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Computation of blowing-up solutions for second-order differential equations using re-scaling techniques

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ABSTRACT

This paper presents a new technique to solve efficiently initial value ordinary differential equations of the second-order which solutions tend to have a very unstable behavior. This phenomenon has been proved by Souplet et al. in [P. Souplet, Critical exponents, special large-time behavior and oscillatory blow-up in nonlinear ode's, Differential and Integral Equations 11 (1998) 147–167; P. Souplet, Etude des solutions globales de certaines équations différentielles ordinaires du second ordre non-linéaires, Comptes Rendus de l'Academie des Sciences Paris Série I 313 (1991) 365–370; P. Souplet, Existence of exceptional growing-up solutions for a class of nonlinear second order ordinary differential equations, Asymptotic Analysis 11 (1995) 185–207; P. Souplet, M. Jazar, M. Balabane, Oscillatory blow-up in nonlinear second order ode's: The critical case, Discrete And Continuous dynamical systems 9 (3) (2003)] for the ordinary differential equation $y'' - b|y'|^{q-1}y' + |y|^{p-1}y = 0, t > 0, p > 0, q > 0$, whereby the time interval of existence of the solution is finite $[0, T_b]$ with $\lim_{t\to T_b^-} |y(t)| = \lim_{t\to T_b^-} |y'(t)| = \infty$. The blow-up of the solution and its derivatives is handled numerically using a re-scaling technique and a time-slices approach that controls the growth of the re-scaled variable through a cut-off value S. The re-scaled models on each time slice obey a criterion of mathematical and computational similarity. We conduct numerical experiments that confirm the accuracy of our re-scaled algorithms.

1. Introduction

In this paper, we consider the computation of solutions to second-order Ordinary Differential Equations of the form:

$$\begin{cases} y'' - b|y'|^{q-1}y' + |y|^{p-1}y = 0, \quad t > 0, p > 0, q > 0 & (1.1)\\ y(0) = y_{1,0}, & (1.2)\\ y'(0) = y_{2,0}. & (1.3)\end{cases} \tag{1}$$

This model describes the motion of a membrane element linked to a spring. The non-linear term in y is related to the rigidity of the spring and that in y' models a "speed-up" of the phenomenon when $b > 0$ and a "slow-down" when $b < 0$. In this last case, the initial-value problem is dissipative and the existence of the solution is global on $[0, \infty)$ [1,2]. Computationally, such a case is not difficult to handle whereas, when $b > 0$, the existence domain of the solution could be finite, with the existence of a finite "blow-up time" $T_b > 0$, at which $y(t)$ and $y'(t)$ "explode", i.e.

$$\lim_{t\to T_b^-} |y(t)| = \lim_{t\to T_b^-} |y'(t)| = \infty.$$

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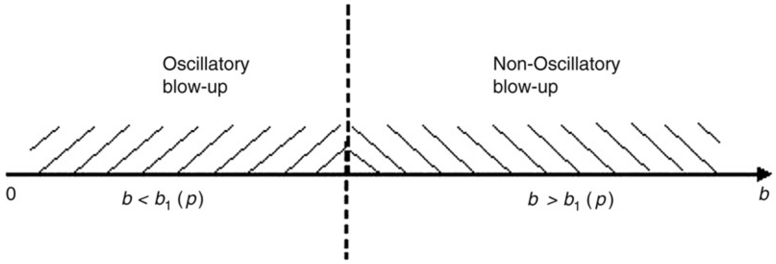

Fig. 1. Behavior of the solutions to (2) for any $b > 0, p > 1, b_1(p) = (p+1)(\frac{p+1}{2p})^{\frac{p}{p+1}}$ and $q = \frac{2p}{p+1}$.

Such a situation can exhibit two types of explosive behavior:

(1) Oscillatory, if:
  (a) $\lim_{t \to T_b^-} |y(t)| = \lim_{t \to T_b^-} |y'(t)| = \infty$, when $t \to T_b^-$, and
  (b) $y(t)$ and $y'(t)$ admit an infinite number of roots in the interval $[0, T_b[$.

(2) Non-Oscillatory, if:
  (a) $\lim_{t \to T_b^-} |y(t)| = \lim_{t \to T_b^-} |y'(t)| = \infty$, when $t \to T_b^-$,
  (b) there exists an interval $[t_1, T_b)$, $T_b > t_1 \geq 0$ in which both $y(t)$ and $y'(t)$ have no roots.

For the case $b = 1$, Souplet [3–5] considered the equation:

$$y'' + |y|^{p-1}y = |y'|^{q-1}y', \quad t \geq 0, \forall p, q > 1, \tag{2}$$

and proved the existence of two critical values $q = p$ and $q = \frac{2p}{p+1}$ in the plane (p, q) with three distinct behaviors of the solution to (1), one of which is focused on in this paper, corresponding to $1 < q \leq \frac{2p}{p+1}$. In this particular case, all non-trivial solutions explode in an oscillatory way in a finite time.

In [6], Balabane, Jazar and Souplet have also studied the critical case where $q = \frac{2p}{p+1}$, $p > 1$, and b is an arbitrary positive number. Their results are summarized in Fig. 1. Specifically:

(1) If $b \geq b_1(p) = (p+1)(\frac{p+1}{2p})^{\frac{p}{p+1}}$, all solutions of (1) are non-oscillatory and explode within a finite time. The asymptotic behavior of $y'(t)$ is given by:

$$C_1'(T_b - t)^{-\frac{p+1}{p-1}} \leq y'(t) \leq C_2'(T_b - t)^{-\frac{p+1}{p-1}}, \quad \text{as } t \to T_b, \tag{3}$$

with:

$$C_1(T_b - t)^{\frac{-2}{p-1}} \leq y(t) \leq C_2(T_b - t)^{\frac{-2}{p-1}} \quad \text{when } t \to T_b. \tag{4}$$

Here C_1, C_1', C_2 and C_2' are positive constants.

(2) If $0 < b < b_1(p) = (p+1)(\frac{p+1}{2})^{\frac{p}{p+1}}$, all solutions of (1) have a finite oscillatory blow-up time with:

$$y(t) = (T_b - t)^{\frac{-2}{p-1}} \omega(\log(T_b - t) + C) \quad \text{when } t \to T_b, \tag{5}$$

where C is a constant and $\omega(.)$ a periodic function that changes sign with $v(t) = (T_b - t)^{\frac{2}{p-1}} y(t)$.

In this work, we present a robust algorithm to efficiently compute the solutions to these singular problems occurring when $q = \frac{2p}{p+1}$. It is based on the idea of "sliced-time" computations introduced in [7]. The basic elements of this method are given in the following simple case.

2. Re-scaling for a case study: $y'' = y^p$, $p > 1$

Consider the initial-value problem:

$$\begin{cases} y'' = y^p, \ t > 0, p > 1 & (6.1) \\ y(0) = y_{1,0} \geq 0, & (6.2) \\ y'(0) = y_{2,0} \geq 0. & (6.3) \end{cases} \tag{6}$$

Multiplication of (6.1) by y' and integration from 0 to t yield:

$$\frac{y'^2(t)}{2} - \frac{y^{p+1}(t)}{p+1} = \frac{y_{2,0}^2}{2} - \frac{y_{1,0}^{p+1}}{p+1}, \quad \forall t \geq 0. \tag{7}$$

This reduces the problem to a first-order initial-value problem:

$$\begin{cases} y' = F(y) = \sqrt{\dfrac{y^{p+1}(t)}{p+1} + \dfrac{y_{2,0}^2}{2} - \dfrac{y_{1,0}^{p+1}}{p+1}}, \quad t > 0, & (8.1) \\ y(0) = y_{1,0} \geq 0. & (8.2) \end{cases} \tag{8}$$

As $F(y) = O(y^{\frac{p+1}{2}})$, one easily shows that this problem has a finite-time blow-up T_b, such that:

$$\lim_{t \to T_b} y(t) = +\infty.$$

Re-scaling techniques generate a **coarse grid** that would subdivide the time interval $[0, T]$ of integration of the differential equation (6). Since the problem under study has a finite-time existence domain $[0, T_b)$, where a priori T_b is unknown, we seek a subdivision of $[0, T_b)$ into an **infinite number of subintervals (slices)** $\{(T_{n-1}, T_n) | n \geq 0\}$ such that:

$$\cup_{n=1}^{\infty} [T_{n-1}, T_n) = [0, T_b), \qquad \lim_{n \to \infty} T_n = T_b \quad \text{and} \quad \lim_{n \to \infty} (T_n - T_{n-1}) = 0. \tag{9}$$

On the nth slice $[T_{n-1}, T_n]$ we let $y_{1,n} = y(T_n)$, $y_{2,n} = y'(T_n)$ and consider the change of variables:

$$t = T_{n-1} + \beta_n s, \qquad y(t) = y_{1,n-1}(1 + z_{1,n}(s)), \qquad y'(t) = y_{2,n-1}(1 + z_{2,n}(s)), \tag{10}$$

with the parameter β_n selected on the basis of allowing the re-scaled systems to become **"similar" on all the slices**. For simplicity of notations, we shall use: $z_1 = z_{1,n}$ and $z_2 = z_{2,n}$ and find out that z_1 verifies:

$$\begin{cases} \dfrac{d^2 z_1}{ds^2} = \beta_n^2 y_{1,n-1}^{p-1} (1 + z_1)^p, \quad s > 0, & (11.1) \\ z_1(0) = 0, & (11.2) \\ z_1'(0) = \beta_n \dfrac{y_{2,n-1}}{y_{1,n-1}}. & (11.3) \end{cases} \tag{11}$$

By selecting:

$$\beta_n = \frac{1}{(y_{1,n-1})^{(p-1)/2}} \tag{12}$$

(11) becomes:

$$\begin{cases} \dfrac{d^2 z_1}{ds^2} = (1 + z_1)^p, \quad s > 0, & (13.1) \\ z_1(0) = 0, & (13.2) \\ z_1'(0) = \omega_n = \dfrac{y_{2,n-1}}{(y_{1,n-1})^{(p+1)/2}}. & (13.3) \end{cases} \tag{13}$$

At that point, we need an **additional constraint** that allows determining the size of the nth slice: $T_n - T_{n-1} = \beta_n s_n$. We refer to it as **the end of slice condition**. This condition depends on the solution behavior and is based, in the considered problem, on the observation that:

$$y(T_n) = y_{1,n} = y_{1,n-1}(1 + z_1(s_n)) > y(T_{n-1}) = y_{1,n-1},$$

leading to the condition $z_1(s_n) = S$, where S is a "cutoff" value that "stops" the growth of $z_1(s)$ on $[0, s_n]$, and therefore that of $y(t)$ on $[T_{n-1}, T_n]$. Such a restriction leads to:

$$y(T_n) = y_{1,n} = y_{1,n-1}(1 + S) = y_{1,0}(1 + S)^n, \quad \text{and} \quad \beta_n = \frac{1}{y_{1,0}^{\frac{p-1}{2}} (1 + S)^{(n-1)(p-1)/2}}.$$

Hence the computation of (6) reduces into solving a sequence of "shooting problems", whereby on the nth slice, one computes an initial-value problem with a stopping criterion:

$$\begin{cases} \dfrac{d^2 z_1}{ds^2} = (1 + z_1)^p, \quad 0 < s \leq s_n, & (14.1) \\ z_1(0) = 0 \geq 0, & (14.2) \\ z_1'(0) = \omega_n, & (14.3) \\ z_1(s_n) = S. & (14.4) \end{cases} \tag{14}$$

Note that in such a problem, the initial and final values of $z_1(s)$ are preset to 0 and S respectively, while $z_1'(0)$ varies with n. One proves the following results:

Theorem 1. *The sequence of problems (14) is "similar", in the sense that, there exist constants c_0, c_1, d_0 and d_1 such that:*

(1) $\forall n, c_0 \le \omega_n \le c_1$,
(2) $\forall n, d_0 \le s_n \le d_1$.

Proof. For the first part of the proof, note from:

$$y' = F(y) = \sqrt{\frac{y^{p+1}(t)}{p+1} + \frac{y_{2,0}^2}{2} - \frac{y_{1,0}^{p+1}}{p+1}}, \quad t > 0,$$

that

$$\frac{y'(t)}{(y(t))^{(p+1)/2}} = \sqrt{\frac{2}{p+1} + \frac{K}{y^{p+1}}},$$

where $K = \frac{y_{2,0}^2}{2} - \frac{y_{1,0}^{p+1}}{p+1}$.

Without loss of generality assume $K \ge 0$. Thus, as $y(t) \ge y_{1,0}$, one has:

$$\sqrt{\frac{2}{p+1}} \le \omega_n \le \sqrt{\frac{2}{p+1} + \frac{K}{y_{1,0}^{p+1}}}.$$

For the second part of the proof, integration of (14.1) gives:

$$s \le z_1'(s) - \omega_n \le (1+S)^p s, \quad \forall s \in [0, s_n].$$

A second integration yields:

$$s^2/2 + \omega_n s \le z_1(s) \le \omega_n s + (1+S)^p s^2/2, \quad \forall s \in [0, s_n].$$

In particular, for $s = s_n$, one has:

$$s_n^2/2 + \omega_n s_n \le S \le \omega_n s_n + (1+S)^p s_n^2/2.$$

The bounds on ω_n would then yield the second part of the theorem. ■

As a consequence, one obtains the following results:

Theorem 2. *The sequence $\{T_n\}$ associated with the similar problems (14) verifies the following results:*

(1) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} T_n = T_b$, *and*
(2) $d_0 g(S) \le T_b \le d_1 g(S)$, *where* $g(S) = \frac{(1+S)^{(p-1)/2}}{(1+S)^{(p-1)/2} - 1}$.

Proof. Formula (12) implies that $\lim_{n\to\infty} \beta_n = 0$. On the other hand the identity:

$$T_N = \Sigma_{n=1}^{N} \beta_n s_n,$$

would imply as $N \to \infty$ the estimate on T_b. ■

Numerically, we deal with (14) by changing it into a first-order system of equations through the variable $z_2(s)$, given by $y'(t) = y_{2,n-1}(1 + z_2(s))$. This yields:

$$\begin{cases} \dfrac{dz_1}{ds} = \omega_n(1+z_2), & (15.1)\\ \dfrac{dz_2}{ds} = \dfrac{1}{\omega_n}(1+z_1)^p, \quad 0 < s \le s_n, & (15.2)\\ z_1(0) = 0, & (15.3)\\ z_2(0) = 0, & (15.4)\\ z_1(s_n) = S. & (15.5) \end{cases} \tag{15}$$

3. A numerical solver on the *n*th slice, $n \le n_0$

By letting: $\mathbf{z}(s) = \binom{z_1(s)}{z_2(s)}$, $\mathbf{z}(0) = \binom{0}{0}$ and $\mathbf{g_{n(z)}} = \begin{pmatrix} \omega_n(1+z_2) \\ \frac{1}{\omega_n}(1+z_1^p) \end{pmatrix}$,
(15) could be rewritten in the vectorial form:

$$\begin{cases} \dfrac{d\mathbf{z}}{ds} = \mathbf{g_{n(z)}}, \quad 0 < s \le s_n, & (16.1) \\ \mathbf{z}(0) = \mathbf{0}, & (16.2) \\ z_1(s_n) = S. & (16.3) \end{cases} \tag{16}$$

- *Total Number of Slices:*
Since $\beta_n = O\left(\frac{T_n - T_{n-1}}{T_1}\right)$, note that for a given computational tolerance of ϵ_{Tol}, the total number of slices n_0, on which we solve (15) is reached when
$$\beta_{n_0} \le \epsilon_{\mathrm{Tol}} < \beta_{n_0-1}. \tag{17}$$
Since re-scaling provides similar models on all the slices, a first advantage consists in implementing a scheme that uses a uniform mesh for numerical integration on each nth slice $1 \le n \le n_0$.
- *Numerical Method:*
We have chosen the standard fourth-order explicit Runge–Kutta method with mesh size τ. On the nth slice, the method would yield $\{\mathbf{Z}^{c,k} \cong \mathbf{z}(s^k)\}$, where $s^k = k\tau$, such that $|\mathbf{Z}_1^c(s^k)| \le S, \ \forall k \le l$, using the formulae:
$$\begin{cases} \mathbf{K}_1 = \tau \mathbf{g}_n(\mathbf{Z}^{c,k}), & (18.1) \\ \mathbf{K}_2 = \tau \mathbf{g}_n\left(\mathbf{Z}^{c,k} + \dfrac{1}{2}\mathbf{K}_1\right), & (18.2) \\ \mathbf{K}_3 = \tau \mathbf{g}_n\left(\mathbf{Z}^{c,k} + \dfrac{1}{2}\mathbf{K}_2\right), & (18.3) \\ \mathbf{K}_4 = \tau \mathbf{g}_n\left(\mathbf{Z}^{c,k} + \mathbf{K}_3\right), & (18.4) \\ \mathbf{Z}^{c,k+1} = \mathbf{Z}^{c,k} + \dfrac{1}{6}\left(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4\right). & (18.5) \end{cases} \tag{18}$$
- *Mesh Size:*
The mesh size τ, is found on the basis of solving (16)
$$\begin{cases} \dfrac{d\mathbf{z}}{ds} = \mathbf{g_{1(z)}}, \quad 0 < s \le s_n, & (19.1) \\ \mathbf{z}(0) = \mathbf{0}, & (19.2) \\ z_1(s_1) = S, & (19.3) \end{cases} \tag{19}$$
using (18) for the used computational tolerance ϵ_{Tol}. One starts with $\tau_1 = \frac{1}{2}$ and compute $\mathbf{Z}_{\tau_1}$ as an approximation to $\mathbf{z}(\tau_1)$. These verify:
$$\mathbf{z}(\tau_1) = \mathbf{Z}_{\tau_1} + \mathbf{a}_1 \tau_1^r + O(\tau_1^{r+1}), \tag{20}$$
with $r = 4$ for (18). Similarly, compute $\mathbf{Z}_{\tau_1/2}$ as an approximation to $\mathbf{z}(\tau_1/2)$ with:
$$\mathbf{z}(\tau_1) = \mathbf{Z}_{\tau_1/2} + \mathbf{a}_1(\tau_1/2)^r + O(\tau_1^{r+1}). \tag{21}$$
Multiplying (21) by 2^r and subtracting (20) from (21), lead to:
$$\mathbf{z}(\tau_1) = \mathbf{Z}_{\frac{\tau_1}{2}} + \frac{\mathbf{Z}_{\frac{\tau_1}{2}} - \mathbf{Z}_{\tau_1}}{2^r - 1} + O(\tau_1^{r+1}). \tag{22}$$
Hence, the step size τ_1 is refined (divided by 2) until achieving τ_0 such that:
$$\frac{\|\mathbf{Z}_{\frac{\tau_0}{2}} - \mathbf{Z}_{\tau_0}\|}{\|\mathbf{Z}_{\frac{\tau_0}{2}}\|} \le (2^r - 1)\epsilon_{\mathrm{Tol}} \le \frac{\|\mathbf{Z}_{\tau_0} - \mathbf{Z}_{2\tau_0}\|}{\|\mathbf{Z}_{\tau_0}\|}. \tag{23}$$
- *Adaptive Procedure:*
At the end of the nth slice an adaptive procedure is adopted to reach the stopping criterion (16.3). The refining process consists of dividing the mesh size of the final time interval $[s^{l-1}, s^l]$ by 2 until:
$$\frac{|\mathbf{Z}_1^{c,l} - \mathbf{Z}_1^{c,l-1}|}{|\mathbf{Z}_1^{c,l}|} \le \epsilon_{\mathrm{Tol}} \text{ with } \mathbf{Z}_1^{c,l-1} < S < \mathbf{Z}_1^{c,l}, \tag{24}$$
so that $\mathbf{Z}_1^{c,l} \cong S$ and $s_n^c = s^l$.

- *Numerical Results for* $\mathbf{y}'' = \mathbf{y}^{\mathbf{p}}$, $p > 1$, $y(0) = 1$ and $y'(0) = \sqrt{\frac{2}{p+1}}$:

The blow-up time is given by: $T_b = \int_0^\infty \frac{dy}{\sqrt{\frac{2}{p+1}(1+y)^{p+1}}} = \frac{\sqrt{2(p+1)}}{p-1}$. The following tables compute the blow-up time T_b of the solution to (6) when $y(0) = 1$ and $y'(0) = \sqrt{\frac{2}{p+1}}$ up to a precision of $\varepsilon = \frac{1}{2}10^{-09}$.

. $p = 7$, $T_b = 6.666666666666666\text{e}{-001}$

Cutoff value	Number of slices	Computed T_b	Relative error
1	12	6.666667233738793e−001	8.506081905501617e−008
2	8	6.666684039412696e−001	2.605911904429714e−006
3	7	6.666845760989344e−001	2.686414840163964e−005
4	6	6.666883118742275e−001	3.246781134130794e−005
5	5	6.666889468937676e−001	3.342034065140220e−005
10	4	6.669083566852009e−001	3.625350278013695e−004

. $p = 5$, $T_b = 8.660254037844386\text{e}{-001}$

Cutoff value	Number of slices	Computed T_b	Relative error
1	17	8.660254109836718e−001	8.312958463015252e−009
2	11	8.660254656535790e−001	7.144032969018153e−008
3	9	8.660256537609148e−001	2.886479716334406e−007
4	8	8.660263614597145e−001	1.105828156626750e−006
5	7	8.660272720618647e−001	2.157300949757250e−006
10	6	8.660390890898021e−001	1.580242947111533e−005

. $p = 3$, $T_b = 1.414213562373095\text{e}{+000}$

Cutoff value	Number of slices	Computed T_b	Relative error
1	32	1.414213576171449e+000	9.756909379156640e−009
2	21	1.414213600578187e+000	2.701507930014423e−008
3	17	1.414213637956183e+000	5.344531391829787e−008
4	15	1.414213706619683e+000	1.019977405431187e−007
5	13	1.414213795921668e+000	1.651437798090409e−007
10	10	1.414214789846060e+000	8.679544568379587e−007

. $p = 2$, $T_b = 2.449489742783178\text{e}{+000}$

Cutoff value	Number of slices	Computed T_b	Relative error
1	63	2.449489747967264e+000	2.116394267254796e−009
2	40	2.449489751835296e+000	3.695511766408810e−009
3	32	2.449489755848351e+000	5.333834518809680e−009
4	28	2.449489760053198e+000	7.050456355510007e−009
5	25	2.449489764323668e+000	8.793868332709786e−009
10	19	2.449489790621602e+000	1.952995495242392e−008

. $p = 1.2$, $T_b = 1.048808848170151\text{e}{+001}$

Cutoff value	Number of slices	Computed T_b	Relative error
1	310	1.048808848408980e+001	2.277152375178535e−010
2	196	1.048808848501172e+001	3.156167214837981e−010
3	156	1.048808848592861e+001	4.030382137628263e−010
4	135	1.048808848676714e+001	4.829891791094239e−010
5	121	1.048808848701219e+001	5.063532910454659e−010
10	91	1.048808848882957e+001	6.796341855088047e−010

4. Case of $y'' - b|y'|^{q-1}y' + |y|^{p-1}y = 0, q = \frac{2p}{p+1}$

The procedure that allows "sliced-time" computations to (1) follows the same steps as for the case study of the previous section.

In this paper, we deal only with the case when $q = \frac{2p}{p+1}$.

Multiplication of (1.1) by y' and integration from 0 to t yield the "Energy equation":

$$\frac{y'^2(t)}{2} + \frac{|y|^{p+1}(t)}{p+1} = \frac{y_{2,0}^2}{2} + \frac{y_{1,0}^{p+1}}{p+1} + \int_0^t b|y'|^{q+1}\mathrm{d}s, \quad \forall t \geq 0. \tag{25}$$

On the basis of the subdivision (9) and the change of variables (10), one finds for (1), the equivalent form to (11):

$$\begin{cases} \dfrac{\mathrm{d}^2 z_1}{\mathrm{d}s^2} = b\beta_n^{2-q}|y_{1,n-1}|^{q-1}|z_1'|^{q-1}z_1', -\beta_n^2|y_{1,n-1}|^{p-1}|1+z_1|^{p-1}(1+z_1) \quad s > 0 & (26.1) \\ z_1(0) = 0, & (26.2) \\ z_1'(0) = \beta_n \dfrac{y_{2,n-1}}{y_{1,n-1}}. & (26.3) \end{cases} \tag{26}$$

To make these slice models similar, one needs to have the coefficients

$$\omega_n = z_1'(0) = \beta_n \frac{y_{2,n-1}}{y_{1,n-1}}, \qquad \gamma_{1,n} = \beta_n^2 |y_{1,n-1}|^{p-1}, \qquad \gamma_{2,n} = \beta_n^{2-q}|y_{1,n-1}|^{q-1}$$

uniformly bounded, independently from n.

Theorem 3. *If $\gamma_{1,n} = 1$ and $q = \frac{2p}{p+1}$, then:*

(1) $\beta_n = \frac{1}{|y_{1,n-1}|^{(p-1)/2}}$ *and in the case of blow-up* $\lim_{n\to\infty} \beta_n = 0$.

(2) $\gamma_{2,n} = 1$.

(3) *If the blow-up is non-oscillatory ($b \geq b_1(p) = (p+1)(\frac{p+1}{2p})^{\frac{p}{p+1}}$), then ω_n has a constant sign and there exist positive constants c_1 and c_2 independent from n such that for large n, $c_1 \leq \omega_n \leq c_2$.*

(4) *If the blow-up is oscillatory ($b < b_1(p) = (p+1)(\frac{p+1}{2p})^{\frac{p}{p+1}}$), then ω_n changes sign and for large n, $\omega_n \leq c_3$, where c_3 is independent from n.*

Proof. The proof of the first part is straightforward.

The second part is proved by applying simple algebra to $\gamma_{2,n} = \left(\frac{1}{|y_{1,n-1}|^{(p-1)/2}}\right)^{2-q} |y_{1,n-1}|^{q-1}$, using $q = \frac{2p}{p+1}$.

For the third part, note first that:

$$\omega_n = \frac{1}{|y_{1,n-1}|^{(p-1)/2}} \frac{y_{2,n-1}}{y_{1,n-1}} = \frac{y_{2,n-1}}{\mathrm{sign}(y_{1,n-1})|y_{1,n-1}|^{(p+1)/2}}.$$

When the blow-up is non-oscillatory, then as $t \to T_b$, $y(t)$ and $y'(t)$ would have the same sign and are governed by (3) and (4) as proved in [6]. If we assume (without loss in generality) that y and y' have positive signs then one obtains:

$$\frac{C_1'}{C_2^{(p-1)/2}} \leq |\omega_n| \leq \frac{C_2'}{C_1^{(p-1)/2}}.$$

For the fourth part, in the same reference [6], the behavior of $y(t)$ as $t \to T_b$, is such that (5) is verified:

$$y(t) = (T_b - t)^{\frac{-2}{p-1}} w(\log(T_b - t) + C) \quad \text{when } t \to T_b,$$

where C is a constant and $w(.)$ a periodic function that changes sign with $v(t) = (T_b - t)^{\frac{2}{p-1}} y(t)$. A similar algebra to that of the third part would give $|\omega_n| = O(1)$. ■

As for the computational approach, we transform (26) into a first-order system, by fixing a cutoff value S that would determine the slice size $T_n - T_{n-1} = \beta_n s_n$. Specifically one computes the initial-value shooting problem:

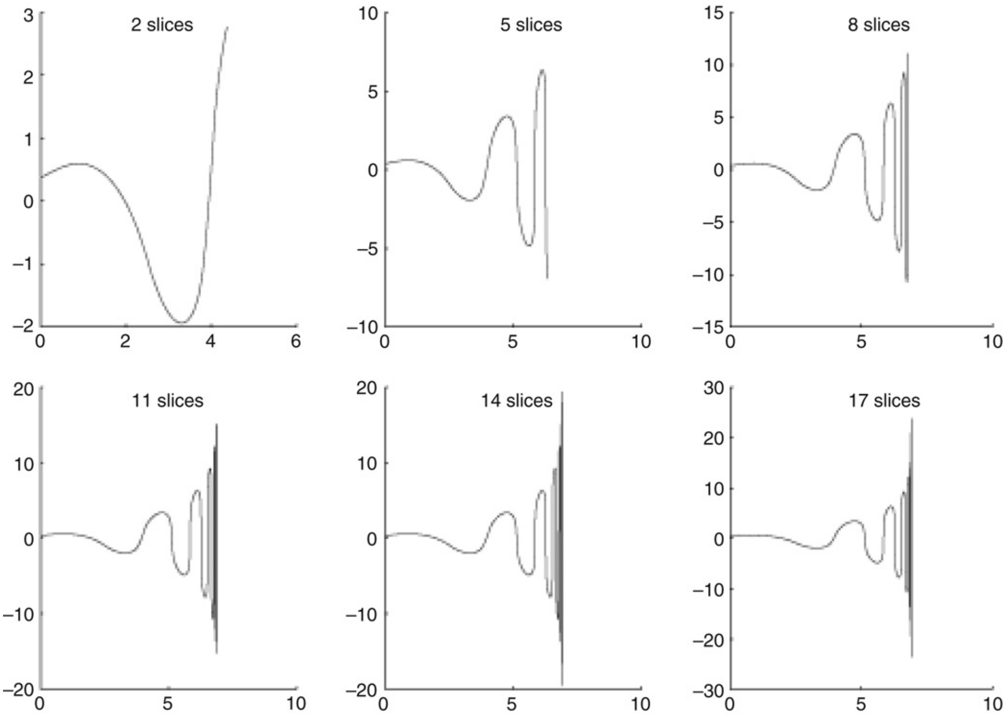

Fig. 2. Example 1: variation of $\ln(y)$ with respect to t for $y'' = -|y|^{0.7}y + |y'|^{0.2592}y'$, $y(0) = 1$ and $y'(0) = 1$, with $S = 5$.

$$\begin{cases} \dfrac{dz_1}{ds} = \omega_n(1+z_2), \quad 0 < s \le s_n, & (27.1) \\ \dfrac{dz_2}{ds} = b|\omega_n|^{q-1}|1+z_2|^{q-1}(1+z_2) - \dfrac{1}{\omega_n}|1+z_1|^{p-1}(1+z_1), & (27.2) \\ z_1(0) = 0, & (27.3) \\ z_2(0) = 0, & (27.4) \\ |z_1(s_n)| = S. & (27.5) \end{cases} \tag{27}$$

Since the sequences $\{y_{1,n}\}$, $\{y_{2,n}\}$, and $\{T_n\}$ verify:

$$\begin{cases} y_{1,n} = y_{1,n-1}[1+z_1(s_n)], & (28.1) \\ y_{2,n} = y_{2,n-1}[1+z_2(s_n)], & (28.2) \\ T_n = T_{n-1} + \beta_n s_n, & (28.3) \end{cases} \tag{28}$$

then, the choice of the cutoff value S is based on the following result:

Theorem 4. (1) *If the blow-up is non-oscillatory, then:*

$$\exists n_0 | \forall n \ge n_0, \quad |y_{1,n}| = |y_{1,n-1}|(1+S).$$

(2) *If the blow-up is oscillatory, then:*

$$\forall n, \quad |y_{1,n-1}|(S-1) \le |y_{1,n}| \le |y_{1,n-1}|(1+S).$$

As a result, one has:

Theorem 5. *If the blow-up is oscillatory, then a sufficient condition for the choice of S is $S > 1$.*

5. Results of numerical tests for $y'' = -|y|^{p-1}y + b|y'|^{q-1}y'$, $y(0) = 1$ and $y'(0) = 1$, $q = \frac{2p}{p+1}$

Results are only given for **oscillatory systems** in which case, values of S close to 1 are to be excluded. Computational tolerance is $\varepsilon_{\mathrm{Tol}} = \frac{1}{2}10^{-006}$.

- *Example* 1: $\mathbf{p = 1.7}$, $q = \frac{2p}{p+1} = 1.2592$, $b_1 = (p+1)(\frac{p+1}{2p})^{(\frac{p}{p+1})} = 2.3352$ (see Fig. 2).

Cutoff S	Computed T_b	Number of slices
$\mathbf{b = 1}$	$\boldsymbol{\tau_0 = 7.8125\text{e}{-003}}$	
2	6.905839905214391	60
3	6.905846065083031	42
4	6.905850775653478	36
5	6.905851250986963	31
$\mathbf{b = 1.5}$	$\boldsymbol{\tau_0 = 7.8125\text{e}{-003}}$	
2	4.741936108013850	49
3	4.741940732163934	37
4	4.741941416493013	31
5	4.741941541394696	27
$\mathbf{b = 2}$	$\boldsymbol{\tau_0 = 1.5625\text{e}{-002}}$	
2	3.669740045525140	42
3	3.669741709731745	32
4	3.669742126114941	27
5	3.669743480338277	25

- *Example* 2: $\mathbf{p = 2}$, $q = \frac{2p}{p+1} = \frac{4}{3}$, $b_1 = (p+1)(\frac{p+1}{2p})^{(\frac{p}{p+1})} = 2.4764$ (see Fig. 3).

Cutoff S	Computed T_b	Number of slices
$\mathbf{b = 1}$	$\boldsymbol{\tau_0 = 7.8125\text{e}{-003}}$	
2	5.209686796510288	44
3	5.209691346707377	31
4	5.209693875703810	26
5	5.209693924947938	22
$\mathbf{b = 1.5}$	$\boldsymbol{\tau_0 = 7.8125\text{e}{-003}}$	
2	3.614915431683971	36
3	3.614916514650755	26
4	3.614919039290611	23
5	3.614918803569950	20
$\mathbf{b = 2}$	$\boldsymbol{\tau_0 = 7.8125\text{e}{-003}}$	
2	2.825331269928384	31
3	2.825331658024978	23
4	2.825331556706841	19
5	2.825331998617265	17

- *Example* 3: $\mathbf{p = 3}$, $q = \frac{2p}{p+1} = \frac{3}{2}$, $b_1 = (p+1)(\frac{p+1}{2p})^{(\frac{p}{p+1})} = 2.5708$ (see Fig. 4).

Cutoff S	Computed T_b	Number of slices
$\mathbf{b = 1}$	$\boldsymbol{\tau_0 = 3.90625\text{e}{-003}}$	
2	3.231011024877603	25
3	3.231014916815904	19
4	3.231014836720568	14
5	3.231014905619674	10
$\mathbf{b = 1.5}$	$\boldsymbol{\tau_0 = 3.90625\text{e}{-003}}$	
2	2.292508822634327	22
3	2.292509254759319	16
4	2.292509352667330	13
5	2.292509393868601	11
$\mathbf{b = 2}$	$\boldsymbol{\tau_0 = 3.90625\text{e}{-003}}$	
2	1.825205554301452	19
3	1.825205857725333	14
4	1.825205895342057	12
5	1.825205800058116	10

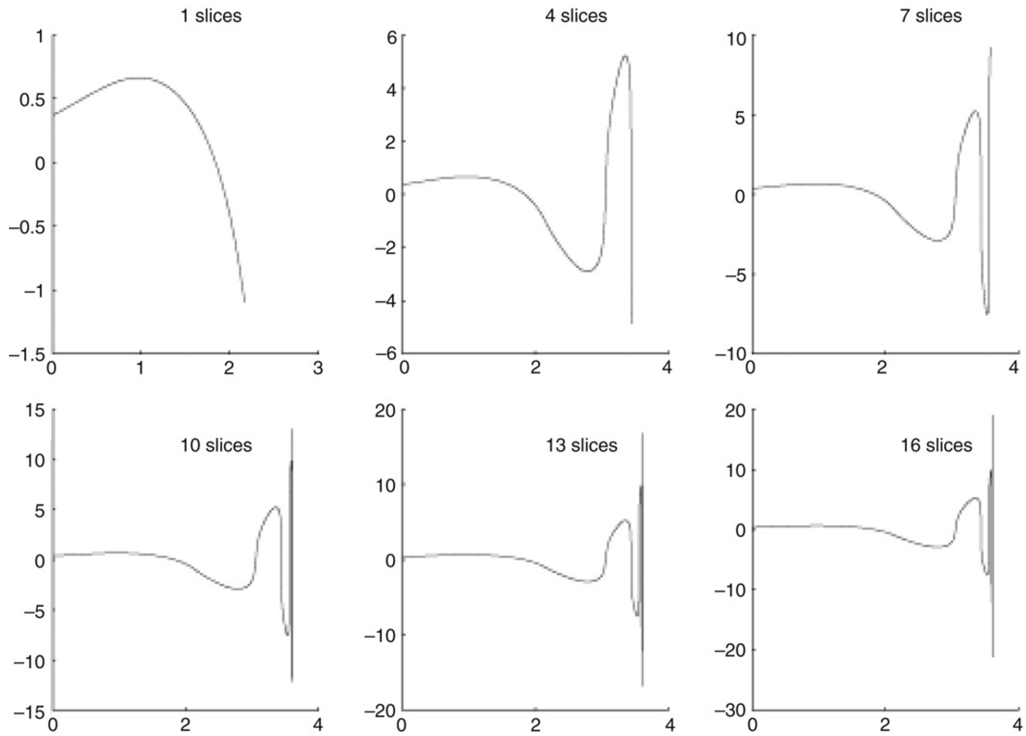

Fig. 3. Example 2: variation of $\ln(y)$ with respect to t for $y'' = -|y|y + 1.5|y'|^{\frac{1}{3}}y'$, $y(0) = 1$, $y'(0) = 1$, with $S = 4$.

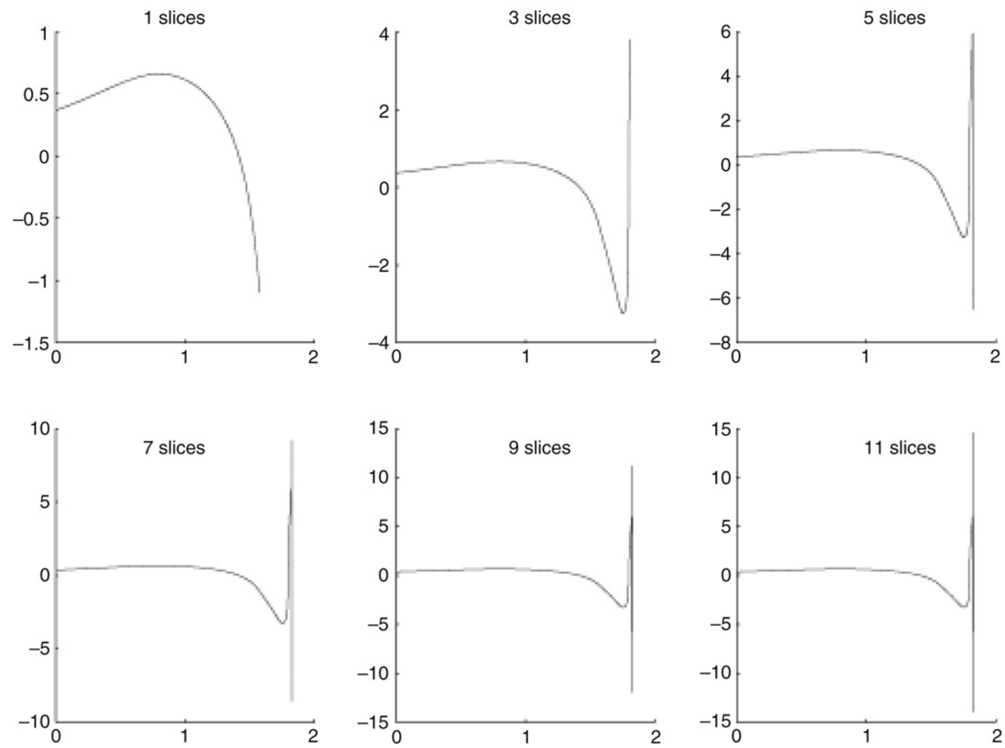

Fig. 4. Example 3: variation of $\ln(y)$ with respect to t for $y'' = -|y|^2y + 2|y'|^{\frac{1}{2}}y'$, $y(0) = 1$, $y'(0) = 1$, with $S = 4$.

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