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Ranks of tensors, secant varieties of Segre varieties and fat points[☆]

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Abstract

A classical unsolved problem of projective geometry is that of finding the dimensions of all the (higher) secant varieties of the Segre embeddings of an arbitrary product of projective spaces. An important subsidiary problem is that of finding the smallest integer t for which the secant variety of projective t-spaces fills the ambient projective space.

In this paper we give a new approach to these problems. The crux of our method is the translation of a well-known lemma of Terracini into a question concerning the Hilbert function of "fat points" in a multiprojective space. Our approach gives much new information on the classical problem even in the case of three factors (a case also studied in the area of Algebraic Complexity Theory).

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0. Introduction

The problem of how to minimally represent certain kinds of tensors as a sum of tensors of a prescribed type (the case of decomposable tensors is what we will consider here) is a problem with a long history (e.g. see [5,10,15,23,28,31]; also [17] for a computational point of view and [11] in the symmetric case). Knowledge of this subject is quite scattered and suffers a bit from the fact that the same type of problem is considered in different areas using different language. We have tried, in this paper, to cite the references from the different areas that we could find.

All of these problems can be considered in the following setting: let $V_1, \dots, V_t$ be finite dimensional vector spaces over the field k (we will always assume that char $k = 0$ and that k is algebraically closed) and let

$$\mathbf{V} = V_1^* \otimes \cdots \otimes V_t^* \simeq (V_1 \otimes \cdots \otimes V_t)^*.$$

If $T \in \mathbf{V}$ one can ask: What is the length of the minimal representation of T as a sum of decomposable tensors? (Recall that T is said to be *decomposable* if we can find vectors $v_i^* \in V_i^*$ such that $T = v_1^* \otimes \cdots \otimes v_t^*$.)

The answer to this question is usually referred to as the *tensor rank* of T. Moreover, since $\mathbf{V}$ is a finite dimensional vector space of dimension $\prod_{i=1}^t (\dim_k V_i)$, which has a basis of decomposable tensors, it is quite trivial to see that every $T \in \mathbf{V}$ is the sum of decomposable tensors (see also Section 1).

It is natural, then, to ask the following three questions:

(1) What is the least integer $D(\mathbf{V})$ such that *every* tensor in $\mathbf{V}$ has tensor rank $\leqslant D(\mathbf{V})$?
(2) What is the least integer $E(\mathbf{V})$ such that there is a dense subset $\mathscr{S} \subset \mathbf{V}$ (dense in the Zariski topology) so that every tensor in $\mathscr{S}$ has tensor rank $\leqslant E(\mathbf{V})$ (this is called the *typical rank* of $\mathbf{V}$ in [5] and the *essential rank* of $\mathbf{V}$ in [10])?
(3) Given an integer r such that $0 < r < E(\mathbf{V})$, what is the dimension of the closure (using the Zariski topology) of the set of all tensors of tensor rank $\leqslant r$?

Our main focus in this paper is on Questions (2) and (3). It is well known that answering these questions is equivalent to solving the problem of determining the dimensions of certain secant varieties to Segre varieties (e.g. see [16] for the case $t = 2$, where everything is well known, or [5] where the higher secant varieties to Segre varieties are discussed for more than 2 factors). The study of higher secant varieties is a very classical subject in Algebraic Geometry, e.g. see [24] or [29], which in recent times, especially after the outstanding work of Zak [32] has received renewed interest, see e.g. [1,6,9,22].

As we mentioned above, Questions (2) and (3) have been considered in several contexts. In the context of Algebraic Complexity Theory (see e.g. [5], especially Chapter 20 and the references there) there are many results in the case $t = 3$ (see our Section 3).

On the other hand, in the context of Representation Theory the emphasis is on Question (3) (for any t) and related problems, such as the singularities of the closure, desingularization, minimal resolution of defining ideal (e.g., see [25]). The Representation Theory approach (at least from the point of view of the higher secant varieties) appears nowhere in the literature and appears to be able to cover in an easy way a very limited, but interesting, number of cases.

Within the context of Algebraic Coding Theory, the emphasis is on Question (2). The results so far from Algebraic Coding Theory show that this approach covers a very limited, but again very interesting, set of cases (see our Section 2).

Our approach is different from all of those above and is inspired by the work of Iarrobino–Kanev [19] and Alexander–Hirschowitz [2] who treated a similar problem which is related to the higher secant varieties to Veronese Varieties.

Using Terracini's Lemma (or the method of Macaulay's Inverse Systems, also classically known, mainly in the case of symmetric tensors, as *apolarity*, see e.g. [30] or [11]), we can convert questions about secant varieties into questions concerning the calculation of a specific value of the Hilbert function of the ideal of a scheme of "2-*fat points*" in a multiprojective space.

We solve the Hilbert function problem in several cases. Our results for the case $t = 3$ cover infinitely many cases not covered by the methods of [28] and [23] with respect to Questions (2) and (3).

Our reinterpretation of the Algebraic Coding Theory results in the language of 2-fat points allows us to extend the observations of Ehrenborg [10] about Question (2) to (3).

As far as the representation theoretic point of view is concerned, the higher secant varieties to Segre varieties are $G = GL(V_1^*) \times \cdots \times GL(V_t^*)$ equivariant, therefore they are, in principle, easy to determine when $\mathbf{V}$ has finitely many G-orbits. This happens only for $t \leqslant 3$. For $t = 2$ it happens always. For $t = 3$ it happens for a very specific family of values of the triples (n_1, n_2, n_3), where $n_i + 1 = \dim V_i$ (see [21]), more precisely for

$$(n_1, n_2, n_3) \in \big\{(1, 1, n), (1, 2, n)\big\}.$$

This same kind of classification may sometimes be made even when there are not finitely many G-orbits, but a classification of all orbits is possible. These are the so-called *tame* cases. In our context the tame cases correspond to the tuples in the set

$$\big\{(2, 2, 2),\ (1, 3, 3),\ (1, 1, 1, 1)\big\}$$

(see e.g. [21, Tables III, IV and I]), see also [26]. All the other cases are called *wild* and are, in principle, difficult to treat by invariant theoretic methods.

Our results properly contain the finite and tame cases: see Theorem 3.1, Example 3.2 (where all cases are wild, except the last one, which is tame), Proposition 3.7 and Example 4.2.

We take this opportunity to warmly thank J. Weyman for his help in interpreting the "folklore" results in this approach to the problem.

The paper is organized in the following way: after a section of preliminaries, in Section 2 we consider schemes of 2-fat points in multiprojective spaces which are built up from co-ordinate points in the factor spaces. In this context the questions we have been considering convert easily into problems about monomial ideals which, in turn, have fascinating combinatorial interpretations. In this way we show how one can use results from coding theory to obtain theorems about secant varieties; moreover this kind of connection suggests problems about monomial ideals in products of polynomial rings which were not under the "spotlight" before. The results of this section owe an enormous debt to the work of Ehrenborg in [10]. The novelty of our approach in this section is in the interpretation of some of Ehrenborg's combinatorial results in the language of monomial ideals. This reinterpretation permits us to extend the results of Ehrenborg, which dealt with Question (2) exclusively, so as to also deal with Question (3).

In Section 3 we consider the general (i.e. non-monomial) case. Here we give our main results concerning Questions (2) and (3) in Theorem 3.1, Proposition 3.3 and Proposition 3.7. The novelty of our approach is evident as we obtain in this section, by elementary arguments, many results already in print, as well as new results.

In Section 4 we review the literature, especially with respect to the case of 3-tensors (since that is where so much work on these Questions has been done) and compare our results to those obtained by others.

There are several people we have consulted during the preparation of this work whom we would like to thank: John Abbott and Ciro Ciliberto for several stimulating conversations about the material of this paper; Tony Iarrobino for bringing the work of Ehrenborg to our attention; Peter Bürgisser for making us aware of the literature (in particular his fascinating book) on the connections between our work and Algebraic Complexity Theory.

1. Preliminaries: secant varieties, Terracini's Lemma

Let $V_1, \ldots, V_t$ be vector spaces of dimensions $n_1 + 1, \ldots, n_t + 1$, respectively. With no loss of generality, we assume that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$.

Let $\mathscr{B}_i^* = \{x_{0,i}, x_{1,i}, \ldots, x_{n_i,i}\}$ be a basis for V_i^*, so that

$$\mathscr{B}^* = \left\{x_{j_1,1} \otimes \cdots \otimes x_{j_t,t} \mid 0 \leqslant j_i \leqslant n_i \text{ for } i = 1, \ldots, t\right\}$$

is a basis for $\mathbf{V} = V_1^* \otimes \cdots \otimes V_t^*$. Thus any $T \in \mathbf{V}$ can be written

$$\begin{aligned} T &= \sum_{\substack{0 \leqslant j_i \leqslant n_i \\ 1 \leqslant i \leqslant t}} \alpha_{j_1,\ldots,j_t} x_{j_1,1} \otimes \cdots \otimes x_{j_{t-1},t-1} \otimes x_{j_t,t} \\ &= \sum_{\substack{0 \leqslant j_i \leqslant n_i \\ 1 \leqslant i \leqslant t-1}} x_{j_1,1} \otimes \cdots \otimes x_{j_{t-1},t-1} \otimes y_{j_1,\ldots,j_{t-1}} \qquad (\dagger) \end{aligned}$$

with $\alpha_{j_1,\ldots,j_t} \in k$ and $y_{j_1,\ldots,j_{t-1}} = \sum_{0 \leqslant j_t \leqslant n_t} \alpha_{j_1,\ldots,j_{t-1},j_t} x_{j_t,t} \in V_t^*$.

From (†) above, we have an easy bound for the tensor rank of every vector in $\mathbf{V}$:

$$D(\mathbf{V}) \leqslant \prod_{i=1}^{t-1} (\dim_k V_i). \tag{††}$$

Notice also that for any $T \in \mathbf{V}$ and any $\lambda \neq 0$ in k, both T and λT have the same tensor rank. Thus it makes sense to speak of the tensor rank of an element in $\mathbb{P}(\mathbf{V})$.

Now, if $T \in \mathbf{V}$ then T corresponds to a multilinear form (abusively also called T) where

$$T : V_1 \times \cdots \times V_t \longrightarrow k.$$

If we choose bases for the V_i (say bases dual to the $\mathscr{B}_i^*$ above) and call them $\mathscr{B}_i$, and write

$$\mathscr{B}_i = \left\{x_{0,i}^*, \ldots, x_{n_i,i}^*\right\},$$

then T is completely described by its values on t-tuples of basis vectors, i.e. T is completely determined by the values

$$T\left(x_{j_1,1}^*, \ldots, x_{j_t,t}^*\right) = \alpha_{j_1,\ldots,j_t}.$$

Those values can be placed in a t-dimensional array (or hypermatrix) of size $(n_1+1) \times \cdots \times (n_t+1)$ which then, in turn, completely describes T. So, after bases are chosen, we have a 1–1 correspondence between t-dimensional hypermatrices of size $(n_1+1) \times \cdots \times (n_t+1)$ and tensors in $\mathbf{V}$. Such hypermatrices are obviously parameterized, up to multiplication by a scalar, by points of $\mathbb{P}^N$, $N = \prod_{i=1}^t (n_i+1) - 1$.

Let $S^j = k[x_{0,j}, \ldots, x_{n_j,j}]$, $j = 1, \ldots, t$, and $A = k[x_{0,1}, \ldots, x_{n_1,1}, \ldots, x_{0,t}, \ldots, x_{n_t,t}]$. We will consider the usual gradation on the S^j, i.e. as $\mathbb{N}$-graded rings. This makes A into an $\mathbb{N}^t$-graded ring in the obvious way.

Clearly each V_i^* can be identified with S_1^i and $\mathbf{V}$ with $A_{\mathbf{1}}$ where $\mathbf{1} = (1, \ldots, 1)$.

With this point of view, we can consider the Segre variety, $V_{\mathbf{n}} \subseteq \mathbb{P}^N$, $\mathbf{n} = (n_1, \ldots, n_t)$, as the image of the embedding

$$\nu_{\mathbf{n}} : (\mathbb{P}^{n_1})^* \times \cdots \times (\mathbb{P}^{n_t})^* = \mathbb{P}S_1^1 \times \cdots \times \mathbb{P}S_1^t \to \mathbb{P}A_{\mathbf{1}},$$

where

$$\nu_{\mathbf{n}}(L_1, \ldots, L_t) = L_1 \otimes L_2 \otimes \cdots \otimes L_t \quad \forall L_j \in S_1^j, \quad j = 1, \ldots, t.$$

Hence we have (e.g. see [16]) that $V_{\mathbf{n}}$ exactly parameterizes the decomposable tensors in $\mathbb{P}^N$.

Now let us consider the notion of secant variety.

Definition 1.1. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)$th *higher secant variety* of X is the closure of the union of all linear spaces spanned by s points of X.

These varieties have been denoted both by $\mathscr{S}ec_{s-1}(X)$ and X^s. We will use the second (more compact) notation.

There is an "*expected dimension*" for X^s, i.e. if $\dim X = n$, one "expects" that

$$\dim X^s = \min\{N, sn + s - 1\},$$

where the number $sn + s - 1$ corresponds to ∞^{sn} choices of s points on X (which is n-dimensional), plus ∞^{s-1} choices of a point on the $\mathbb{P}^{s-1}$ spanned by the s points. When this number is too big, we should just get that $X^s = \mathbb{P}^N$.

Since it is not always the case that X^s has the "expected dimension", whenever

$$\dim X^s < \min\{N, sn + s - 1\},$$

then X^s is said to be *defective*. A measure of this "defectiveness" is given by the quantity

$$\min\{N, sn + s - 1\} - \dim X^s.$$

Let us go back to the Segre varieties $V_{\mathbf{n}} \subset \mathbb{P}^N$. Since Segre varieties parameterize the decomposable tensors in $\mathbb{P}^N$, their secant varieties $V_{\mathbf{n}}^s$ are exactly the closure of the locus of tensors of tensor rank s. Hence we have:

Fact. A description of the number $E(\mathbf{V})$ for a k-vector space $\mathbf{V} = V_1^* \otimes \cdots \otimes V_t^*$, with $\dim V_i = n_i + 1$, given in terms of secant varieties to Segre varieties, is

$$E = E(\mathbf{V}) = \min\left\{s \mid V_{\mathbf{n}}^s = \mathbb{P}^N\right\}.$$

By a slight abuse of notation we will sometimes write

$$E(\mathbf{V}) = E(V_{\mathbf{n}}) = E(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}).$$

Notice that, from (††), for $n_t \leqslant m = \prod_{i=1}^{t-1}(n_i + 1) - 1$, we have

$$E(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}) = E(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{t-1}} \times \mathbb{P}^m). \tag{‡}$$

A classical result about secant varieties is Terracini's Lemma (see [29], or, e.g. [1]), which we give here in the following form:

Terracini's Lemma. *Let $X \subset \mathbb{P}^N$ be a non-singular variety. Then*

$$T_P(X^s) = \left\langle T_{P_1}(X), \ldots, T_{P_s}(X)\right\rangle$$

hence

$$\dim X^s = \dim\left\langle T_{P_1}(X), \ldots, T_{P_s}(X)\right\rangle,$$

where $P_1, \ldots, P_s$ are s generic points on X, P is generic in $\langle P_1, \ldots, P_s\rangle$ and $T_{P_i}(X)$ is the projectivized tangent space of X in $\mathbb{P}^N$ at P_i.

Notice that, if $(X, \mathscr{L})$ is an integral, non-singular, polarized scheme, and $\mathscr{L}$ embeds X into $\mathbb{P}^N = \mathbb{P}H^0(X, \mathscr{L})^*$, we can view the elements of $H^0(X, \mathscr{L})$ as hyperplanes in $\mathbb{P}^N$. Those hyperplanes which contains a space $T_{P_i}(X)$ correspond to elements in $H^0(X, \mathscr{I}^2_{P_i}(\mathscr{L}))$, since they intersect X in a subscheme containing the first infinitesimal neighbourhood of P_i.

Hence there will be a bijection between hyperplanes of the space $\mathbb{P}^N$ containing the subspace $\langle T_{P_1}(X), \dots, T_{P_s}(X)\rangle$ and the elements of $H^0(X, \mathscr{I}_Z(\mathscr{L}))$, where Z is the scheme defined by the ideal sheaf $\mathscr{I}_Z = \mathscr{I}^2_{P_1} \cap \cdots \cap \mathscr{I}^2_{P_s} \subset \mathscr{O}_X$. This 0-scheme is what we will call a scheme of s generic 2-*fat points* in X.

By what we have just observed, we get the following consequence of Terracini's Lemma:

Corollary 1.2. *Let X, $\mathscr{L}$, be as above. Then*

$$\dim X^s = \dim\big\langle T_{P_1}(X), \dots, T_{P_s}(X)\big\rangle = N - \dim H^0(X, \mathscr{I}_Z(\mathscr{L})),$$

where Z is a subscheme of s generic 2-fat points in X.

Now, applying Corollary 1.2 to the case of the Segre varieties

$$(X, \mathscr{L}) = \big(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}, \mathscr{O}_X(1, \dots, 1)\big),$$

we get that $\dim V^s_{\mathbf{n}} = N - \dim H^0(X, \mathscr{I}_Z(\mathscr{L}))$.

We observe also that, instead of using Terracini's Lemma, we can derive the relation between $\dim V^s_{\mathbf{n}}$ and $H(Z, \mathbf{1})$ via Macaulay's theory of "inverse systems" (see [13,19]). Our reason for mentioning this alternative view is that we were able to use it in [7] to speak about secant varieties in a context where Terracini's Lemma was not useful.

Lemma 1.3. *The following three numbers are equal*:

(1) *the dimension of the closure of the locus of tensors of tensor rank $\leqslant s$ in $\mathbb{P}^N$*;

(2) *the dimension of the variety $V^s_{\mathbf{n}} \subset \mathbb{P}^N$*;

(3) *the value $H(Z, \mathbf{1}) - 1$, where $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is a set of s generic 2-fat points in X, and where $\forall \mathbf{j} \in \mathbb{N}^t$, $H(Z, \mathbf{j})$ is the Hilbert function of Z, i.e.*

$$H(Z, \mathbf{j}) = \dim A_{\mathbf{j}} - \dim H^0(X, \mathscr{I}_Z(\mathbf{j})).$$

Proof. The equality between (1) and (2) is well known (see Introduction). We now give our alternate proof for the equality between (2) and (3).

Recall that we are considering $V_{\mathbf{n}}$ as given by the embedding

$$\nu_{\mathbf{n}} : \mathbb{P}S^1_1 \times \cdots \times \mathbb{P}S^t_1 \to \mathbb{P}A_{\mathbf{1}},$$

where

$$\begin{aligned}\nu_{\mathbf{n}}(L_1, \dots, L_t) &= L_1 \otimes L_2 \otimes \cdots \otimes L_t \\ &= L_1 L_2 \cdots L_t \quad \forall L_j \in S^j_1, \quad j = 1, \dots, t.\end{aligned}$$

Recall also that we are identifying $S^1_1 \otimes \cdots \otimes S^t_1$ with $A_{\mathbf{1}}$.

With this point of view it is not hard to determine $T_{L_1\cdots L_t}(V_{\mathbf{n}})$, i.e the projectivized tangent space to $V_{\mathbf{n}}$ at the point $L_1\cdots L_t$. We will first pass to the affine (so we are viewing $\nu_{\mathbf{n}}$ as a map $S_1^1\times\cdots\times S_1^t\to A_{\mathbf{1}}$) and consider the differential map

$$\mathrm{d}\nu_{\mathbf{n}}:T_{(L_1,\ldots,L_t)}\big(S_1^1\times\cdots\times S_1^t\big)\to T_{L_1\cdots L_t}(A_{\mathbf{1}}).$$

If we choose a direction through $(L_1,\ldots,L_t)$ in $S_1^1\times\cdots\times S_1^t$, say $(L_1,\ldots,L_t)+\lambda(M_1,\ldots,M_t)$, we get that the image of the corresponding tangent vector in $T_{L_1\cdots L_t}(A_{\mathbf{1}})$ is given by

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(\nu_{\mathbf{n}}((L_1,\ldots,L_t)+\lambda(M_1,\ldots,M_t))\big)\\
&\quad=\lim_{\lambda\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\big((L_1+\lambda M_1)\cdots(L_t+\lambda M_t)\big)\\
&\quad=\lim_{\lambda\to 0}\big[M_1(L_2+\lambda M_2)\cdots(L_t+\lambda M_t)+\cdots\\
&\qquad\quad+(L_1+\lambda M_1)(L_2+\lambda M_2)\cdots(L_{t-1}+\lambda M_{T-1})M_t\big]\\
&\quad=\sum_{j=1}^{t}L_1\cdots L_{j-1}M_jL_{j+1}\cdots L_t.
\end{aligned}$$

Then, since $V_{\mathbf{n}}$ is smooth, we have an isomorphism

$$\mathrm{d}\nu_{\mathbf{n}}:S_1^1\times\cdots\times S_1^t\to T_{L_1\cdots L_t}\big(\nu_{\mathbf{n}}(S_1^1\times\cdots\times S_1^t)\big),$$

given by $(M_1,\ldots,M_t)\to\sum_{j=1}^{t}L_1\cdots M_jL_{j+1}\cdots L_t$, where we view

$$\begin{aligned}
&T_{L_1\cdots L_t}\big(\nu_{\mathbf{n}}(S_1^1\times\cdots\times S_1^t)\big)\\
&\quad\simeq\left\{\sum_{j=1}^{t}L_1\ldots M_jL_{j+1}\ldots L_t\;\middle|\;M_j\in S_1^j,\ j=1,\ldots,t\right\},
\end{aligned}$$

which is the subspace of $A_{\mathbf{1}}$ given by the multidegree $\mathbf{1}$ part of the ideal generated by

$$\big\{L_1\cdots L_{j-1}M_jL_{j+1}\cdots L_t\big\}_{j=1,\ldots,t}.$$

Note that this subspace of $A_{\mathbf{1}}$ has dimension $(n_1+1)+\cdots+(n_t+1)-(t-1)=\sum_{j=1}^{t}n_j+1$ (since when $M_j=L_j$ we get $L_1\cdots L_t$ in all cases), i.e. the projective dimension is $n_1+\cdots+n_t$, as expected.

We will consider this vector space in more detail: as each M_j varies in S_1^j, we can write it as

$$W_{\mathbf{1}}=\big\langle S_1^1(L_2\cdots L_t);\ldots;S_1^t(L_1\cdots L_{t-1})\big\rangle=\mathrm{Im}(d\nu_{\mathbf{n}})\subset A_{\mathbf{1}}.$$

Let $B = k[y_{0,1}, \dots, y_{n_1,1}; \dots; y_{0,t}, \dots, y_{n_t,t}] \simeq A$, and consider the action of B on A defined by (see [13] or [19] for details in the $\mathbb{N}$-graded case):

$$y_{a,j} \circ x_{b,k} = (\partial/\partial x_{a,j})(x_{b,k}),$$

where we use the standard properties of differentiation to extend this action to all of

$$B_{\mathbf{a}} \times A_{\mathbf{i}} \to A_{\mathbf{i}-\mathbf{a}}.$$

In this way, if I is a multihomogeneous ideal in B, we can define the *inverse system* of I, denoted I^{-1}, as the B-submodule (multigraded) of A consisting of all elements of A annihilated by I (note that I^{-1} is almost never an ideal in A).

If $L_1, \dots, L_t$ are generic, we can choose coordinates with $L_j = x_{0,j}$, so that

$$W_{\mathbf{1}} = \left\langle S_1^1(x_{0,2} \cdots x_{0,t}); \dots; S_1^t(x_{0,1} \cdots x_{0,t-1}) \right\rangle.$$

Now consider the space $I_{\mathbf{1}} = W_{\mathbf{1}}^{\perp} \subset B_{\mathbf{1}}$. It is easy to check that if we put

$$I = (y_{1,1}, y_{2,1}, \dots, y_{n_1,1}; \dots.; y_{1,t}, y_{2,t}, \dots, y_{n_t,t})^2,$$

then $W_{\mathbf{1}} = (I^{-1})_{\mathbf{1}}$.

Note that I represents a scheme $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ given by the second infinitesimal neighbourhood of the point $(1:0:\cdots:0) \times \cdots \times (1:0:\cdots:0)$.

We will call such a 0-dimensional scheme a 2-*fat point* in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$.

We have that $\dim W_{\mathbf{1}} + \dim I_{\mathbf{1}} = \dim B_{\mathbf{1}} = N + 1$, hence $\dim W_{\mathbf{1}} = H(Z; \mathbf{1})$, where $H(Z; \bullet) = \dim_k B_{\bullet}/(I_Z)_{\bullet}$ is the (multi)-Hilbert function of Z. Note that this shows that Z is a degree $n_1 + n_2 + \cdots + n_t + 1$ structure on $(1:0:\cdots:0) \times \cdots \times (1:0:\cdots:0)$.

If we want to consider $V_{\mathbf{n}}^s$, we can study the map

$$\phi_s : (S_1^1 \times \cdots \times S_1^t)^s \to V_{\mathbf{1}}^s,$$

where

$$\begin{aligned}&\phi_s(L_{1,1}, \dots, L_{t,1}; \dots; L_{1,s}, \dots, L_{t,s})\\ &\quad = (L_{1,1} \cdots L_{t,1} + L_{1,2} \cdots L_{t,2} + \cdots + L_{1,s} \cdots L_{t,s}).\end{aligned}$$

For a generic choice of $L_{1,1}, \dots, L_{t,s}$, the dimension of $\mathrm{im}(\mathrm{d}\phi_s)$ will tell us the dimension of $V_{\mathbf{n}}^s$.

With the same procedure as before, we get the (affine) space

$$\mathrm{im}(\mathrm{d}\phi_s) \cong W_{\mathbf{1}}^1 + W_{\mathbf{1}}^2 + \cdots + W_{\mathbf{1}}^s = \mathbf{W}_{\mathbf{1}}.$$

where $W_{\mathbf{1}}^i = \langle S_1^1(L_{2,i} \cdots L_{t,i}); \dots; S_1^t(L_{1,i} \cdots L_{t-1,i})\rangle$.

We know that $W_{\mathbf{1}}^i = (I_i^{-1})_{\mathbf{1}}$, where I_i is the ideal of a 2-fat point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. Let $I_i = \mathfrak{p}_i^2$. Then

$$\mathbf{W}_{\mathbf{1}} = (\mathfrak{p}_1^2)_{\mathbf{1}}^{-1} + \cdots + (\mathfrak{p}_s^2)_{\mathbf{1}}^{-1} = (\mathfrak{p}_1^2 \cap \cdots \cap \mathfrak{p}_s^2)_{\mathbf{1}}^{-1} = (I^{-1})_{\mathbf{1}}$$

is the multidegree **1** part of $I^{-1} = (\mathfrak{p}_1^2 \cap \cdots \cap \mathfrak{p}_s^2)^{-1}$, where $\mathfrak{p}_i$, $i = 1, \ldots, s$, are the multihomogeneous ideals of s generic points P_i in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$, i.e. I is the ideal of a scheme Z which is the union of s 2-fat points.

We have that $\dim \mathbf{W_1} + \dim I_\mathbf{1} = \dim B_\mathbf{1}$, and so $\dim \mathbf{W_1} = H(Z; \mathbf{1})$, and we get that the problem of determining $\dim V_\mathbf{n}^s$ amounts to determining $H(Z; \mathbf{1})$. This, then, gives the equality between (2) and (3) above. □

The expected value for $H(Z, \mathbf{1})$ is $\min\{N+1, s(n_1 + \cdots + n_t + 1)\}$ (if all the $\mathfrak{p}_i^2$ impose independent conditions to hypersurfaces of multidegree **1**); so we expect this value for $\dim \mathbf{W_1}$. This agrees with the expected dimension for $V_\mathbf{n}^s \subset \mathbb{P}^N$:

$$\operatorname{expdim} V_\mathbf{n}^s = \min\left\{N, s(n_1 + \cdots + n_t + 1) - 1\right\}.$$

In particular, the typical rank $E = E(\mathbf{V})$ (for $\mathbf{V}$ above) is the smallest value of s for which there are no $(1, \ldots, 1)$-forms in the ideal of s generic 2-fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$.

Remark 1.4. Since $\deg Z = s(n_1 + \cdots + n_t + 1)$, then $\dim V_\mathbf{n}^s = H(Z, \mathbf{1}) - 1 \leqslant s(n_1 + \cdots + n_t + 1) - 1$. A lower bound for E is then easily given by

$$E \geqslant \frac{\prod_{i=1}^t (n_i + 1)}{n_1 + \cdots + n_t + 1}.$$

Remark 1.5. We can notice that, proceeding in an analogous way, we find that $H(Z, \mathbf{j}) - 1$, for an arbitrary multiindex $\mathbf{j} = (j_1, \ldots, j_t) \in \mathbb{N}^t$, represents the dimension of X^s, where X is given by the embedding

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \to \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_t} \to \mathbb{P}^{N_\mathbf{j}},$$

where the first map is given by the product of the (Veronese) j_ith embeddings $\mathbb{P}^{n_i} \to \mathbb{P}^{N_i}$, and the second is the Segre embedding.

2. On the dimension of secant varieties to Segre varieties: the monomial case

We will give here some results about the dimension of the varieties $V_\mathbf{n}^s$ (notation as in Section 1) for some particular values of s.

We saw above that questions about $\dim V_\mathbf{n}^s$ can be translated into questions about the Hilbert function of 2-fat points. We now investigate the Hilbert function of a special family of 2-fat points, namely those fat points whose support is a product of co-ordinate points. We will see that even these special points can give us interesting results about secant varieties.

In order to discuss products of co-ordinate points, we introduce some notation. Let

$$\mathbf{J} = \left\{\mathbf{r} = (r_1, \ldots, r_t) \mid 0 \leqslant r_i \leqslant n_i\right\}.$$

A *co-ordinate point* of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is a point $P_{\mathbf{r}} = P_{r_1} \times \cdots \times P_{r_t}$, where P_{r_j} is the r_jth co-ordinate point of $\mathbb{P}^{n_j}$.

Definition. Given $\mathbf{r}_1 = (r_{1,1}, \ldots, r_{t,1})$ and $\mathbf{r}_2 = (r_{1,2}, \ldots, r_{t,2})$ in $\mathbf{J}$, we say that the *Hamming distance* between $\mathbf{r}_1$ and $\mathbf{r}_2$ is l if $(r_{1,1} - r_{1,2}, \ldots, r_{t,1} - r_{t,2})$ has exactly l non-zero entries.

Proposition 2.1. *Let $P_{\mathbf{r}_1}, \ldots, P_{\mathbf{r}_s}$ be a set of co-ordinate points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. Let $\mathfrak{p}_i$ be the ideal of $P_{\mathbf{r}_i}$, and let Z be the scheme defined by $\bigcap_{i=1}^{s} \mathfrak{p}_i^2$. Then*

$$H(Z, \mathbf{1}) = \Big|\Big\{\mathbf{r} \in \mathbf{J} \mid \mathbf{r} \textit{ has Hamming distance} \leqslant 1 \\ \textit{from at least one of } \mathbf{r}_1, \ldots, \mathbf{r}_s\Big\}\Big|.$$

Proof. Let us start with $s = 1$. Consider $P_{\mathbf{r}} = P_{r_1} \times \cdots \times P_{r_t}$, where $P_{r_j} \in \mathbb{P}^{n_j}$ is the r_jth coordinate point in $\mathbb{P}^{n_j}$, i.e. $P_{r_j} = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)$, with 1 in the r_jth position.

Then $I_{P_{\mathbf{r}}}^2 = (y_{0,1}, \ldots, \widehat{y_{r_1,1}}, \ldots, y_{n_1,1}; \ldots; y_{0,t}, \ldots, \widehat{y_{r_t,t}}, \ldots, y_{n_t,t})^2$. Let Z be the scheme defined by this ideal.

We have that I_Z is a monomial ideal, and so computing $H(Z, \mathbf{1})$ amounts to counting the monomials of degree $(1, \ldots, 1)$ which are not in I_Z.

Now it is quite immediate to see that a monomial $y_{\mathbf{j}} \notin I_{P_r}^2$ if and only if at most one entry in $\mathbf{j} = (j_1, \ldots, j_t)$ differs from the entries in $\mathbf{r} = (r_1, \ldots, r_t)$, which is exactly what the statement of the theorem says.

When $s > 1$, let $R = \{\mathbf{r_1}, \ldots, \mathbf{r_s}\}$, and $I_Z = \bigcap_{\mathbf{r} \in R} I_{P_{\mathbf{r}}}^2$. We have that a monomial $y_{\mathbf{j}} \notin I_Z$ if and only if there is at least one $\mathbf{r} \in R$ such that $y_{\mathbf{j}} \notin I_{P_{\mathbf{r}}}^2$, and the statement immediately follows from what we have already seen for $s = 1$. $\square$

There is a simple way to visualize this result, by "playing with rooks on a t-dimensional chessboard". To be more precise, if we define $A_r = \{0, 1, \ldots, r\}$, our "chessboard" will be the set $A = A_{n_1} \times \cdots \times A_{n_t}$. We will associate to the set $\mathbf{X} = \{P_{\mathbf{r}_1}, \ldots, P_{\mathbf{r}_s}\}$ of co-ordinate points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$, the set of places $R = \{\mathbf{r}_1, \ldots, \mathbf{r}_s\}$ in A, which we will call the *rook set* associated to $\mathbf{X}$ (see also [10]).

Definition. Let A be as above, and let $R \subset A$. We define the subset *generated by* R (and write $\langle R\rangle$) to be the set of all the elements in A that can be obtained by changing at most one coordinate of an element of R (these are the places in A which are "attacked" by rooks situated in R).

Proposition 2.1 above can now be reformulated as follows:

Proposition 2.1a. *Let $\mathbf{X} = \{P_{\mathbf{r}_1}, \ldots, P_{\mathbf{r}_s}\}$ be a set of co-ordinate points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$, and let R be the rook set associated to $\mathbf{X}$. Let Z be as in Proposition 2.1. Then $H(Z, \mathbf{1}) = |\langle R\rangle|$.*

Now we want to show that when a rook set R has nice properties, Proposition 2.1a allows us to say something useful about the secant varieties $V_{\mathbf{n}}^s$.

Definitions

(1) A rook set R is said to be *perfect* if every element in $\langle R\rangle$ comes from exactly one element of R.
(2) A rook set R is a *rook covering* if $\langle R\rangle = A$.
(3) A rook set R is a *perfect rook covering* if both (1) and (2) hold for R.

It immediately follows from Proposition 2.1a that:

Corollary 2.2. *Let $R \subseteq A$ be a rook set with $|R| = s$. Then*:
(1) *If R is a rook covering, then $E(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}) \leqslant s$.*
(2) *If R is a perfect rook set, then* $\dim V_{\mathbf{n}}^{s'} = s'(n_1 + \cdots + n_t + 1) - 1$ *for all* $s' \leqslant s$.
(3) *If R is a perfect rook covering, then we have $E = s$ (so $V_{\mathbf{n}}^s = \mathbb{P}^N$).*

Recall our convention that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$. We want to prove the following:

Proposition 2.3. *Let $V_{\mathbf{n}}^s \subset \mathbb{P}^N$ be defined as in Section 1. Then*:
(i) *if $t = 2$, and $s \leqslant n_1 + 1$, then* $\dim V_{\mathbf{n}}^s = N$;
(ii) *for $t = 2$, and $s \leqslant n_1$,* $\dim V_{\mathbf{n}}^s = s(n_1 + n_2 + 1) - s^2 + s - 1$;
(iii) *for $t \geqslant 3$ and $s \leqslant n_1 + 1$,* $\dim V_{\mathbf{n}}^s = s(n_1 + n_2 + \cdots + n_t + 1) - 1$.

Proof. Cases (i) and (ii) are actually well known since they correspond to ordinary matrices. We can easily prove them by observing that for $s = n_1 + 1$ there is always a trivial rook covering (the main diagonal) with $|R| = s$, so we get (i); while for $s \leqslant n_1$ use s places on the main diagonal to form R: then there are $s(s-1)$ positions that are covered by two of them, so they generate a set made by $s(n_1 + \cdots + n_t + 1) - s(s-1)$ elements, and we get (ii) from Proposition 2.1a (notice that every set of s points can viewed as a set of co-ordinate points in this case).

When $t \geqslant 3$ and $s \leqslant n_1 + 1$, a perfect rook set of s elements can always be obtained taking s places on the main diagonal, so, by Corollary 2.2, we get case (iii). □

When $n_1 + 1 = n_2 + 1 = \cdots = n_t + 1 = q$, the problems about rook sets have an interpretation in coding theory (e.g. see [10,27]). If we consider an alphabet made of q letters and words of length t, then a code can be obtained by taking as its words the elements of a rook set in $A = A_{q-1}^t$. In this setting a perfect rook set R with $|R| = s$ corresponds to what is called a “1-correcting code” of s elements, denoted as “$(t, s, 3)$-code”, i.e. a set $R \subset A$ such that the Hamming distance between any two words in R is $\geqslant 3$.

To determine the maximum size $s = A_q(t, 3)$ for which there is a $(t, s, 3)$-code in $A = A_q^t$ is what is called *the main coding theory problem*. Many bounds are known for $A_q(t, 3)$, but even for $q = 2$ there is no general formula computing this value; a table of values for $A_2(t, 3)$ can be found in [27, p. 173] for $t \leqslant 16$.

Perfect rook coverings correspond to what are called *perfect codes*, and such codes are quite rare: the only known ones of type $(t, s, 3)$ are the *Hamming codes*, which are of type $(t = (q^k - 1)/(q - 1), s = q^{t-k}, 3)$, where q is a prime power and $k \geqslant 2$ (a computer check showed that for $q \leqslant 100$, $t \leqslant 1000$, there are no others, see [27]).

Let us see how we may apply these results from coding theory to our problems.

Example 2.4. Let $V_{\mathbf{n}}$ be the Segre embedding

$$\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{t} \to \mathbb{P}^{2^t-1}.$$

Then for $t = 2^k - 1, k \geqslant 2$, we get $\dim V_{\mathbf{n}}^s = s(t + 1) - 1 = 2^k s - 1$ for all $2 \leqslant s \leqslant E = 2^{2^k-k-1}$ and $V_{\mathbf{n}}^E = \mathbb{P}^{2^t-1}$.

The example comes from the Hamming codes with $q = 2$ and $t = 2^k - 1$.

Example 2.5. Let $V_{\mathbf{2}}$ be the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^{80}$. Then we have $\dim V_{\mathbf{2}}^s = 9s - 1$ for $2 \leqslant s \leqslant 9$ and $V_{\mathbf{2}}^9 = \mathbb{P}^{80}$.

The example comes from using the Hamming code with $q = 3$ and $k = 2$.

Example 2.6. Let $V_{(3,3,11)}$ be the Segre embedding $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^{11} \to \mathbb{P}^{191}$. Then $\dim V_{(3,3,11)}^4 = 4(3 + 3 + 11 + 1) - 1 = 71$ (the expected value, from Proposition 2.3(iii)) but (as we will see in Theorem 3.1(2)) $E = 12$ (not the expected value). So, somewhere between the secant 3 spaces and the secant 11 spaces, something goes wrong!

As we anticipated, this kind of procedure is useful only when we can reduce to the case of co-ordinate points (from the "2-fat points" point of view); in other cases, when we have to consider larger values of s, other ways to attack the problem have to be found (e.g. see Proposition 3.3). This is what we do in the following section.

3. Higher secant varieties to Segre varieties (the general case)

As we pointed out in Section 2, the dimension of $V_{\mathbf{n}}^s$ cannot be studied in all cases using rook coverings (i.e. monomial ideals of 2-fat points).

Our main result about typical rank is the following theorem.

Let us establish the notation for this theorem. Let $\mathbf{V} = V_1^* \otimes \cdots \otimes V_t^* \otimes W^*$, where $\dim V_i = n_i + 1$, $i = 1, \ldots, t$, and $\dim W = n + 1$ with $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t \leqslant n$.

Theorem 3.1. *The typical rank $E = E(\mathbf{V})$ is*:
(1) *for* $n > \prod_{i=1}^t (n_i + 1) - 1$, *exactly* $E = \prod_{i=1}^t (n_i + 1)$;
(2) *for* $\prod_{i=1}^t (n_i + 1) - \sum_{i=1}^t n_i - 1 \leqslant n \leqslant \prod_{i=1}^t (n_i + 1) - 1$, *exactly* $E = n + 1$;
(3) *while for* $n_t \leqslant n \leqslant \prod_{i=1}^t (n_i + 1) - \sum_{i=1}^t n_i - 1$, *we have*

$$n + 1 \leqslant E \leqslant \prod_{i=1}^{t} (n_i + 1) - \sum_{i=1}^{t} n_i, \qquad E \geqslant \frac{(n+1)\prod_{i=1}^t (n_i + 1)}{n + \sum_{i=1}^t n_i + 1}.$$

Proof. (1) This is obvious from (‡), Section 1 and (2).

(2) Let $N = (n+1)\prod_{i=1}^t (n_i + 1) - 1$, $m = \prod_{i=1}^t (n_i + 1) - 1$, $\mathbf{n} = (n_1, \ldots, n_t)$ and let $V_{\mathbf{n}} \subseteq \mathbb{P}^m$ be the Segre variety image of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. We have the following embeddings:

$$(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}) \times \mathbb{P}^n \to V_{\mathbf{n}} \times \mathbb{P}^n \to \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$$

(where the composite map is also the Segre embedding).

Since $n \leqslant m$, $E(\mathbb{P}^m \times \mathbb{P}^n) = n + 1$, and it follows that $E(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n) \geqslant n + 1$.

Now consider a generic point $P \in \mathbb{P}^N$; we can write

$$P = A_1 \times (1 : 0 : \cdots : 0) + \cdots + A_{n+1} \times (0 : \cdots : 0 : 1),$$

where $A_1, \ldots, A_{n+1}$ can be viewed as $n + 1$ generic points in $\mathbb{P}^m$ (they are the "t-dimensional slices" of the tensor).

The n-dimensional linear space L generated by the A_i's in $\mathbb{P}^m$ will intersect $V_{\mathbf{n}}$, and the dimension of the intersection will be

$$n + \sum_{i=1}^{t} n_i - \prod_{i=1}^{t} (n_i + 1) + 1 \geqslant 0.$$

Moreover, because of the genericity of L, and since $V_{\mathbf{n}}$ is integral and non-degenerate, the intersection $L \cap V_{\mathbf{n}}$ will contain enough distinct points, say $P_{1,1} \times \cdots \times P_{1,t}; \ldots; P_{n+1,1} \times \cdots \times P_{n+1,t}$ with $P_{j,i} \in \mathbb{P}^{n_i}$, $i = 1, \ldots, t$, in order to span the linear space $L = \langle A_1, \ldots, A_{n+1} \rangle$.

Let $A_k = \sum_{j=1}^{n+1} \lambda_{k,j} P_{j,1} \times \cdots \times P_{j,t}$. Then we get

$$\begin{aligned} P &= A_1 \times (1 : 0 : \cdots : 0) + \cdots + A_{n+1} \times (0 : \cdots : 0 : 1) \\ &= \left(\sum_{j=1}^{n+1} \lambda_{1,j} P_{j,1} \times \cdots \times P_{j,t} \right) \times (1 : 0 : \cdots : 0) + \cdots \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\sum_{j=1}^{n+1} \lambda_{n+1,j} P_{j,1} \times \cdots \times P_{j,t}\right) \times (0 : \cdots : 0 : 1) \\
&= P_{1,1} \times \cdots \times P_{1,t} \times (\lambda_{1,1} : \cdots : \lambda_{n+1,1}) + \cdots \\
&\quad + P_{n+1,1} \times \cdots \times P_{n+1,t} \times (\lambda_{1,n+1} : \cdots : \lambda_{n+1,n+1})
\end{aligned}
$$

which expresses P as the sum of $n+1$ decomposable tensors, as required.

(3) The bound

$$E \geqslant \frac{(n+1)\prod_{i=1}^{t}(n_i+1)}{n+\sum_{i=1}^{t} n_i + 1}$$

is obvious (see Remark 1.4), while the bound $E \leqslant n+1$ follows from the argument at the beginning of (2).

To prove that

$$E \leqslant \prod_{i=1}^{t}(n_i+1) - \sum_{i=1}^{t} n_i,$$

we proceed as in (2), but we do not work on the space $L = \langle A_1, \ldots, A_{n+1}\rangle$ (because now $L \cap V_{\mathbf{n}} = \emptyset$): but rather, we consider a linear space $L' \subseteq \mathbb{P}^m$, with

$$\dim L' = \prod_{i=1}^{t}(n_i+1) - \left(\sum_{i=1}^{t} n_i\right) - 1 \quad \text{and} \quad L' \supseteq L.$$

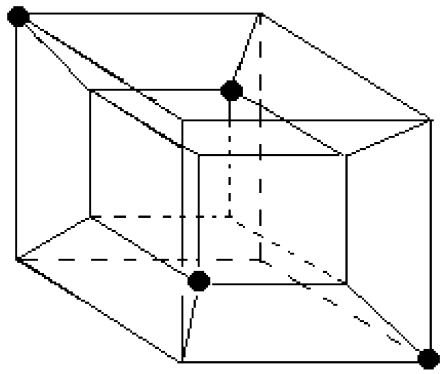
We have $L' \cap V_{\mathbf{n}} \neq \emptyset$, and we can span L', and hence L, with

$$\prod_{i=1}^{t}(n_i+1) - \sum_{i=1}^{t} n_i$$

points in $V_{\mathbf{n}}$. We then continue as in 2). □

Example 3.2. Consider the case $(n_1, n_2, n_3, n) = (1, 1, 1, n)$:
(a) for tensors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$, with $n \geqslant 7$, we have $E = 8$;
(b) for tensors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$, with $4 \leqslant n \leqslant 7$, we have $E = n+1$;
(c) for tensors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$, we have $E = 5$;
(d) for tensors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, we have $E = 4$;
(e) for tensors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have $E = 4$.

Cases (a)–(c) come directly from Theorem 3.1. Case (d) is dealt with by a direct computation using CoCoA [8]. As for (e), by Theorem 3.1(3) we get $E \geqslant 4$, and it is not hard to find a (non-perfect) rook covering of a $2 \times 2 \times 2 \times 2$ hypercube made with four rooks:



Proposition 3.3. *Let V be the Segre embedding of* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$ *and let s be such that*

$$\prod_{i=1}^{t}(n_i + 1) - \left(\sum_{i=1}^{t} n_i\right) + 1 \leqslant s \leqslant \min\left\{n, \prod_{i=1}^{t}(n_i + 1) - 1\right\}.$$

Then V^s is defective.

Proof. The expected dimension for V^s is $s(1 + n + \sum_{i=1}^{t} n_i) - 1$. Thus the expected number of independent forms of degree $(1, 1, \ldots, 1)$ in the ideal of s generic 2-fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$ is

$$(n + 1)\prod_{i=1}^{t}(n_i + 1) - s\left(1 + n + \sum_{i=1}^{t} n_i\right).$$

On the other hand, there will be $(\prod_{i=1}^{t}(n_i + 1) - s)$ forms of degree $(1, \ldots, 1)$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ passing through s generic points, and $(n + 1 - s)$ linear forms in $\mathbb{P}^n$ passing through s generic points there, hence (just making products) we can find at least $(\prod_{i=1}^{t}(n_i + 1) - s)(n + 1 - s)$ forms passing doubly through s generic points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$ (this number is > 1 by our bound on s). So, whenever

$$(n + 1)\prod_{i=1}^{t}(n_i + 1) - s\left(1 + n + \sum_{i=1}^{t} n_i\right) < \left(\prod_{i=1}^{t}(n_i + 1) - s\right)(n + 1 - s),$$

we have that V^s is defective. A straightforward computation shows that the above inequality amounts to $s \geqslant \prod_{i=1}^{t}(n_i + 1) - (\sum_{i=1}^{t} n_i) + 1$, as required. □

Notice that, in the context of Proposition 3.3, in general we do not know how defective V^s is. Let us check what happens in an example:

Example 3.4. Consider $\mathbf{n} = (1, 1, 3)$, i.e. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^{15}$. We have $E = 4$ (use Theorem 3.1), while $V^2_{\mathbf{n}}$ has the right dimension by Proposition 2.3(iii). Proposition 3.3 gives us that $V^3_{\mathbf{n}}$ is defective; more precisely that $\dim V^3_{\mathbf{n}} \leqslant 15 - 1 = 14$, because there is at least one form in the ideal of 3 generic 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$, while the expected dimension of $V^3_{\mathbf{n}}$ should be $\min\{17, 15\} = 15$.

In this case it is not too hard to check that we actually have $\dim V^3_{\mathbf{n}} = 14$ (the three points $P_1 = (1:0) \times (1:0) \times (1:0:0:0)$, $P_2 = (1:0) \times (0:1) \times (0:1:0:0)$, $P_3 = (0:1) \times (0:1) \times (0:0:0:1)$ have exactly one (1,1,1)-form passing through them).

From the proof of Proposition 3.3 we can immediately deduce the following:

Corollary 3.5. *Let $s, n_1, \ldots, n_t, n$ be integers which satisfy the bounds in Proposition* 3.3. *Then every rook set $R \subset A = A_{n_1} \times \cdots \times A_{n_t} \times A_n$ with $|R| = s$ is such that*

$$|A - \langle R \rangle| \geqslant \left(\prod_{i=1}^{t} (n_i + 1) - s \right) (n + 1 - s).$$

The following result will give us a bound in order to have that V^s has the expected dimension. We first give a useful lemma.

Lemma 3.6. *Let $Z = 2P_1 + \cdots + 2P_m + P_{m+1} + \cdots + P_{m+r} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ be the scheme given by the union of m 2-fat points and r simple points, with support on $m + r$ generic points P_i. Let $t \leqslant 2$, $n_1 + \cdots + n_t \leqslant 2m + r$, and $n_1 + \cdots + n_{t-1} \leqslant m$.*

Then there exists a form $f \neq 0$ in $I(Z)_{\mathbf{1}}$.

Proof. Let

$$P_i = P_{i,1} \times \cdots \times P_{i,t}, \quad P_{i,j} \in \mathbb{P}^{n_j}.$$

We work by induction on t. Let $t = 2$. For $n_1 \leqslant m + r$, let $\{g_1 = 0\} \subset \mathbb{P}^{n_1}$ be a hyperplane through $P_{1,1}, \ldots, P_{n_1,1}$. Since $n_1 \leqslant m$, and $n_1 + n_2 \geqslant 2m + r$, we can find a hyperplane $\{g_2 = 0\} \subset \mathbb{P}^{n_2}$ through $P_{1,2}, \ldots, P_{m,2}, P_{n_1+1,2}, \ldots, P_{m+r,2}$. For $n_1 > m + r$, since $n_2 \geqslant n_1$, let $\{g_1 = 0\} \subset \mathbb{P}^{n_1}$ be a hyperplane through $P_{1,1}, \ldots, P_{m+r,1}$ and $\{g_2 = 0\} \subset \mathbb{P}^{n_2}$ be a hyperplane through $P_{1,2}, \ldots, P_{m+r,2}$.

Then $g_1 g_2$ is the required form.

For $t > 2$, let $\{g_t = 0\} \subset \mathbb{P}^{n_t}$ be a hyperplane through $P_{1,t}, \ldots, P_{n_t,t}$. By the inductive hypothesis, there exists a form $g \in I(Z^*) \subset k[y_{0,1}, \ldots, y_{n_1,1}; \ldots; y_{0,t-1}, \ldots, y_{n_{t-1},t-1}]$ of multidegree (1,...,1), where Z^* is the projection of

$$Z-(P_1+\cdots+P_{n_t})$$
$$=\begin{cases} P_1+\cdots+P_{n_t}+2P_{n_t+1}+\cdots & \\ \quad +2P_m+P_{m+1}+\cdots+P_{m+r} & \text{for } n_t<m,\\ P_1+\cdots+P_m+P_{n_t+1}+\cdots+P_{m+r} & \text{for } n_t\geqslant m\end{cases}$$

into $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_{t-1}}$.

Then gg_t is the required form. □

Now we can prove the following proposition.

Proposition 3.7. *Let* $\mathbf{n}=(n_1,\ldots,n_t)$ *and let* $t\geqslant 3$,

$$\left\lceil\frac{n_1+n_2+\cdots+n_t+1}{2}\right\rceil\geqslant\max\{n_t+1,s\}.$$

Then $\dim V_{\mathbf{n}}^s=s(n_1+n_2+\cdots+n_t+1)-1$.

Proof. By induction on s. For $s\leqslant n_1+1$, the result follows from Proposition 2.3. Let $s>n_1+1$. As in the lemma above, let

$$P_i=P_{i,1}\times\cdots\times P_{i,t},\quad P_{i,j}\in\mathbb{P}^{n_j},\ \ 1\leqslant i\leqslant s,\ \ 1\leqslant j\leqslant t.$$

For $s\geqslant n_j+1$, we may assume that

$$\begin{aligned}P_1&=P_{1,1}\times\cdots\times P_{1,j}\times\cdots\times P_{1,t}\\&=P_{1,1}\times\cdots\times(1:0:\cdots:0)\times\cdots\times P_{1,t},\\&\ \ \vdots\\P_{n_j+1}&=P_{n_j+1,1}\times\cdots\times P_{n_j+1,j}\times\cdots\times P_{t,n_j+1}\\&=P_{n_j+1,1}\times\cdots\times(0:\cdots:0:1)\times\cdots\times P_{n_j+1,t}.\end{aligned}$$

For $s<n_j+1$, we may assume that

$$\begin{aligned}P_1&=P_{1,1}\times\cdots\times P_{1,j}\times\cdots\times P_{1,t}\\&=P_{1,1}\times\cdots\times(1:0:\cdots:0)\times\cdots\times P_{1,t},\\&\ \ \vdots\\P_s&=P_{s,1}\times\cdots\times P_{s,j}\times\cdots\times P_{s.t}\\&=P_{s,1}\times\cdots\times(0:\cdots:0:1:0:\cdots)\times\cdots\times P_{s,t}.\end{aligned}$$

Now, $y_{1,j},\ldots,y_{n_j,j}\in I(P_{1,j})\subset k[y_{0,j},\ldots,y_{n_j,j}]$ are linearly independent forms (i.e. $\{y_{1,j}=0\},\ldots,\{y_{n_j,j}=0\}\subset\mathbb{P}^{n_j}$ are independent hyperplanes through $P_{1,j}=(1,0,\ldots,0)$ in $\mathbb{P}^{n_j}$).

For any $2 \leqslant l \leqslant n_j + 1$, let Z' be the projection of the scheme

$$
\begin{aligned}
&(2P_2 + \cdots + 2P_s) - (P_2 + \cdots + \hat{P}_l + \cdots + P_{\min(s,n_j+1)})\\
&\quad = \begin{cases} P_2 + \cdots + P_{l-1} + 2P_l + P_{l+1} + \cdots \\ \quad + P_{n_j+1} + 2P_{n_j+2} + \cdots + 2P_s & \text{for } s \geqslant n_j + 1, \\ P_2 + \cdots + P_{l-1} + 2P_l + P_{l+1} + \cdots + P_s & \text{for } l \leqslant s < n_j + 1, \\ P_2 + \cdots + P_s & \text{for } s < l \leqslant n_j + 1 \end{cases}
\end{aligned}
$$

on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{\hat{n}_j} \times \cdots \times \mathbb{P}^{n_t}$. Now apply Lemma 3.6 to Z'. We have

$$
m = \begin{cases} s - n_j & \text{for } s \geqslant n_j + 1, \\ 1 & \text{for } l \leqslant s < n_j + 1, \\ 0 & \text{for } s < l \leqslant n_j + 1, \end{cases} \quad r = \begin{cases} n_j - 1 & \text{for } s \geqslant n_j + 1, \\ s - 2 & \text{for } l \leqslant s < n_j + 1, \\ s - 1 & \text{for } s < l \leqslant n_j + 1. \end{cases}
$$

Hence

$$
2m + r = \begin{cases} 2s - n_j - 1 & \text{for } s \geqslant n_j + 1, \\ s & \text{for } l \leqslant s < n_j + 1, \\ s - 1 & \text{for } s < l \leqslant n_j + 1, \end{cases}
$$

and from

$$
\left\lfloor \frac{n_1 + n_2 + \cdots + n_t + 1}{2} \right\rfloor \geqslant \max\{n_t + 1, s\},
$$

it follows, by an easy computation, that

$$
2m + r \leqslant n_1 + \cdots + \hat{n_j} + \cdots + n_t
$$

and

$$
m \leqslant \begin{cases} n_1 + \cdots + \hat{n_j} + \cdots + n_{t-1} & \text{for } j \leqslant t - 1, \\ n_1 + \cdots + n_{t-2} & \text{for } j = t. \end{cases}
$$

So, by Lemma 3.6, there exists a form $f_l \in I(Z')$ of multidegree $(1, \ldots, 1)$ and, since P_1 is generic, we may assume that $f_l \notin I(P_{1,j})$. Then

$$
y_{1,j} f_2, \ldots, y_{n_j,j} f_{n_j+1} \in A_{\mathbf{1}}, \quad 1 \leqslant j \leqslant t,
$$

are $n_1 + \cdots + n_t$ linearly independent forms in $I(P_1 + 2P_2 + \cdots + 2P_s)$, not in $I(Z)$.

Moreover, applying Lemma 3.6 to the scheme $2P_2 + \cdots + 2P_s \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$, we can find a form in $I(2P_2 + \cdots + 2P_s)$, and, since P_1 is generic, not in $I(P_1)$.

Then

$$
H(Z, \mathbf{1}) = H(Z', \mathbf{1}) + n_1 + \cdots + n_s + 1,
$$

where $Z' = 2P_2 + \cdots + 2P_s$. The conclusion follows from the inductive hypothesis. $\square$

4. The tridimensional case

Although our results cover cases for any number of factors, the most intensive study of Questions (1)–(3) in the literature occur in the case of three factors, i.e. for $\mathbf{V} = V_1^* \otimes V_2^* \otimes W^*(\dim V_i = n_i + 1, \dim W = n, n_1 \leqslant n_2 \leqslant n \leqslant n_1 n_2 + n_1 + n_2$, see (‡)).

In this case, because of the connection between these questions and the search for fast algorithms for matrix multiplication (see [5] for details), there are several results in the area of Algebraic Complexity Theory.

We summarize those results here and explain where our results fit into this literature.

• *Case* 1. In the special case of three factors, our Theorem 3.1 gives the value of E in all cases (n_1, n_2, n) for which $n \geqslant n_1 n_2$.

More precisely, let $n = n_1 n_2 + k$, $k \geqslant 0$. The expected value for E is

$$e = \left\lceil \frac{(n_1+1)(n_2+1)(n+1)}{n_1+n_2+n+1} \right\rceil.$$

But, by Theorem 3.1, we know that

$$E = \begin{cases} n+1 & \text{for } 0 \leqslant k \leqslant n_1+n_2, \\ (n_1+1)(n_2+1) & \text{for } k \geqslant n_1+n_2. \end{cases}$$

It is an easy computation to check that:
(i) When either $k = 0, 1$ or $k > (n_1+1)(n_2+1)(n_1+n_2-1)$, we have $E = e$;
(ii) For $2 \leqslant k \leqslant (n_1+1)(n_2+1)(n_1+n_2-1)$, we have $E > e$. Hence in this case the varieties V^s are defective for $e \leqslant s \leqslant E-1$. In these cases we should have $V^s = \mathbb{P}^N$, but this does not happen until $s = E$.

This remark shows that the values of E given in Theorem 3.1 are, in many cases, not the expected values.

This covers earlier work of Ja'Ja [20] who considered the cases $(1, n_2, n)$ and work of Atkinson and Stephens [4] and Atkinson and LLoyd [3] who considered the cases

$$(n_1, n_2, n) = (n_1, n_2, n_1 n_2 + n_1 + n_2 - p), \quad \text{where } p = 1, 2.$$

• *Case* 2. (n_1, n_2, n) are all odd and $Q = \frac{(n_1+1)(n_2+1)}{n_1+n_2+n+1}$ is an integer.

In this case Strassen [28] and Lickteig [23] have the very nice result that $E = (n+1)Q$, and this is precisely the "expected" value. Moreover, the divisibility condition implies that $\dim V^s_{(n_1,n_2,n)}$ (for $s < E$) is also exactly as expected. This intersects our work only for the case $Q = 1$.

• *Case* 3. $(n_1, n_2, n) = (2, n, n)$.

This has been studied by Strassen [28], see also [5] for details.

(A) (n even). We have that

$$E = \frac{3}{2}n + 2$$

which is one more than the expected value.

Moreover, for $2 \leqslant s \leqslant \frac{3}{2}n$, dim $V^s_{(2,n,n)}$ is exactly as expected.

When $s = \frac{3}{2}n + 1$, then $V^s_{(2,n,n)}$ is a hypersurface in $\mathbb{P}^{3n^2+6n+2}$. Strassen gives also an expression for the equation of this hypersurface.

(B) (n odd). We have that

$$E = \frac{3(n+1)}{2}.$$

This is the expected value.

Interestingly enough, in this case the same expression which gives the hypersurface $V^s_{(2,n,n)}$, for $s = \frac{3}{2}n + 1$, here gives the closure of the locus of tensors which do not have rank *exactly* E.

- *Case* 4. $(n_1, n_2, n) = (3, n, n)$.

In this case (see [5, p. 15]) dim $V^s_{(3,n,n)}$ is exactly as expected for all n and for all s. In particular, $E = 2n + 1$.

- *Case* 5. $(n_1, n_2, n) = (n, n, n),\ n \neq 2$.

This "cubic" case was treated by Lickteig [23], who showed that $E = \lceil (n+1)^3/(3n+1) \rceil$. This is the expected value for E.

Note that $(n+1)^3/(3n+1)$ is an integer only for (the positive integer) $n = 1$, hence this result, unfortunately, does not give information about $V^s_{(n,n,n)}$ for $s < E$. When $s \leqslant n + 1$, our Proposition 2.3 gives that the dimension of this variety is the expected one.

We remark that Cases 3–5 are not covered by our theorems. We can, however, offer a proof for Case 3 and $n = 2$.

Example 4.1. Consider the variety $V_{(2,2,2)} \subset \mathbb{P}^{26}$; in this case we are looking at $3 \times 3 \times 3$ hypermatrices.

By Proposition 2.3 we have dim $V^2_{(2,2,2)} = 13$ and dim $V^3_{(2,2,2)} = 20$, hence the hypermatrices of tensor rank 3 form only a 20-dimensional subvariety in $\mathbb{P}^{26}$. Remark 1.4 gives that $E \geqslant 4$, but we want to check that we actually have that $V^4_{(2,2,2)} \neq \mathbb{P}^{26}$, i.e. that this variety is defective, and so $E = 5$ (since, by Theorem 3.1, we know that $E \leqslant 5$).

In order to check that $V^4_{(2,2,2)} \neq \mathbb{P}^{26}$, it is enough to show that 4 generic 2-fat points in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ are contained in a hypersurface of degree $(1, 1, 1)$. So let us consider the four points $P_i \times P_i \times P_i, i = 1, \ldots, 4$, where $P_1 = (1 : 0 : 0)$;

$P_2 = (0 : 1 : 0)$; $P_3 = (0 : 0 : 1)$; $P_4 = (1 : 1 : 1)$ (we can always choose coordinates so that this is the case). If the multihomogeneous coordinates are $\{x_0, x_1, x_2; y_0, y_1, y_2; z_0, z_1, z_2\}$, we have that the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix}$$

has rank 1 at each point $P_i \times P_i \times P_i$, i.e. its determinant gives us a form of degree $(1, 1, 1)$ which passes doubly through our points. Hence $\dim V^4_{(2,2,2)} \leqslant 25$.

Thus the generic $3 \times 3 \times 3$ hypermatrix has tensor rank 5 (note that the actual maximum value for the tensor rank could even be bigger than 5).

We conclude this section with one last observation in the case of three factors, comparing what we have just seen above and the ideas of Section 2. This observation shows how Theorem 3.1 could not be obtained by working only with monomial ideals.

Remark 4.2. *Let* $1 \leqslant n_1 \leqslant n_2$, $n = n_1 n_2 + k$ *with* $k \geqslant 0$ *and* E *as in Case* 1 *above. Let* $A = A_{n_1} \times A_{n_2} \times A_n$. *Then*:

(a) *for* $n_1 = n_2 = 1$, *there is a rook covering with* E *rooks, which is perfect only in case* $n = 1$;

(b) *for* $k \geqslant n_1 + n_2$, *there is a* (*never perfect*) *rook covering of* A *made of* E *rooks* (*here* $E = (n_1 + 1)(n_2 + 1)$);

(c) *for* $0 \leqslant k < n_1 + n_2$, $n_2 \neq 1$, *there are no rook coverings of* A *with* E *rooks* (*here* $E = n + 1$).

Proof. Case (a) can be easily seen. In Case (b), a rook covering R with $|R| = E$ can be easily found by choosing all the places on an $A_{n_1} \times A_{n_2}$ slice of A. Moreover, a covering with that cardinality can never be perfect since $E > n_1 + 1$ and so there must be an $A_{n_2} \times A_n$ slice of A containing at least two rooks.

Case (c). Suppose there is a rook covering R with $|R| = E = n + 1$. Then there cannot be an $A_{n_1} \times A_{n_2}$ slice of A containing no rooks, since every place in such a slice would have to be "covered" by a rook of R from outside the slice, and this would imply $|R| \geqslant (n_1 + 1)(n_2 + 1)$. But $(n_1 + 1)(n_2 + 1) > E = n + 1$.

Thus each of the $(n + 1)$ $A_{n_1} \times A_{n_2}$ slices contains exactly one rook.

If for all $(i, j) \in A_{n_1} \times A_{n_2}$ we have at least one rook in place $a_{i,j,k}$ for some k, then we must have $E \geqslant (n_1 + 1)(n_2 + 1)$, and we get again a contradiction. Hence there is at least one pair $(i_0, j_0) \in A_{n_1} \times A_{n_2}$ such that no rook of R is in position $a_{i_0,j_0,k}$ for all $k = 0, \ldots, n$. Without loss of generality we can suppose $i_0 = j_0 = 0$. Hence all elements $r_{i,j,k} \in R$ must have either $i = 0$ or $j = 0$ in order to "cover" the places $a_{0,0,k}$. But in this case, in order to cover all the places $a_{1,1,k}$, every $r_{i,j,k} \in R$ should have $i \neq j \in \{0, 1\}$.

Now, since $n_2 > 1$, it follows that R cannot be a covering (e.g. the place $a_{1,2,0}$ is not covered). □

References

[1] B. Ådlandsvik, Varieties with an extremal number of degenerate higher secant varieties, J. Reine Angew. Math. 392 (1988) 16–26.
[2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Alg. Geom. 4 (1995) 201–222.
[3] M.D. Atkinson, S. LLoyd, The ranks of $m \times n \times (mn - 2)$ tensors, SIAM J. Comput. 12 (1983) 1–8.
[4] M.D. Atkinson, N.M. Stephens, On the maximal multiplicative complexity of a family of bilinear forms, Linear Algebra Appl. 27 (1979) 611–615.
[5] P. Bürgisser, M. Clausen, M.A. Shokrollahi, in: Algebraic Complexity Theory, vol. 315, Grund. der Math. Wiss., Springer, Berlin, 1997.
[6] M.L. Catalano-Johnson, The possible dimension of the higher secant varieties, Am. J. Math. 118 (1996) 355–361.
[7] M.V. Catalisano, A.V. Geramita, A. Gimigliano, On the secant varieties to the tangential varieties of a Veronesean, Proc. AMS 130 (4) (2001) 975–985.
[8] A. Capani, G. Niesi, L. Robbiano, CoCoA, a system for doing Computations in Commutative Algebra. Available via anonymous ftp from: cocoa.dima.unige.it.
[9] L. Chiantini, C. Ciliberto, Weakly defective varieties, preprint.
[10] R. Ehrenborg, On apolarity and generic canonical forms, J. Algebra 213 (1999) 167–194.
[11] R. Ehrenborg, G.-C. Rota, Apolarity and canonical forms for homogeneous polynomials, European J. Combin. 14 (1993) 157–181.
[13] A.V. Geramita, Inverse systems of fat points, Queens Papers in Pure Applied Mathematics, vol. 102, The Curves Seminar at Queens', vol. X, 1998.
[15] R. Grone, Decomposable tensors as a quadratic variety, Proc. Am. Math. Soc. 64 (1977) 227–230.
[16] J. Harris, Algebraic Geometry, a First Course, Springer, New York, 1993.
[17] J. Hstad, Tensor rank is NP-complete, J. Algorithms 11 (1990) 644–654.
[19] A. Iarrobino, V. Kanev, in: Power Sums, Gorenstein algebras, and determinantal loci, Lecture Notes in Math., vol. 1721, Springer, Berlin, 1999.
[20] J. Ja'Ja, Optimal evaluation of pairs of bilinear forms, SIAM J. Comput. 8 (1979) 443–462.
[21] V.G. Kac, Some remark on nilpotent orbits, J. Algebra 64 (1980) 190–213.
[22] V. Kanev, Chordal varieties of veronese varieties and catalecticant matrices, J. Math. Sci. (New York) 94 (1999) 1114–1125.
[23] T. Lickteig, Typical tensorial rank, Linear Algebra Appl. 69 (1985) 95–120.
[24] F. Palatini, Sulle varietà algebriche per le quali sono di dimensione minore dell' ordinario, senza riempire lo spazio ambiente, una o alcuna delle varietà formate da spazi seganti, Atti Accad. Torino Cl. Scienze Mat. Fis. Nat. 44 (1909) 362–375.
[25] O. Porras, Rank varieties and their resolution, J. Algebra 186 (1996) 677–723.
[26] C.M. Ringel, The development of the representation theory of finite dimensional algebras 1968–1975, in: Representation Theory and Algebraic Geometry, London Math. Soc., Lecture Note Ser., vol. 238, Cambrige University Press, Cambridge, 1997, pp. 89–115.
[27] S. Roman, Coding and Information Theory, Springer, New York, 1992.
[28] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra Appl. 52 (1983) 645–685.
[29] A. Terracini, Sulle V_k per cui la varietàdegli $S_h(h+1)$-seganti ha dimensione minore dell'ordinario, Rend. Circ. Mat. Palermo 31 (1911) 392–396.
[30] K. Wakeford, On canonical forms (2), Proc. London Math. Soc. 18 (1919/20) 403–410.
[31] W. Watkins, Linear maps and tensor rank, J. Algebra 38 (1976) 75–84.
[32] F.L. Zak, Tangents and Secants of Algebraic Varieties (Translations of Math. Monographs), vol. 127, AMS, Providence, RI, 1993.