JOURNAL OF COMBINATORIAL THEORY, Series B **48**, 111–116 (1990)

Hamiltonian Degree Conditions Which Imply a Graph Is Pancyclic

DOUGLAS BAUER

Stevens Institute of Technology, Hoboken, New Jersey 07030

AND

EDWARD SCHMEICHEL*

San Jose State University, San Jose, California 95192

Communicated by the Managing Editors

Received November 24, 1986; revised March 29, 1988

We use a recent cycle structure theorem to prove that three well-known hamiltonian degree conditions (due to Chvátal, Fan, and Bondy) each imply that a graph is either pancyclic, bipartite, or a member of an easily identified family of exceptions. © 1990 Academic Press, Inc.

INTRODUCTION

We consider only finite, undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated. A good reference for undefined terms is [3]. We mention only that we will use $d(x)$ for the degree of a vertex x, $\mathrm{dist}(x, y)$ to denote the distance between vertices x and y, and $\beta(G)$, $\chi(G)$ and $\Delta(G)$ to denote (respectively) the independence number, chromatic number, and maximum vertex degree of a graph G. Indices throughout the paper are to be taken modulo n.

Beginning with a classical theorem of Dirac [7], various sufficient conditions for a graph to be hamiltonian have been given in terms of the vertex degrees of the graph. Three well-known such conditions (due to Chvátal, Fan, and Bondy) are given below.

PROPOSITION 1 [6]. *Let G be a graph on $n \geqslant 3$ vertices with vertex degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. If $d_k \leqslant k < n/2$ implies $d_{n-k} \geqslant n-k$, then G is hamiltonian.*

* Supported in part by the National Science Foundation under Grant ECS 85-11211.

0095-8956/90 $3.00

PROPOSITION 2 [8]. *Let G be a 2-connected graph on n vertices. If* $\operatorname{dist}(x, y)=2$ *implies* $\max\{d(x), d(y)\} \geqslant n/2$ *for all vertices x and y, then G is hamiltonian.*

PROPOSITION 3 [4]. *Let G be a 2-connected graph on n vertices. If for every set of three independent vertices x, y, and z we have* $d(x)+d(y)+d(z) \geqslant 3n/2-1$, *then G is hamiltonian.*

The goal of this paper is to show that each of these three degree conditions implies substantially more about the cycle structure of G than the mere fact that G is hamiltonian. We call an n-vertex graph *pancyclic* if it contains an l-cycle for every l such that $3 \leqslant l \leqslant n$. We will prove the following results.

THEOREM 1. *Let G be a graph on* $n \geqslant 3$ *vertices with degree sequence* $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. *If* $d_k \leqslant k < n/2$ *implies* $d_{n-k} \geqslant n-k$, *then G is pancyclic or bipartite.*

THEOREM 2. *Let G be a 2-connected graph on n vertices. If* $\operatorname{dist}(x, y)=2$ *implies* $\max\{d(x), d(y)\} \geqslant n/2$ *for all vertices x and y, then G is pancyclic,* $K_{n/2, n/2}$, $K_{n/2, n/2}-e$ *or the graph* F_n *in Fig.* 1.

THEOREM 3. *Let G be a 2-connected graph on n vertices. Suppose that for every set of three independent vertices x, y, and z, we have* $d(x)+d(y)+d(z) \geqslant 3n/2-1$. *Then G is pancyclic,* $K_{n/2, n/2}$, $K_{n/2, n/2}-e$ *or* C_5.

Although proofs of Theorems 1 and 2 have appeared previously (in [9] and [1], respectively), the proofs were somewhat ad hoc. By contrast, the proofs of Theorems 1, 2, and 3 given below are all straightforward applications of a recent cycle structure theorem (Lemma 3 below), which may find other applications as well.

Before giving the proofs of the main theorems, we need a few preliminary results.

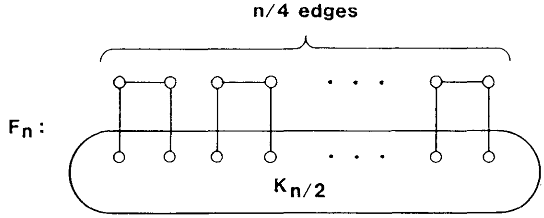

FIGURE 1

PRELIMINARY RESULTS

LEMMA 1 [1, Lemma 1]. *Let G be a graph on n vertices. Suppose G contains an* $(n-1)$*-cycle which does not contain the vertex x. If* $d(x) \geqslant n/2$*, then G is pancyclic.*

LEMMA 2 [2]. *Let G be a graph on n vertices containing a hamiltonian cycle* Γ*. If* Γ *contains consecutive vertices x, y such that* $d(x)+d(y)>n$*, then G is pancyclic.*

Unfortunately, the degree conditions in Propositions 1, 2, and 3 do not readily guarantee the existence of a hamiltonian cycle having a consecutive pair of vertices x, y satisfying $d(x)+d(y)>n$. This difficulty is largely overcome by the following recent result.

LEMMA 3 [10]. *Let G be a graph on n vertices with hamiltonian cycle* $\Gamma = v_1 v_2 \cdots v_n v_1$*. Suppose that* $d(v_j)+d(v_{j+1}) \geqslant n$ *for some j. Then G is pancyclic or bipartite unless all the following conditions are true for G:*

(i) *the only cycle length missing in G is* $n-1$*;*

(ii) v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3} *are independent except for the edges of* Γ*;*

(iii) $d(v_{j-2})$, $d(v_{j-1})$, $d(v_{j+2})$, $d(v_{j+3})<n/2$ *(this implies G contains at most* $n/2$ *vertices of degree at least* $n/2$*);*

(iv) *if* $d(v_j)=d(v_{j+1})=n/2$*, then* $v_j v_{j-4}$, $v_j v_{j-3}$, $v_{j+1} v_{j+4}$*, and* $v_{j+1} v_{j+5}$ *are all edges of G.*

Lemma 3 is the key tool needed to give unified proofs of Theorems 1, 2, and 3.

PROOFS OF THE MAIN THEOREMS

Before proving Theorem 1, we state the following result.

LEMMA 4 [9, Lemma 2]. *If G satisfies the degree condition in Theorem 1 with n odd, or with n even and* $d_{n/2} \neq n/2$*, then G is pancyclic.*

Proof of Theorem 1. By Proposition 1, G contains a hamiltonian cycle $\Gamma = v_1 v_2 \cdots v_n v_1$. By Lemma 4, we may assume that n is even and $d_{n/2} = n/2$. So G has more than $n/2$ vertices of degree at least $n/2$. Thus $d(v_j)$, $d(v_{j+1}) \geqslant n/2$, for some j. But then by Lemma 3(iii), G is pancyclic or bipartite.

This completes the proof of Theorem 1. ∎

Proof of Theorem 2. By Proposition 2, G contains a hamiltonian cycle $\Gamma = v_1 v_2 \cdots v_n v_1$. Clearly, by the degree condition, $\Delta(G) \geqslant n/2$. Without loss of generality we assume $d(v_1) \geqslant n/2$.

Consider the vertex pair $\{v_2, v_n\}$. If $v_2 v_n \in E(G)$, then $v_2 v_3 \cdots v_n v_2$ is an $(n-1)$-cycle in G. Since $d(v_1) \geqslant n/2$, G would be pancyclic by Lemma 1. But if $v_2 v_n \notin E(G)$, then $\mathrm{dist}(v_2, v_n) = 2$ and so $\max\{d(v_2), d(v_n)\} \geqslant n/2$. Without loss of generality, we may assume $d(v_n) \geqslant n/2$. If either inequality $d(v_1), d(v_n) \geqslant n/2$ were strict, then $d(v_1) + d(v_n) > n$ and G would be pancyclic by Lemma 2. Hence we assume $d(v_1) = d(v_n) = n/2$. But then by Lemma 3 with $j = n$, G is pancyclic or bipartite unless conditions (i)–(iv) in Lemma 3 *all* hold. Moreover, it is easy to verify that if G is bipartite, the degree condition together with the assumption that $d(v_1) = d(v_n) = n/2$ implies that G must be $K_{n/2,\, n/2}$ or $K_{n/2,\, n/2} - e$. To complete the proof, it suffices to show that the degree condition together with conditions (i)–(iv) in Lemma 3 imply that G is the graph F_n in Fig. 1.

By (i), G does not contain an $(n-1)$-cycle and so $v_j v_{j+2} \notin E(G)$ for any j. But then $\mathrm{dist}(v_j, v_{j+2}) = 2$, and so $\max\{d(v_j), d(v_{j+2})\} \geqslant n/2$. Since this holds for every j, it follows that at least $n/2$ of the vertices of G have degree at least $n/2$. But by Lemma 3(iii), at most $n/2$ vertices in G have degree at least $n/2$. So exactly $n/2$ vertices in G have degree at least $n/2$. Using Lemma 2 and Lemma 3(iii), it follows that $n \equiv 0 \pmod 4$ and (since $d(v_1) = d(v_n) = n/2$) that $d(v_k) = n/2$ if $k \equiv 0, 1 \pmod 4$ while $d(v_k) < n/2$ otherwise.

Let $A = \{v_i \mid d(v_i) < n/2\} = \{v_i \mid i \equiv 2, 3 \pmod 4\}$ and $B = \{v_i \mid d(v_i) = n/2\} = \{v_i \mid i \equiv 0, 1 \pmod 4\}$. We now show

(a) If $v_i, v_j \in A$, then $v_i v_j \notin E(G)$ unless $j = i \pm 1$.

For suppose $v_i, v_j \in A$ and $v_i v_j \in E(G)$ but $j \neq i \pm 1$. Either v_{i-1} or v_{i+1} belongs to A; without loss of generality, suppose $v_{i+1} \in A$. Then $v_{i+1} v_j \in E(G)$, since otherwise $\mathrm{dist}(v_{i+1}, v_j) = 2$ and so $\max\{d(v_{i+1}), d(v_j)\} \geqslant n/2$, which contradicts $v_{i+1}, v_j \in A$. Now either v_{j-1} or v_{j+1} belongs to A. If $v_{j-1} \in A$, then $v_{j-2} v_{j+1} \in E(G)$ by Lemma 3(iv) and G contains the $(n-1)$-cycle $v_i v_j v_{i+1} v_{i+2} \cdots v_{j-2} v_{j+1} v_{j+2} \cdots v_i$. This contradicts Lemma 3(i). An analogous contradiction arises if $v_{j+1} \in A$. This proves (a).

We next show

(b) If $v_i \in B$, $v_j \in A$, then $v_i v_j \notin E(G)$ unless $j = i \pm 1$.

For suppose $v_i \in B$, $v_j \in A$ and $v_i v_j \in E(G)$ but $j \neq i \pm 1$. Either v_{i-1} or v_{i+1} belongs to A; without loss of generality suppose $v_{i-1} \in A$. By an argument similar to that used in (a) we conclude $v_{i-1} v_j \in E(G)$. Now (a) implies that $v_j = v_{i-2}$. This contradiction with Lemma 3(i) proves (b).

From (a), (b), and the definition of B, we see that the graph induced by B is $K_{n/2}$. It is then immediate that G must be the graph F_n.

This completes the proof of Theorem 2. ▮

Proof of Theorem 3. By Proposition 3, G contains a hamiltonian cycle $\Gamma = v_1 v_2 \cdots v_n v_1$.

We first establish the following fact which will be needed later in the proof.

If $d(v_j) + d(v_{j+1}) \geqslant n$ for some j, then G is pancyclic, $K_{n/2,\,n/2}$ or $K_{n/2,\,n/2} - e$. $(*)$

To prove $(*)$, note that if $d(v_j) + d(v_{j+1}) > n$, then G would be pancyclic by Lemma 2. Thus we assume $d(v_j) + d(v_{j+1}) = n$, with say $d(v_j) \leqslant n/2$. But then by Lemma 3, G will be pancyclic or bipartite unless conditions (i)–(iv) all hold for G. But if conditions (i)–(iv) held for G, then by (ii) $\{v_{j-2}, v_j, v_{j+2}\}$ would be an independent set and by (iii) $d(v_{j-2})$, $d(v_{j+2}) < n/2$, so that $d(v_{j-2}) + d(v_j) + d(v_{j+2}) \leqslant 3n/2 - 3/2 < 3n/2 - 1$, a contradiction. We conclude that G is pancyclic or bipartite. If G is bipartite, it follows easily from the degree condition that G must be $K_{n/2,\,n/2}$ or $K_{n/2,\,n/2} - e$. This proves $(*)$.

We assume henceforth that G is not C_5, and show next that $\Delta(G) \geqslant n/2$. If $\beta(G) = 1$ then G is complete and the result is immediate. If $\beta(G) \geqslant 3$, the result follows from the degree condition. Hence we may assume $\beta(G) = 2$. In particular, $\chi(G) \geqslant n/2$, since any color class contains at most $\beta(G) = 2$ vertices. Since $\beta(G) = 2$ and G is not C_5, G is not an odd cycle. But if G is neither complete nor an odd cycle, then $\Delta(G) \geqslant \chi(G) \geqslant n/2$ by a theorem of Brooks [5].

Let x be a vertex of G with $d(x) = \Delta \geqslant n/2$, where $\Delta = \Delta(G)$. Let y, z denote the vertices immediately preceding and succeeding x on Γ. If $yz \in E(G)$ then (since $d(x) \geqslant n/2$) G would be pancyclic by Lemma 1. Hence we assume $yz \notin E(G)$. We may also assume $d(y), d(z) \leqslant n - \Delta - 1$, since otherwise either $d(x) + d(y) \geqslant n$ or $d(x) + d(z) \geqslant n$ and G would be pancyclic, $K_{n/2,\,n/2}$ or $K_{n/2,\,n/2} - e$ by $(*)$. But then $d(y) + d(z) \leqslant 2(n - \Delta) - 2 \leqslant n - 2$, and so there exists a vertex $u \neq y, z$ such that $\{u, y, z\}$ is an independent set. Since $d(u) + d(y) + d(z) \geqslant 3n/2 - 1$, we obtain $d(u) \geqslant 3n/2 - 1 - (d(y) + d(z)) \geqslant 3n/2 - 1 - (2(n - \Delta) - 2) = (\Delta - n/2) + \Delta + 1 \geqslant \Delta + 1$, a contradiction.

This completes the proof of Theorem 3. ▮

Acknowledgment

We thank the referees for strengthening the result of Theorem 3.

References

1. A. Benhocine and A. P. Wojda, The Geng-Hua Fan conditions for pancyclic or Hamilton-connected graphs, *J. Combin. Theory Ser. B* **42** (1987), 167–180.
2. J. A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser. B* **11** (1971), 80–84.
3. J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," Macmillan Co., New York, 1976.
4. J. A. Bondy, "Longest Paths and Cycles in Graphs of High Degree," Research Report CORR 80-16, University of Waterloo, Waterloo, Ontario, 1980.
5. R. L. Brooks, On colouring the nodes of a graph, *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.
6. V. Chvátal, On hamilton's ideals, *J. Combin. Theory Ser. B* **12** (1972), 163–168.
7. G. A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc.* **2** (1952), 69–81.
8. Fan Geng-Hua, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984), 221–227.
9. E. Schmeichel and S. L. Hakimi, Pancyclic graphs and a conjecture of Bondy and Chvátal, *J. Combin. Theory Ser. B* **17** (1974), 22–34.
10. E. Schmeichel and S. L. Hakimi, A cycle structure theorem for hamiltonian graphs, *J. Combin. Theory Ser. B* **45** (1988), 99–107.