^∞ senses. However since

$$\frac{\|f_1\|_\infty}{\|f_1\|_2}=O(h^{-1/2}),$$

only the L^2 result (5) is of particular value. See note 13 for details.

2. A FOURTH ORDER OPERATOR IN ONE DIMENSION

Our approach to analysis of the biharmonic operator acting in $L^2(R_h)$ involves attempting to separate variables. Eigenfunctions of $(\Delta^2)_{\mathrm{DIR}}$ are not separable functions, a fact confirmed by oscillations at the corners, so it is remarkable that we obtain such good estimates for the first eigenvalue. The method is successful because the eigenfunctions are close to being separable. The following sections will rely heavily upon spectral analysis of the self-adjoint operator

$$H(h,\alpha)=\frac{d^4}{dx^4}-2\alpha\,\frac{d^2}{dx^2} \tag{6}$$

acting in $L^2([0,h])$, with quadratic form domain $W_0^{2,2}([0,h])$. Since $H(h,\alpha)$ is bounded below and has compact resolvent we may order the eigenvalues as an increasing list.

Let σ: $(0, \infty) \times \mathbf{R} \times \mathbf{N} \to \mathbf{R}$ be the function which associates to each α the nth eigenvalue of the operator $H(h, \alpha)$ acting in $L^2([0, h])$, and let $\rho_n(\alpha) = \sigma(1, \alpha, n)$.

THEOREM 1. (i) *The first eigenvalue ρ_1 of $H(1, \alpha)$ is an increasing and concave function of α. The functions σ and ρ are related by the equation*

$$\sigma(h, \alpha, n) = \rho_n(h^2\alpha)\, h^{-4}. \tag{7}$$

For $\alpha > 0$, ρ_1 is analytic;

(ii) *For $\alpha > 0$ the Green's function of $H(h, \alpha)$ is positive. It follows that for such α the first eigenvalue is of multiplicity one and the groundstate is positive;*

(iii) *For $\alpha > 0$ let f be the nth eigenfunction of $H(h, \alpha)$ and have unit L^2 norm. Then*

$$\|f'\|_2^2 = \tfrac{1}{2}\rho_n'(h^2\alpha)\, h^{-2}; \tag{8}$$

(iv) *For $\alpha > 0$ let $\beta > \alpha$ be the nth solution of the transcendental equation*

$$\cosh\sqrt{\beta+\alpha}\cos\sqrt{\beta-\alpha} - \frac{\alpha}{\sqrt{\beta^2-\alpha^2}}\sinh\sqrt{\beta+\alpha}\sin\sqrt{\beta-\alpha} = 1. \tag{9}$$

Then $\rho_n(\alpha) = \beta^2 - \alpha^2$;

(v)

$$\begin{aligned}
\rho_n(\alpha) &= 2n^2\pi^2\alpha + 4\sqrt{2}\, n^2\pi^2\alpha^{1/2} + O(1) && \textit{as} \quad \alpha \to \infty;\\
\rho_n'(\alpha) &= 2n^2\pi^2 + 2\sqrt{2}\, n^2\pi^2\alpha^{-1/2} + O(\alpha^{-1}) && \textit{as} \quad \alpha \to \infty;\\
\rho_n(\alpha) &= c_n^4 + 2d_n c_n^2\alpha + O(\alpha^2) && \textit{as} \quad \alpha \to 0;\\
\rho_n'(\alpha) &= 2d_n c_n^2 + O(\alpha) && \textit{as} \quad \alpha \to 0;
\end{aligned} \tag{10}$$

where c_n is the nth positive solution of the equation $\cosh c \cos c = 1$, *and*

$$d_n = \frac{2\tanh c_n \tan c_n - c_n \tanh c_n - c_n \tan c_n}{c_n \tanh c_n - c_n \tan c_n};$$

(vi)

$$2\pi^2 \leqslant \rho_1'(\alpha) \leqslant 2d_1 c_1^2 \qquad \forall \alpha \in (0, \infty). \tag{11}$$

Figure 1 shows the first four eigenvalues of $H(1, \alpha)$ and was plotted by *Mathematica* using the implicit formula (9) in Theorem 1 (iv) for positive ρ, with a similar formula for negative ρ. The portion of the graph for

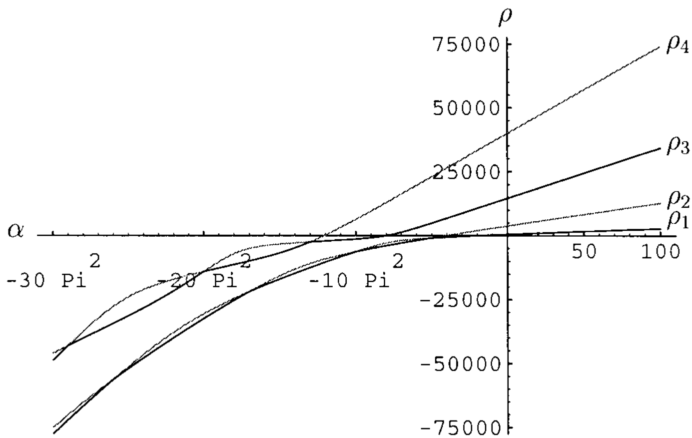

FIG. 1. Plot of the first four eigenvalues of $H(1, \alpha)$.

negative α is not relevant in this paper but has been included to demonstrate that even simple fourth order operators have completely different eigenvalue behaviour to the second order theory; coincidence of ρ_{2n-1} and ρ_{2n} occurs for

$$(\alpha, \rho) = (-(m^2+n^2)\,\pi^2, -(m^2-n^2)^2\,\pi^4) \qquad m-n = 1, 2, 3, ...$$

Proof of Theorem 1 (i). The function ρ_1 is increasing because the perturbing operator is positive. Using the Rayleigh–Ritz formula,

$$\rho_1(\alpha) = \inf\{\langle H(1, \alpha)\, g, g\rangle_1 : \|g\|_2 = 1, g \in \mathrm{Dom}(H(1, \alpha))\}.$$

Suppose that $\alpha = \lambda\alpha_1 + (1-\lambda)\,\alpha_0$ where $0 < \lambda < 1$. Then

$$\langle H(1, \alpha)\, g, g\rangle_1 = \lambda\langle H(1, \alpha_1)\, g, g\rangle_1 + (1-\lambda)\langle H(1, \alpha_0)\, g, g\rangle_1.$$

Let $\varepsilon > 0$ and choose $g \in \mathrm{Dom}(H(1, \alpha))$ such that $\|g\|_2 = 1$ and $\langle H(1, \alpha)\, g, g\rangle_1 < \rho_1(\alpha) + \varepsilon$. Then

$$\begin{aligned}\rho_1(\alpha) + \varepsilon &> \lambda\langle H(1, \alpha_1)\, g, g\rangle_1 + (1-\lambda)\langle H(1, \alpha_0)\, g, g\rangle_1 \\ &\geqslant \lambda\rho_1(\alpha_1) + (1-\lambda)\,\rho_1(\alpha_0).\end{aligned}$$

Since this holds for all positive ε, $\rho_1(\alpha)$ is concave.

For fixed α let $f_n \in L^2([0, h])$ be the nth eigenfunction of $H(h, \alpha)$. Define $g_n \in L^2([0, 1])$ by $g_n(y) = f_n(hy)$. By the chain rule,

$$H(1, h^2\alpha)\, g_n = \frac{d^4 g_n}{dy^4} - 2h^2\alpha \frac{d^2 g_n}{dy^2} = h^4\sigma(h, \alpha, n)\, g_n, \tag{12}$$

so g_n is an eigenfunction of $H(1, h^2\alpha)$ with eigenvalue $h^4\sigma(h, \alpha, n)$. It follows that $h^4\sigma(h, \alpha, n) \geqslant \rho_n(h^2\alpha)$. By a similar reverse argument we obtain equality.

The family $H(1, \alpha)$ of differential operators indexed by α is a holomorphic family of type B and so ρ_n are analytic except where the eigenvalues swap. See [8, Chapter VII §4]. We shall see in part (ii) that swapping does not occur when α is positive. ∎

Proof of Theorem 1 (ii). For $a > 0$ define $G\colon [0, 1]^2 \to \mathbf{R}$ by

$$G(x, y) = \begin{cases} \dfrac{1}{c} k(1-x, y), & y \leqslant x \\ \dfrac{1}{c} k(x, 1-y), & x < y \end{cases} \tag{13}$$

where

$$c = a^3(2(1-\cosh a) + a \sinh a) \tag{14}$$

and

$$\begin{aligned} k(x, y) = {} & (\sinh a - a)(\cosh ax - 1)(\cosh ay - 1) \\ & - (\cosh a - 1)(\cosh ax - 1)(\sinh ay - ay) \\ & - (\cosh a - 1)(\sinh ax - ax)(\cosh ay - 1) \\ & + \sinh a(\sinh ax - ax)(\sinh ay - ay). \end{aligned} \tag{15}$$

For $g \in L^2([0, 1])$ define f by

$$f(x) = \int_0^1 G(x, y)\, g(y)\, dy.$$

Putting

$$k^{(r, s)}(x, y) = \frac{\partial^{r+s} k}{\partial x^r\, \partial y^s}(x, y),$$

the identities

$$\begin{aligned} k(1-x, x) - k(x, 1-x) &\equiv 0 & k^{(4, 0)}(x, y) &\equiv a^2 k^{(2, 0)}(x, y) \\ -k^{(1, 0)}(1-x, x) - k^{(1, 0)}(x, 1-x) &\equiv 0 & & \\ k^{(2, 0)}(1-x, x) - k^{(2, 0)}(x, 1-x) &\equiv 0 & k(0, y) &\equiv 0 \\ -k^{(3, 0)}(1-x, x) - k^{(3, 0)}(x, 1-x) &\equiv c & k^{(1, 0)}(0, y) &\equiv 0 \end{aligned} \tag{16}$$

imply that

$$f(0) = f'(0) = f(1) = f'(1) = 0$$

and

$$\frac{d^4 f}{dx^4} - a^2 \frac{d^2 f}{dx^2} = g,$$

so G is the Green's function of the operator

$$\frac{d^4}{dx^4} - a^2 \frac{d^2}{dx^2}$$

acting in $L^2([0, 1])$. For positivity of $G(x, y)$ it is sufficient to prove that

$$k(x, y) - a(\sinh ax - ax)(\sinh ay - ay)$$

is positive for $x, y > 0$ and $x + y < 1$ because the term subtracted is positive. In order to do this we introduce the function

$$\phi(x) = \tanh^{-1}\left(\frac{\sinh x - x}{\cosh x - 1}\right). \tag{17}$$

The reader should verify that

$$\lim_{x \to 0^+} \phi'(x) = \frac{1}{3}$$

and

$$\phi''(x) = \int_0^{x/2} 4 \sinh 2y[\sinh^2 y \tanh y - y^3]\, dy \bigg/ \left(\int_0^x y(\cosh y - 1)\, dy\right)^2 > 0$$

for $x > 0$, which imply that ϕ is convex and increasing for positive x. Let (x, y) lie in the triangular region $x, y > 0$, $x + y < 1$. Then

$$\phi(a) > \phi(ax) + \phi(ay). \tag{18}$$

Hence using inequality (18) and a two angle tanh identity,

$$\begin{aligned}
\frac{\sinh a - a}{\cosh a - 1} &+ \frac{\sinh a - a}{\cosh a - 1}\,\frac{\sinh ax - ax}{\cosh ax - 1}\,\frac{\sinh ay - ay}{\cosh ay - 1} \\
&= \tanh\{\phi(a)\}[1 + \tanh\{\phi(ax)\} \tanh\{\phi(ay)\}] \\
&> \tanh\{\phi(ax) + \phi(ay)\}[1 + \tanh\{\phi(ax)\} \tanh\{\phi(ay)\}] \\
&= \tanh\{\phi(ax)\} + \tanh\{\phi(ay)\} \\
&= \frac{\sinh ax - ax}{\cosh ax - 1} + \frac{\sinh ay - ay}{\cosh ay - 1}.
\end{aligned}$$

Thus

$$\begin{aligned}&(\sinh a - a)(\cosh ax - 1)(\cosh ay - 1)\\&\quad - (\cosh a - 1)(\cosh ax - 1)(\sinh ay - ay)\\&\quad - (\cosh a - 1)(\sinh ax - ax)(\cosh ay - 1)\\&\quad + (\sinh a - a)(\sinh ax - ax)(\sinh ay - ay) > 0\end{aligned}$$

as required.

It follows that the Green's function of $H(h, \alpha)$ for all $h, \alpha > 0$ is positive because of the relationship (12) between $H(h, \alpha)$ and $H(1, h^2\alpha)$ established in part (i).

See [10, Theorem XIII.44] for a proof that the groundstate of $H(h, \alpha)$ is positive and the associated eigenvalue is multiplicity one. ∎

Proof of Theorem 1 (iii). For $\alpha > 0$ let $f = f_\alpha \in W_0^{2,2}([0, h]) \cap C^\infty$ satisfy

$$\frac{d^4 f}{dx^4} - 2\alpha \frac{d^2 f}{dx^2} = \sigma(h, \alpha, n)\, f \tag{19}$$

and have unit L^2-norm. Then f_α is a critical point of the functional

$$\mathscr{E}_\alpha(\psi) = \left\langle \frac{d^2\psi}{dx^2}, \frac{d^2\psi}{dx^2} \right\rangle - 2\alpha \left\langle \frac{d^2\psi}{dx^2}, \psi \right\rangle \tag{20}$$

in the sense that if $\psi(t)$ is C^1 with respect to t, $\|\psi(t)\| = 1\ \forall t$ and $\psi(0) = f_\alpha$ then

$$\frac{d}{dt} \mathscr{E}_\alpha(\psi(t))\bigg|_{t=0} = 0.$$

Letting $\psi(t) = f_{\alpha + t}$ and differentiating $\mathscr{E}_\alpha(f_\alpha)$,

$$\begin{aligned}\rho_n'(h^2\alpha)\, h^{-2} &= \frac{d}{d\alpha} \sigma(h, \alpha, n)\\&= \frac{d}{d\alpha} \mathscr{E}_\alpha(f_\alpha)\\&= \frac{d}{dt} \mathscr{E}_t(f_\alpha)\bigg|_{t=\alpha} + \frac{d}{dt} \mathscr{E}_\alpha(\psi(t))\bigg|_{t=0}\\&= -2 \left\langle \frac{d^2 f}{dx^2}, f \right\rangle_h + 0. \quad \blacksquare\end{aligned}$$

Proof of Theorem 1 (iv). Since $H(1, \alpha)$ is positive for α positive, eigenfunctions may be found by solving the auxiliary equation $y^4 - 2\alpha y^2 - \rho = 0$ where $\rho > 0$.

Since $\alpha - \sqrt{\alpha^2 + \rho} < 0 < \alpha + \sqrt{\alpha^2 + \rho}$, there are four distinct roots, two real and two imaginary. Denoting these roots by a, $-a$, ib, $-ib$, we see that $a^2 - b^2 = 2\alpha$ and $a^2 b^2 = \rho$.

There exist a combination of functions $\cosh ax$, $\sinh ax$, $\cos bx$, $\sin bx$ which satisfy the boundary conditions of $H(1, \alpha)$ if and only if

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & a & 0 & b \\ \cosh a & \sinh a & \cos b & \sin b \\ a \sinh a & a \cosh a & -b \sin b & b \cos b \end{pmatrix} = 0.$$

Simplifying this determinant, the equation becomes

$$2ab \cosh a \cos b - (a^2 - b^2) \sinh a \sin b = 2ab,$$

so $\rho = \beta^2 - \alpha^2$ is an eigenvalue of the operator if and only if

$$\cosh \sqrt{\beta + \alpha} \cos \sqrt{\beta - \alpha} - \frac{\alpha}{\sqrt{\beta^2 - \alpha^2}} \sinh \sqrt{\beta + \alpha} \sin \sqrt{\beta - \alpha} = 1. \quad \blacksquare$$

Proof of Theorem 1 (v). The proof of asymptotic formulae for ρ and ρ' as $\alpha \to \infty$ is given in the Appendix. It is possible to find the asymptotic formula for ρ as $\alpha \to 0$ by a similar method. The formula for ρ' then follows by differentiation because ρ is analytic at 0. Here we give a sketch of an alternative proof of the case $\alpha \to 0$ for interest:

Let $\gamma_n = \cosh c_n - \cos c_n$ and $\delta_n = \sinh c_n - \sin c_n$. Define

$$g_n(x) = \cosh c_n x - \frac{\gamma_n}{\delta_n} \sinh c_n x - \cos c_n x + \frac{\gamma_n}{\delta_n} \sin c_n x. \tag{21}$$

We claim that $(g_n)_{n \in \mathbf{N}}$ is an orthonormal sequence of eigenfunctions of $H(1, 0)$, and so $\rho_n(0) = c_n^4$. Moreover we claim that $\| g_n' \|_2^2 = d_n c_n^2$, and so from (8) we see that $\rho_n'(0) = 2 d_n c_n^2$.

Verification of these claims is not trivial. Indeed a lengthy calculation is needed even to establish that $\| g_n \|_2 = 1$ for each n. This task is left to the reader. $\blacksquare$

Proof of Theorem 1 (vi). Since ρ_1 is concave, ρ_1' is decreasing. The result now follows immediately from part (v). $\blacksquare$

3. LOWER BOUNDS ON EIGENVALUES

We find lower bounds $\lambda_1(h)$, $\lambda_3(h)$ for $\mu_1(h)$, $\mu_3(h)$ respectively, by elementary algebraic manipulation of the biharmonic operator. This method would be referred to as a finite renormalisation procedure in the physics literature.

THEOREM 2.

$$\lambda_1(h) := \rho_1(\pi^2 h^2)\, h^{-4} + \rho_1(\pi^2 h^{-2}) - 2\pi^4 h^{-2} \leqslant \mu_1(h), \tag{22}$$

$$\lambda_3(h) := \min \begin{Bmatrix} \rho_1(\pi^2 h^2)\, h^{-4} + \rho_2(\pi^2 h^{-2}) - 2\pi^4 h^{-2} \\ \rho_3(\pi^2 h^2)\, h^{-4} + \rho_1(\pi^2 h^{-2}) - 2\pi^4 h^{-2} \end{Bmatrix} \leqslant \mu_3(h). \tag{23}$$

where ρ_n are defined in Theorem 1.

Proof. In this proof we consider, where relevant, restrictions of operators to $C_c^\infty([0, h])$. Let A_h denote the biharmonic operator acting in $L^2([0, h])$, and let B_h denote the Dirichlet Laplacian acting in $L^2([0, h])$. Then

$$\begin{aligned} \Delta^2 &= A_h \otimes 1_1 + 1_h \otimes A_1 + 2B_h \otimes B_1 \\ &= (A_h + 2\pi^2 B_h) \otimes 1_1 + 1_h \otimes (A_1 + 2\pi^2 h^{-2} B_1) \\ &\quad + 2(B_h - \pi^2 h^{-2} 1_h) \otimes (B_1 - \pi^2 1_1) - 2\pi^4 h^{-2} 1_h \otimes 1_1. \end{aligned}$$

The operator $(B_h - \pi^2 h^{-2} 1_h) \otimes (B_1 - \pi^2 1_1)$ has eigenvalues

$$\pi^4 h^{-2}(m^2 - 1)(n^2 - 1) \qquad m, n = 1, 2, \ldots$$

with the corresponding complete orthonormal sequence of eigenfunctions

$$2 \sin m\pi h^{-1} x \sin n\pi y,$$

and so is non-negative. Hence

$$(\Delta^2)_{\mathrm{DIR}} \geqslant (A_h + 2\pi^2 B_h) \otimes 1_1 + 1_h \otimes (A_1 + 2\pi^2 h^{-2} B_1) - 2\pi^4 h^{-2} 1_h \otimes 1_1.$$

It now follows that for h large enough

$$\begin{aligned} \mu_n(h) &\geqslant \sigma(h, \pi^2, n) + \sigma(1, \pi^2 h^{-2}, 1) - 2\pi^4 h^{-2} \\ &= \rho_n(\pi^2 h^2)\, h^{-4} + \rho_1(\pi^2 h^{-2}) - 2\pi^4 h^{-2} \\ &= \lambda_n(h). \quad \blacksquare \end{aligned}$$

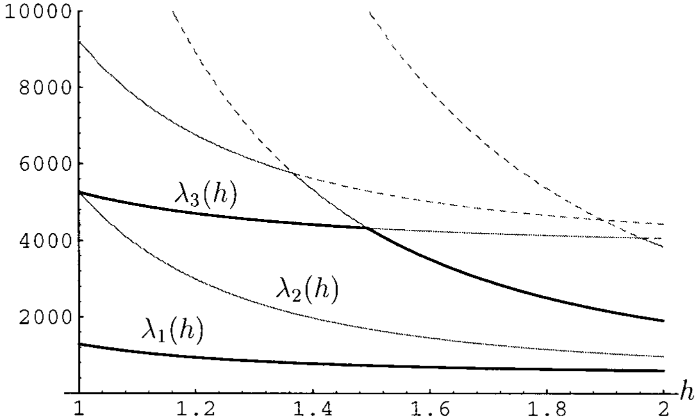

FIG. 2. Graph of $\lambda_1(h)$ and $\lambda_3(h)$.

The functions λ_n may be explicitly calculated using formulae (9), (22) and (23). See Figs. 2 and 3 and Table I. Despite the fact that the lower bounds λ_n cross each other, swapping of eigenvalues is not actually a genuine feature of the increasing size of the rectangle. In [4], Behnke and Mertins show that the eigenvalues veer away from each other just before the points where one might expect them to cross.

4. UPPER BOUND ON THE FIRST EIGENVALUE

We shall find an upper bound $v_1(h)$ for $\mu_1(h)$ by approximation of the groundstate eigenfunction with separable functions. This is called the Hartree method in the physics/chemistry literature and was used in [5] to estimate the groundstate energy of a somewhat similar but simpler problem. Define the functional

$$\mathscr{E}\colon L^2([0, h]) \times L^2([0, 1]) \to \mathbf{R}$$

by

$$\mathscr{E}(f, g) = Q(f \otimes g) = \begin{cases} \displaystyle\int_0^h \int_0^1 |\Delta(f(x)\, g(y))|^2 \, dx\, dy, & \\ \qquad \text{if } \ f \in W_0^{2,2}([0, h]) \text{ and } g \in W_0^{2,2}([0, 1]); & \\ \infty, \qquad \text{otherwise.} & \end{cases} \tag{24}$$

Let

$$v_1(h) = \inf\{\mathscr{E}(f, g) \colon f \in L^2([0, h]),\ g \in L^2([0, 1]),\ \|f\|_2 = \|g\|_2 = 1\}, \tag{25}$$

where the two norms are taken in $L^2([0, h])$ and $L^2([0, 1])$ respectively.

LEMMA 3. *The infimum in the expression* (25) *for* v_1 *is attained.*

Proof. Let A_h denote the biharmonic operator acting in $L^2([0, h])$. Since A_h has compact resolvent, it has a complete orthonormal sequence of eigenfunctions whose corresponding eigenvalues form a divergent sequence. Using this orthonormal sequence we see that the set

$$S_h := \{f \in L^2([0, h]) \colon \|f\|_2^2 + Q_h(f) \leqslant 1\}$$

is compact. The set

$$\begin{aligned} S := \{f \in L^2([0, h]) \colon \|f\|_2^2 + Q_h(f) \leqslant 1,\ \|f\|_2^2 \geqslant 1/(2v_1 + 1)\} \\ \times \{g \in L^2([0, 1]) \colon \|g\|_2^2 + Q_1(g) \leqslant 1,\ \|g\|_2^2 \geqslant 1/(2v_1 + 1)\} \end{aligned} \tag{26}$$

is a closed subset of $S_h \times S_1 \subseteq L^2(R_h)$, separated from the origin, and so is a compact subset of $L^2(R_h) \backslash \{0\}$. Since the map

$$f \mapsto Q(f)/\|f\|_2^2 \tag{27}$$

is lower semicontinuous on $L^2(R_h) \backslash \{0\}$ it attains its infimum when restricted to S.

Suppose that $\mathscr{E}(f, g) \leqslant 2v_1$ where f and g have unit norm. We may rescale f and g so that $\|f\|_2^2 + Q_h(f) = 1$ and $\|g\|_2^2 + Q_1(g) = 1$. Now

$$\frac{1 - \|f\|_2^2}{\|f\|_2^2} = \frac{Q_h(f)}{\|f\|_2^2} \leqslant \frac{\mathscr{E}(f, g)}{\|f\|_2^2\, \|g\|_2^2} \leqslant 2v_1,$$

so $\|f\|_2^2 \geqslant 1/(2v_1 + 1)$. A similar argument for g shows that $(f, g) \in S$. It follows that

$$\min\left\{\frac{\mathscr{E}(f, g)}{\|f\|_2\, \|g\|_2} \colon (f, g) \in S\right\} \leqslant v_1(h).$$

Let $(\tilde{f}, \tilde{g}) \in S$ take the minimum value of the map (27) restricted to S. We may rescale so that $\|\tilde{f}\|_2 = \|\tilde{g}\|_2 = 1$. Now

$$v_1(h) \leqslant \mathscr{E}(\tilde{f}, \tilde{g}) = \min\left\{\frac{\mathscr{E}(f, g)}{\|f\|_2\, \|g\|_2} \colon (f, g) \in S\right\} \leqslant v_1(h). \quad \blacksquare$$

From this point onwards, we shall assume that $\mathscr{E}$ is only applied to functions of unit norm. We may rewrite the formula for $\mathscr{E}$ as

$$\mathscr{E}(f, g) = \left\langle \frac{d^2 f}{dx^2}, \frac{d^2 f}{dx^2} \right\rangle_h + 2\left\langle \frac{d^2 f}{dx^2}, f \right\rangle_h \left\langle \frac{d^2 g}{dy^2}, g \right\rangle_1 + \left\langle \frac{d^2 g}{dy^2}, \frac{d^2 g}{dy^2} \right\rangle_1, \tag{28}$$

where $\langle\cdot,\cdot\rangle_h$ denotes the inner product on $L^2([0,h])$. We shall identify f and g by searching for the critical points of $\mathscr{E}$.

LEMMA 4. *Let $f\in W_0^{2,2}([0,h])$ and $g\in W_0^{2,2}([0,1])$ minimise $\mathscr{E}(f,g)$. Then $f\in C^\infty([0,h])\cap W_0^{2,2}([0,h])$ is the groundstate of the operator $H(h,\alpha_g)$, where*

$$\alpha_g := \|g'\|_2^2. \tag{29}$$

Proof. Let $\tilde{f}\in C_c^\infty([0,h])$ be such that $\langle f,\tilde{f}\rangle_h=0$. Define

$$f(t) := \frac{f+t\tilde{f}}{\|f+t\tilde{f}\|}. \tag{30}$$

By differentiating we see that

$$f(0)=f \qquad \text{and} \qquad \frac{df}{dt}(0)=\tilde{f}.$$

The minimum of $\mathscr{E}$ will be a critical point so

$$0=\frac{d\mathscr{E}}{dt}(f(t),g)\bigg|_{t=0}=2\operatorname{Re}\left\langle\frac{d^4f}{dy^4}-2\alpha_g\frac{d^2f}{dy^2},\tilde{f}\right\rangle_h$$

where the fourth derivative has been taken in the distributional sense. Replacing $\tilde{f}$ by $i\tilde{f}$ we see that

$$\left\langle\frac{d^4f}{dy^4}-2\alpha_g\frac{d^2f}{dy^2},\tilde{f}\right\rangle_h=0 \tag{31}$$

for all such $\tilde{f}$. It now follows that

$$\frac{d^4f}{dy^4}-2\alpha_g\frac{d^2f}{dy^2}=\mu f \qquad \mu\in\mathbf{R}. \tag{32}$$

For suppose otherwise, then there exist $f_1,f_2\in C_c^\infty$ with $\langle f,f_1\rangle_h\neq 0$ and $\langle f,f_2\rangle_h\neq 0$, and $\mu_1\neq\mu_2$ such that

$$\left\langle\frac{d^4f}{dy^4}-2\alpha_g\frac{d^2f}{dy^2},f_i\right\rangle_h=\mu_i\langle f,f_i\rangle_h, \qquad i=1,2.$$

Now let $\tilde{f}=\langle f,f_2\rangle_h f_1-\langle f,f_1\rangle_h f_2$. Then

$$\langle f,\tilde{f}\rangle_h=0$$

and

$$\left\langle \frac{d^4 f}{dy^4} - 2\alpha_g \frac{d^2 f}{dy^2}, \tilde{f} \right\rangle_h = \langle f, f_2 \rangle_h \mu_1 \langle f, f_1 \rangle_h - \langle f, f_1 \rangle_h \mu_2 \langle f, f_2 \rangle_h$$
$$= (\mu_1 - \mu_2) \langle f, f_1 \rangle_h \langle f, f_2 \rangle_h$$
$$\neq 0,$$

which contradicts (31).

It follows from equation (32) that $f \in W^{n,2}$ for all $n \in \mathbf{N}$, and consequently f is smooth. From (28) the value of $\mathscr{E}$ at the critical point is

$$\mu + \left\langle \frac{d^2 g}{dx^2}, \frac{d^2 g}{dx^2} \right\rangle_1,$$

In order for this to be the minimum, f must be the groundstate of $H(h, \alpha_g)$. ∎

THEOREM 5.

$$v_1(h) = \rho_1(h^2\alpha)\, h^{-4} + \rho_1(\tfrac{1}{2}\rho_1'(h^2\alpha)\, h^{-2}) - \rho_1'(h^2\alpha)\, h^{-2}\alpha. \tag{33}$$

where $\alpha \in [\pi^2, dc^2]$ *is a solution of the equation*

$$2\alpha = \rho_1'(\tfrac{1}{2}\rho_1'(h^2\alpha)\, h^{-2}). \tag{34}$$

Proof. Let (f, g) minimise $\mathscr{E}$. Then by lemma 4, f is the groundstate of the operator $H(h, \alpha_g)$. Using (8) we see that

$$\alpha_f = \tfrac{1}{2}\rho_1'(h^2\alpha_g)\, h^{-2}. \tag{35}$$

By an identical argument,

$$\alpha_g = \tfrac{1}{2}\rho_1'(\alpha_f), \tag{36}$$

so

$$2\alpha_g = \rho_1'(\tfrac{1}{2}\rho_1'(h^2\alpha_g)\, h^{-2}).$$

Using equations (19), (25), (28), and (35) we see that

$$v_1(h) = \left\langle \frac{d^4 f}{dx^4}, f \right\rangle_h + 2 \left\langle \frac{d^2 f}{dx^2}, f \right\rangle_h \left\langle \frac{d^2 g}{dy^2}, g \right\rangle_1 + \left\langle \frac{d^4 g}{dy^4}, g \right\rangle_1$$
$$= \sigma(h, \alpha_g, 1) - 2\alpha_f\alpha_g + 2\alpha_f\alpha_g + \rho_1(\alpha_f) - 2\alpha_f\alpha_g$$
$$= \rho_1(h^2\alpha_g)\, h^{-4} - \rho_1'(h^2\alpha_g)\, h^{-2}\alpha_g + \rho_1(\tfrac{1}{2}\rho_1'(h^2\alpha_g)\, h^{-2}).$$

Inequality (11) implies that

$$\rho_1'(\tfrac{1}{2}\rho_1'(h^2\alpha)\,h^{-2}) \geqslant 2\pi^2 > 2\alpha$$

for $\alpha < \pi^2$ and

$$\rho_1'(\tfrac{1}{2}\rho_1'(h^2\alpha)\,h^{-2}) \leqslant 2dc^2 < 2\alpha$$

for $\alpha > dc^2$, so there is at least one solution of equation (34) in the interval $[\pi^2, dc^2]$, and there are no solutions outside. ∎

Note 6. Numerical evidence strongly suggests that there is a unique solution of equation (34) for any value of h. Plots of $\rho_1''(\alpha)$ suggest that

$$-0.0603 < \rho_1''(\alpha) < 0$$

for all positive α. The fact that $\rho_1''(\alpha)$ is negative is clear because we showed that ρ is concave in Theorem 1 (i). A unique solution of equation (34) is guaranteed however under the weaker requirement that

$$-2 < \rho_1''(\alpha) < 0$$

for all positive α. For then

$$\frac{d}{d\alpha}(2\alpha) = 2 < \tfrac{1}{2}\,\rho_1''(\tfrac{1}{2}\,\rho_1'(h^2\alpha)\,h^{-2})\,\rho_1''(h^2\alpha)$$

$$= \frac{d}{d\alpha}(\rho_1'(\tfrac{1}{2}\,\rho_1'(h^2\alpha)\,h^{-2}))$$

and so

$$\rho_1'(\tfrac{1}{2}\rho_1'(h^2\alpha)\,h^{-2}) - 2\alpha$$

is a strictly increasing function.

Figure 3 has been plotted by *Mathematica* using formulae (9), (22), (33), and (34). For every value of h taken there was, as expected, only one solution of equation (34). Note that in Theorem 10 we show that both λ_1 and v_1 converge to $c^4 \approx 500.564$ and the guaranteed percentage error

$$\left(\frac{v_1 - \lambda_1}{\lambda_1}\right) \times 100$$

on μ_1 is of order h^{-3}. This allows us to prove the asymptotic formula (3) correct to the same order. As in [5], the Hartree approximation $f \otimes g$ gives the worst approximation when there is an extra rotational symmetry, as for $h = 1$.

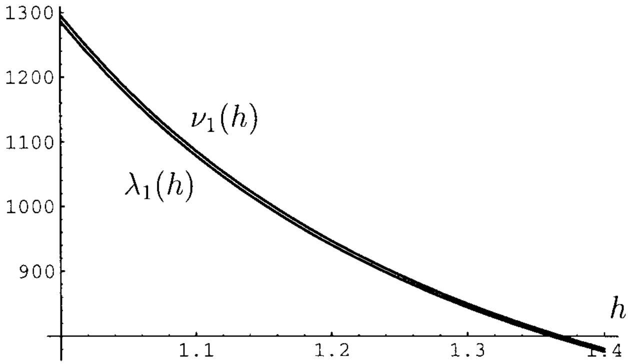

FIG. 3. Graph of $\lambda_1(h)$ and $v_1(h)$.

5. THE GROUNDSTATE

THEOREM 7.

$$\frac{\|f_1^-\|_2}{\|f_1\|_2} < \frac{(v_1-\lambda_1)^{1/2}}{(\lambda_3-v_1)^{1/2}} \tag{37}$$

where $\lambda_1(h)$, $\lambda_3(h)$ *and* $v_1(h)$ *are given by Theorem* 2 *and Equation* (25).

We give an asymptotic expansion of the above bound in Corollary 12, but it is possible to evaluate this bound for smaller values of h giving the results in Fig. 4 and Table II.

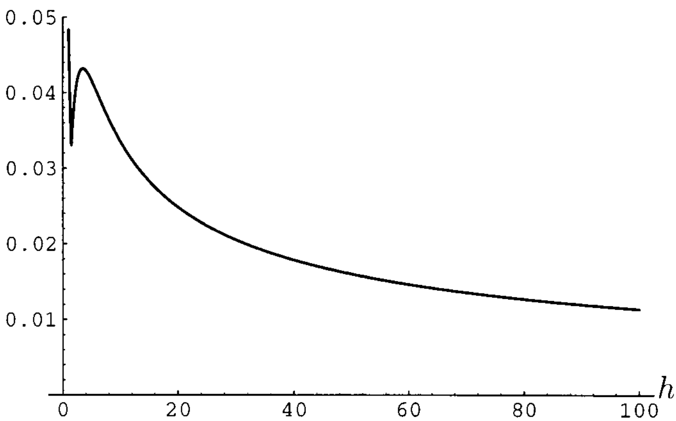

FIG. 4. Graph of $(v_1-\lambda_1)^{1/2}/(\lambda_3-v_1)^{1/2}$.

Proof. There exists a complete orthonormal sequence of eigenfunctions f_n of the biharmonic operator acting in $L^2(R_h)$ with corresponding eigenvalues μ_n listed in increasing order. Let (f, g) minimise $\mathscr{E}$. Then

$$\|f \otimes g\|_2 = 1 \qquad \text{and} \qquad \lambda_1 \leqslant \mu_1 = \|\Delta f_1\|_2^2 < \|\Delta(f \otimes g)\|_2^2 = \nu_1. \tag{38}$$

By writing $f \otimes g = \sum_{n=1}^{\infty} \alpha_n f_n$ with $\alpha_1 > 0$, we see that

$$\sum_{n=1}^{\infty} \alpha_n^2 = 1, \qquad \alpha_2 = 0 \qquad \text{and} \qquad \lambda_1 \leqslant \mu_1 < \sum_{n=1}^{\infty} \alpha_n^2 \mu_n = \nu_1. \tag{39}$$

Using the above information,

$$\begin{aligned} \nu_1 - \mu_1 &= \sum_{n=3}^{\infty} \alpha_n^2 \mu_n - (1 - \alpha_1^2)\, \mu_1 \\ &\geqslant \lambda_3 \sum_{n=3}^{\infty} \alpha_n^2 - (1 - \alpha_1^2)\, \mu_1 \\ &= (\lambda_3 - \mu_1)(1 - \alpha_1^2). \end{aligned} \tag{40}$$

Rearranging (40), we see that

$$\begin{aligned} 1 + \alpha_1 = 2 - (1 - \alpha_1) &\geqslant 2 - \frac{\nu_1 - \mu_1}{\lambda_3 - \mu_1} \\ &> \frac{2(\lambda_3 - \nu_1)}{\lambda_3 - \mu_1}. \end{aligned}$$

Using (40) again,

$$1 - \alpha_1 < \frac{\nu_1 - \lambda_1}{2(\lambda_3 - \nu_1)}. \tag{41}$$

Now

$$\begin{aligned} \|f \otimes g - f_1\|_2^2 &= (1 - \alpha_1)^2 + \sum_{n=3}^{\infty} \alpha_n^2 \\ &= \sum_{n=1}^{\infty} \alpha_n^2 + (1 - 2\alpha_1) \\ &= 2(1 - \alpha_1) \\ &< \frac{\nu_1 - \lambda_1}{\lambda_3 - \nu_1}. \end{aligned} \tag{42}$$

Using theorem 1 (ii) we see that $f \otimes g$ is positive. It follows that

$$|f_1^-| \leqslant |f \otimes g - f_1|. \quad \blacksquare \tag{43}$$

The following Sobolev embedding theorem is used to convert the L^2 bound above to an L^∞ bound.

LEMMA 8. *Let $f \in W^{2,2}(\mathbf{R}^2)$. Then*

$$\|f\|_\infty^2 \leqslant \tfrac{1}{4} \|f\|_2 \|\Delta f\|_2. \tag{44}$$

Proof. Let $g \in L^2(\mathbf{R}^2)$ be the function whose Fourier transform is

$$\hat{g}(\xi) = (2\pi)^{-1} (\gamma + \gamma^{-1} |\xi|^4)^{-1/2}. \tag{45}$$

Then

$$\begin{aligned} \|g\|_2^2 = \|\hat{g}\|_2^2 &= \frac{1}{4\pi^2} \int_{\mathbf{R}^2} \frac{d\xi}{(\gamma + \gamma^{-1} |\xi|^4)} = \frac{1}{4\pi} \int_0^\infty \frac{2r\, dr}{\gamma + \gamma^{-1} r^4} \\ &= \frac{1}{4\pi} \int_0^{\pi/2} \frac{\gamma \sec^2 \theta\, d\theta}{\gamma(1 + \tan^2 \theta)} = \frac{1}{8}. \end{aligned} \tag{46}$$

Now

$$(\gamma + \gamma^{-1} \Delta^2)^{-1/2} f = g * f \qquad \forall f \in L^2(\mathbf{R}^2), \tag{47}$$

so

$$\|(\gamma + \gamma^{-1} \Delta^2)^{-1/2} f\|_\infty^2 = \|g * f\|_\infty^2 \leqslant \|g\|_2^2 \|f\|_2^2 = \tfrac{1}{8} \|f\|_2^2. \tag{48}$$

Hence

$$\|f\|_\infty^2 \leqslant \tfrac{1}{8} \|(\gamma + \gamma^{-1} \Delta^2)^{1/2} f\|_2^2 = \tfrac{1}{8}(\gamma \|f\|_2^2 + \gamma^{-1} \|\Delta f\|_2^2).$$

The result is obtained by setting

$$\gamma = \frac{\|\Delta f\|_2}{\|f\|_2}. \quad \blacksquare$$

THEOREM 9.

$$\frac{\|f_1^-\|_\infty}{\|f_1\|_2} < \frac{(\nu_1 - \lambda_1)^{1/2} \lambda_3^{1/4}}{2(\lambda_3 - \nu_1)^{1/2}}. \tag{49}$$

Proof. Let f, g, and f_1 be as in Theorem 7. Then

$$\begin{aligned} \|\Delta(f \otimes g - f_1)\|_2^2 &= \sum_{n=1}^\infty \alpha_n^2 \mu_n - \mu_1 + 2(1 - \alpha_1) \mu_1 \\ &< \nu_1 - \lambda_1 + \left(\frac{\nu_1 - \lambda_1}{\lambda_3 - \nu_1}\right) \nu_1 \\ &= \frac{(\nu_1 - \lambda_1)}{(\lambda_3 - \nu_1)} \lambda_3. \end{aligned}$$

Lemma 8 implies that

$$\|f\otimes g-f_1\|_\infty^4 \leqslant \frac{1}{16}\|f\otimes g-f_1\|_2^2\,\|\Delta(f\otimes g-f_1)\|_2^2$$

$$<\frac{(\nu_1-\lambda_1)^2\,\lambda_3}{16(\lambda_3-\nu_1)^2}. \quad ∎$$

6. ASYMPTOTIC ESTIMATES

THEOREM 10. *The first eigenvalue of the biharmonic operator acting in $L^2(R_h)$ has the asymptotic formula*

$$\mu_1(h)=c^4+2dc^2\pi^2h^{-2}+O(h^{-3}) \tag{50}$$

as $h\to\infty$.

Proof. Let $\alpha(h)$ be the solution of (34) which makes (33) valid. It follows from inequality (11) that

$$\tfrac{1}{2}\rho_1'(h^2\alpha(h))\,h^{-2}=O(h^{-2})$$

as $h\to\infty$. Hence by equation (34),

$$\alpha(h)=dc^2+O(h^{-2}) \tag{51}$$

as $h\to\infty$.

Substituting this expression into the formula (33) for ν_1 we see that

$$\begin{aligned}\nu_1(h)&=h^{-4}\rho_1(dc^2h^2+O(1))-h^{-2}\rho_1'(dc^2h^2+O(1))(dc^2+O(h^{-2}))\\&\quad+\rho_1(\tfrac{1}{2}\rho_1'(dch^2+O(1))\,h^{-2})\\&=h^{-4}(2\pi^2dc^2h^2+4\sqrt{2}\,\pi^2d^{1/2}ch+O(1))\\&\quad-h^{-2}(2\pi^2+2\sqrt{2}\,\pi^2d^{-1/2}c^{-1}h^{-1}+O(h^{-2}))(dc^2+O(h^{-2}))\\&\quad+\rho(\pi^2h^{-2}+\sqrt{2}\,\pi^2d^{-1/2}c^{-1}h^{-3}+O(h^{-4}))\\&=2\pi^2dc^2h^{-2}+4\sqrt{2}\,\pi^2d^{1/2}ch^{-3}-2\pi^2dc^2h^{-2}-2\sqrt{2}\,\pi^2d^{1/2}ch^{-3}+c^4\\&\quad+2dc^2\pi^2h^{-2}+2\sqrt{2}\,\pi^2d^{1/2}ch^{-3}+O(h^{-4})\\&=c^4+2dc^2\pi^2h^{-2}+4\sqrt{2}\,\pi^2d^{1/2}ch^{-3}+O(h^{-4}),\end{aligned} \tag{52}$$

as $h\to\infty$.

Also, by substituting the asymptotic formulae (10) for ρ_1 into the formula (22) for λ_1, we see that

$$\begin{aligned}\lambda_1(h) &= c^4 + 2dc^2\pi^2h^{-2} + h^{-4}(2\pi^2h^2\pi^2 + 4\sqrt{2}\,\pi^2h\pi + O(1)) \\ &\quad - 2\pi^4h^{-2} + O(h^{-4}) \\ &= c^4 + 2dc^2\pi^2h^{-2} + 4\sqrt{2}\,\pi^3h^{-3} + O(h^{-4}),\end{aligned} \tag{53}$$

as $h \to \infty$. ▮

Note 11. For long thin rectangles a good approximation to the groundstate of the biharmonic operator is

$$\sqrt{2}\,h^{-1/2}\sin\left(\frac{\pi x}{h}\right)g_1(y)$$

where g_1 is defined by formula (21). Intuitively one expects this function to be a fairly good approximation because the boundary conditions at the ends of the rectangle become less influential on the eigenfunction. Note that the above function does not actually lie in the quadratic form domain. The energy of this function may be computed however if we ignore this fact, and we see that

$$Q(f) = c^4 + 2dc^2\pi^2h^{-2} + \pi^4h^{-4}.$$

This compares well with the asymptotic formula (50) for μ_1 as $h \to \infty$, differing only by terms of order h^{-3}.

Corollary 12. *The bounds* (37) *and* (49) *have asymptotic formulae*

$$\frac{\|f_1^-\|_2}{\|f_1\|_2} < \frac{(\nu_1 - \lambda_1)^{1/2}}{(\lambda_3 - \nu_1)^{1/2}} = \frac{2^{1/4}(d^{1/2}c - \pi)^{1/2}}{2\pi}h^{-1/2} + O(h^{-3/2}) \tag{54}$$

and

$$\frac{\|f_1^-\|_\infty}{\|f_1\|_2} < \frac{(\nu_1 - \lambda_1)^{1/2}\lambda_3^{1/4}}{2(\lambda_3 - \nu_1)^{1/2}} = \frac{2^{1/4}(d^{1/2}c - \pi)^{1/2}c}{4\pi}h^{-1/2} + O(h^{-3/2}) \tag{55}$$

as $h \to \infty$.

Proof. The asymptotic formula of λ_3,

$$\lambda_3(h) = c^4 + (2dc^2\pi^2 + 16\pi^4)\,h^{-2} + O(h^{-3}) \tag{56}$$

as $h \to \infty$ is found by using formula (23) and the asymptotic formulae (10) for ρ_1 and ρ_3. The corollary follows by using the formulae (52), (53) and (56). ▮

Note 13. It is conjectured that

$$\frac{\|f_1\|_2}{\|f_1\|_\infty} = \frac{\cosh(c/2)\sinh(c/2) - \cos(c/2)\sin(c/2)}{2\sqrt{2}\,[\cosh^2(c/2)\sin(c/2) + \sinh(c/2)\cos^2(c/2)]}\, h^{1/2} + O(h^{-1/2}) \tag{57}$$

as $h \to \infty$ so

$$\begin{aligned}\frac{\|f_1^-\|_\infty}{\|f_1\|_\infty} &= \frac{\|f_1^-\|_\infty}{\|f_1\|_2}\frac{\|f_1\|_2}{\|f_1\|_\infty}\\ &< \frac{2^{1/4}(d^{1/2}c-\pi)^{1/2}\, c[\cosh(c/2)\sinh(c/2)-\cos(c/2)\sin(c/2)]}{8\sqrt{2}\,\pi[\cosh^2(c/2)\sin(c/2)+\sinh(c/2)\cos^2(c/2)]}\\ &\quad + O(h^{-1})\\ &< 0.121 \end{aligned} \tag{58}$$

for h large enough. Comparing this expression with (54) we see that bound (49) is of some use, but is a lot weaker than (37). An improved bound would be desirable.

7. APPENDIX

Proof of Theorem 1 (v). Preliminary calculations show that the roots β_n of equation (9) are of the form $\alpha + n^2\pi^2 + o(1)$ as $\alpha \to \infty$. Define

$$\beta_+(\alpha) = \alpha + n^2\pi^2 + 2\sqrt{2}\,n^2\pi^2\alpha^{-1/2} + 6n^2\pi^2\alpha^{-1} + \tfrac{1}{6}(-1)^n\, 5\sqrt{2}\,n^4\pi^2\alpha^{-3/2}.$$

Then

$$\begin{aligned}&\cos\sqrt{\beta_+-\alpha} - \frac{\alpha}{\sqrt{\beta_+^2-\alpha^2}}\tanh\sqrt{\beta_++\alpha}\,\sin\sqrt{\beta_+-\alpha}\\ &\quad= \cos(n^2\pi^2 + 2\sqrt{2}\,n^2\pi^2\alpha^{-1/2} + O(\alpha^{-1}))^{1/2}\\ &\qquad - \frac{\sqrt{2}\,\alpha^{1/2}}{2n\pi}\left(1 + 2\sqrt{2}\,\alpha^{-1/2} + \frac{(12+n^2\pi^2)}{2}\alpha^{-1} + O(\alpha^{-3/2})\right)^{-1/2}\\ &\qquad \times \tanh(2\alpha + O(1))^{1/2}\\ &\qquad \times \sin(n^2\pi^2 + 2\sqrt{2}\,n^2\pi^2\alpha^{-1/2} + 6n^2\pi^2\alpha^{-1}\\ &\qquad + \tfrac{1}{6}(-1)^n\, 5\sqrt{2}\,n^4\pi^4\alpha^{-3/2})^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n (1 - n^2\pi^2\alpha^{-1} + O(\alpha^{-3/2})) \\
&\quad - \frac{\sqrt{2}\,\alpha^{1/2}}{2n\pi} (1 - \sqrt{2}\,\alpha^{-1/2} - \tfrac{1}{4}n^2\pi^2\alpha^{-1} + O(\alpha^{-3/2}))(1 + O(\alpha^{-3/2})) \\
&\quad \times (-1)^n (\sqrt{2}\, n\pi\alpha^{-1/2} + 2n\pi\alpha^{-1} - 2\sqrt{2}\, n\pi\alpha^{-3/2} - \tfrac{1}{3}\sqrt{2}\, n^3\pi^3\alpha^{-3/2} \\
&\quad + \tfrac{1}{12}(-1)^n 5\sqrt{2}\, n^3\pi^3\alpha^{-3/2} + O(\alpha^{-2})) \\
&= (-1)^n [4 - \tfrac{5}{12} n^2\pi^2(1 + (-1)^n)]\, \alpha^{-1} + O(\alpha^{-3/2}) \\
&< \operatorname{sech} \sqrt{\beta_+ + \alpha}
\end{aligned}
$$

for α large enough.

Similarly, defining

$$
\beta_-(\alpha) = \alpha + n^2\pi^2 + 2\sqrt{2}\, n^2\pi^2\alpha^{-1/2} + 6n^2\pi^2\alpha^{-1} - \tfrac{1}{6}(-1)^n 5\sqrt{2}\, n^4\pi^4\alpha^{-3/2},
$$

we see that

$$
\begin{aligned}
&\cos\sqrt{\beta_- - \alpha} - \frac{\alpha}{\sqrt{\beta_-^2 - \alpha^2}} \tanh\sqrt{\beta_- + \alpha}\, \sin\sqrt{\beta_- - \alpha} \\
&\qquad = (-1)^n [4 - \tfrac{5}{12} n^2\pi^2(1 - (-1)^n)]\, \alpha^{-1} + O(\alpha^{-3/2}) \\
&\qquad > \operatorname{sech}\sqrt{\beta_- + \alpha}
\end{aligned}
$$

for α large enough.

It follows that for α large enough, β_n lies between β_- and β_+ so

$$
\beta_n(\alpha) = \alpha + n^2\pi^2 + 2\sqrt{2}\, n^2\pi^2\alpha^{-1/2} + 6n^2\pi^2\alpha^{-1} + O(\alpha^{-3/2}). \tag{59}
$$

Thus

$$
\begin{aligned}
\rho_n(\alpha) &= \beta_n(\alpha)^2 - \alpha^2 \\
&= 2n^2\pi^2\alpha + 4\sqrt{2}\, n^2\pi^2\alpha^{1/2} + O(1).
\end{aligned}
$$

Let $F(\alpha, \beta)$ be the left hand side of equation (9). Then differentiating (9), we see that

$$
\beta_n'(\alpha) = -\frac{F_1(\alpha, \beta_n)}{F_2(\alpha, \beta_n)} = 1 - \left(\frac{F_1(\alpha, \beta_n) + F_2(\alpha, \beta_n)}{F_2(\alpha, \beta_n)}\right). \tag{60}
$$

By explicit differentiation of F and substitution of the asymptotic formula (59) of β_n, we see that

$$\begin{aligned}
&\frac{2(F_1(\alpha, \beta_n) + F_2(\alpha, \beta_n))(\beta_n^2 - \alpha^2)^{3/2}}{\cosh\sqrt{\beta_n + \alpha}\cos\sqrt{\beta_n - \alpha}} \\
&\quad = 2(\beta_n - \alpha)(\beta_n + \alpha)^{1/2}\tanh\sqrt{\beta_n + \alpha} - 2\alpha(\beta_n - \alpha)^{1/2}\tan\sqrt{\beta_n - \alpha} \\
&\qquad - 2\beta_n(\beta_n - \alpha)^{1/2}(\beta_n + \alpha)^{-1/2}\tanh\sqrt{\beta_n + \alpha}\tan\sqrt{\beta_n - \alpha} \\
&\quad = \cdots \\
&\quad = -2n^2\pi^2 + O(\alpha^{-1/2}).
\end{aligned}$$

Also,

$$\begin{aligned}
&\frac{2F_2(\alpha, \beta_n)(\beta_n^2 - \alpha^2)^{3/2}}{\cosh\sqrt{\beta_n + \alpha}\cos\sqrt{\beta_n - \alpha}} \\
&\quad = (\beta_n - 2\alpha)(\beta_n - \alpha)^{1/2}(\beta_n + \alpha)\tanh\sqrt{\beta_n + \alpha} \\
&\qquad - (\beta_n + 2\alpha)(\beta_n - \alpha)(\beta_n + \alpha)^{1/2}\tan\sqrt{\beta_n - \alpha} \\
&\qquad + 2\alpha\beta_n\tanh\sqrt{\beta_n + \alpha}\tan\sqrt{\beta_n - \alpha} \\
&\quad = \cdots \\
&\quad = -\sqrt{2}\,\alpha^{3/2} + O(\alpha).
\end{aligned}$$

Hence using (60)

$$\begin{aligned}
\beta_n'(\alpha) &= 1 - \left(\frac{-2n^2\pi^2 + O(\alpha^{-1/2})}{-\sqrt{2}\,\alpha^{3/2} + O(\alpha)}\right) \\
&= 1 - \sqrt{2}\,n^2\pi^2\alpha^{-3/2} + O(\alpha^{-2}). \qquad (61)
\end{aligned}$$

Therefore

$$\begin{aligned}
\rho_n'(\alpha) &= 2\beta_n\beta_n' - 2\alpha \\
&= 2n^2\pi^2 + 2\sqrt{2}\,n^2\pi^2\alpha^{-1/2} + O(\alpha^{-1}). \quad \blacksquare
\end{aligned}$$

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