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Inversion of a block Löwner matrix

M. Van Barel[a,*], Z. Vavřín[b]

[a] *Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Heverlee, Belgium*
[b] *Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague, Czech Republic*

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Abstract

In this paper, we give a fast algorithm to compute the parameters of an inversion formula for any nonsingular block Löwner matrix. The connection with matrix rational interpolation is given.

Keywords: (Block) Löwner matrix; Inversion formula; (Matrix) Rational interpolation; (Un)Attainable points; (In)Accessible points

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1. Introduction

The present paper gives some results on block Löwner matrices, i.e. matrices of the form

$$\left[\frac{C_i - D_j}{y_i - z_j}\right]_{i=0,1,\ldots,m-1}^{j=0,1,\ldots,n-1},$$

the C_i's, D_j's being $p \times q$ blocks. The investigation is restricted to square nonsingular matrices. Particularly, this means that $mp = nq$. The method of UV-reduction proposed in [17, Part II, p.136] for Toeplitz-like operators proves to be very useful here giving a simple inversion formula (and a criterion of nonsingularity). Generalization of Löwner's well-known results leads to an interpolation interpretation of the parameters of the inversion formula. More exactly, four couples of matrix polynomials $[V(x), U(x)]$, $[\tilde{V}(x), \tilde{U}(x)]$, $[Q(x), P(x)]$ and $[\tilde{Q}(x), \tilde{P}(x)]$ appear, the first and third satisfying the linearized conditions for a set of interpolation nodes $\{\bar{x}\}$ and a set of

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corresponding (matrix) values $\{F_{\bar{x}}\}$:

$$V(\bar{x}) - U(\bar{x})F_{\bar{x}} = 0,$$

$$Q(\bar{x}) - P(\bar{x})F_{\bar{x}} = 0,$$

$$V(x) \in \mathbb{F}^{p\times q}[x], \qquad U(x) \in \mathbb{F}^{p\times p}[x],$$

$$Q(x) \in \mathbb{F}^{q\times q}[x], \qquad P(x) \in \mathbb{F}^{q\times p}[x],$$

$$\deg U(x) = m, \qquad \deg Q(x) = m,$$

$$\deg V(x) < m, \qquad \deg P(x) < m.$$

Thus, the first system gives a solution of the rational interpolation problem

$$U^{-1}(\bar{x})V(\bar{x}) = F_{\bar{x}} \tag{1}$$

if the values $U(\bar{x})$ are nonsingular. Similarly, the second and fourth couples satisfy

$$\tilde{V}(\bar{x}) - F_{\bar{x}}\tilde{U}(\bar{x}) = 0, \qquad \tilde{Q}(\bar{x}) - F_{\bar{x}}\tilde{P}(\bar{x}) = 0.$$

Löwner matrices (by some authors called divided differences or interpolation matrices) with 1×1 blocks were introduced in the inspiring paper [19] as a tool to investigate monotone matrix functions (see also [11]) and to solve the scalar rational interpolation problem (see also [6, 3]). Starting from a Löwner matrix, one can investigate the connection to Hankel, Toeplitz, Bézout matrices and to rational interpolation as was done in [12, 27]. In [26], an inversion formula is given for a Löwner matrix. In [1], the block Löwner matrix is used as a tool to construct a minimal state-variable realization from interpolation data (see also [4]).

In Section 2, a criterion of invertibility and an inversion formula for the block Löwner matrix is constructed based on the UV-reduction. Section 3 shows the connection with matrix rational interpolants. In Section 4, we find interesting properties of the matrices

$$T(x) = \begin{bmatrix} U(x) & -V(x) \\ -P(x) & Q(x) \end{bmatrix} \quad \text{and} \quad \tilde{T}(x) = \begin{bmatrix} \tilde{Q}(x) & \tilde{V}(x) \\ \tilde{P}(x) & \tilde{U}(x) \end{bmatrix}.$$

Section 5 is an application of the results of [24] where a unified approach to solve a wide class of interpolation problems in $O(n^2)$ operations is given (n denotes the number of interpolation data). We use it to find $T(x)$ and $\tilde{T}(x)$ and thus also to compute L^{-1}. We also give the connection to the polynomial approach used in linear system theory to solve rational interpolation problems, more specifically, to the behavioral approach to linear exact modelling described in [5]. In Section 6, we give the connection with matrix continued fractions. Section 7 deals with the rational interpolation problem (1) in more detail, studying the (un)attainability or (in)accessibility of interpolation points (for the scalar case, see [6, 21] and for multiple points [27]).

2. Inversion formula

Consider an $(m \times n)$ block Löwner matrix

$$L = \left[\frac{C_i - D_j}{y_i - z_j} \right]_{i=0,1,2,\dots,m-1}^{j=0,1,2,\dots,n-1}$$

with $C_i, D_j \in \mathbb{F}^{p \times q}$ and $y_i, z_j \in \mathbb{F}$ such that $Y = \{y_0, y_1, \dots, y_{m-1}\}$ and $Z = \{z_0, z_1, \dots, z_{n-1}\}$ have m, respectively n different elements and $Y \cap Z = \emptyset$.

We assume that the block Löwner matrix is square, i.e., $mp = nq$. In this section, we give an invertibility criterion for such a square block Löwner matrix. If the inverse exists, we construct an inversion formula. To this end, we use the method of UV-reduction proposed in [17, Part II, p.136] for Toeplitz-like operators.

Theorem 2.1 (UV-reduction). *Given a block Löwner matrix*

$$L = \left[\frac{C_i - D_j}{y_i - z_j} \right]_{i=0,1,2,\dots,m-1}^{j=0,1,2,\dots,n-1},$$

then

$$\operatorname{diag}(y_i)L - L\operatorname{diag}(z_j) = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{m-1} \end{bmatrix} [I_q \ I_q \ \dots \ I_q] - \begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix} [D_0 \ D_1 \ \dots \ D_{n-1}] \tag{2}$$

with

$$\operatorname{diag}(y_i) = \operatorname{diag}(y_0 I_p, y_1 I_p, \dots, y_{m-1} I_p)$$

and

$$\operatorname{diag}(z_j) = \operatorname{diag}(z_0 I_q, z_1 I_q, \dots, z_{n-1} I_q).$$

Proof. Evident by direct computation of both sides. □

Using the UV-reduction, we get the following inversion formula and invertibility criterion for a block Löwner matrix.

Theorem 2.2. *Given the block Löwner matrix $L = [(C_i - D_j)/(y_i - z_j)]$. Consider the equations*

$$[P_0 \ P_1 \ \dots \ P_{m-1}]L = [I_q \ I_q \ \dots \ I_q], \tag{3}$$

$$[U_0 \ U_1 \ \dots \ U_{m-1}]L = [D_0 \ D_1 \ \dots \ D_{n-1}], \tag{4}$$

$$L\begin{bmatrix}\tilde{P}_0\\ \tilde{P}_1\\ \vdots\\ \tilde{P}_{n-1}\end{bmatrix}=\begin{bmatrix}I_p\\ I_p\\ \vdots\\ I_p\end{bmatrix}, \tag{5}$$

$$L\begin{bmatrix}\tilde{U}_0\\ \tilde{U}_1\\ \vdots\\ \tilde{U}_{n-1}\end{bmatrix}=\begin{bmatrix}C_0\\ C_1\\ \vdots\\ C_{m-1}\end{bmatrix}. \tag{6}$$

Eqs. (3) *and* (4) *are solvable* (*similarly*, (5) *and* (6) *are solvable*) *iff the block Löwner matrix*

$$L=\left[\frac{C_i-D_j}{y_i-z_j}\right]_{i=0,1,2,\ldots,m-1}^{j=0,1,2,\ldots,n-1}$$

is nonsingular.

If L *is nonsingular, its inverse* L^{-1} *can be written as*

$$L^{-1}=\left[\frac{\tilde{U}_iP_j-\tilde{P}_iU_j}{y_j-z_i}\right]_{i=0,1,2,\ldots,n-1}^{j=0,1,2,\ldots,m-1}.$$

Note that

$$P_j\in\mathbb{F}^{q\times p},\quad j=0,1,\ldots,m-1,$$

$$U_j\in\mathbb{F}^{p\times p},\quad j=0,1,\ldots,m-1,$$

$$\tilde{P}_i\in\mathbb{F}^{q\times p},\quad i=0,1,\ldots,n-1,$$

$$\tilde{U}_i\in\mathbb{F}^{q\times q},\quad i=0,1,\ldots,n-1.$$

Proof. It is clear that if L is invertible, Eqs. (3) and (4) are solvable. Suppose now that (3) and (4) are solvable. If $c\in\operatorname{Ker}L$, we get from (3),

$$[I_q\ I_q\ \ldots\ I_q]c=0$$

and from (2)–(4) that

$$L\operatorname{diag}(z_j)c=0\quad\text{or}\quad\operatorname{diag}(z_j)c\in\operatorname{Ker}L.$$

Repeating the same reasoning, we derive

$$[z_j^iI_q]_{i=0,1,\ldots,n-1}^{j=0,1,\ldots,n-1}c=0.$$

Because the block Vandermonde matrix $[z_j^iI_q]$ is nonsingular (since $z_i\neq z_j, j\neq i$), it follows that $c=0$. Hence, L is invertible.

Assume now that L is invertible. Multiplying (2) to the left and right by L^{-1}, we get

$$L^{-1}\operatorname{diag}(y_i) - \operatorname{diag}(z_j)L^{-1}$$

$$= L^{-1}\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{m-1} \end{bmatrix}[I_q \ I_q \ \dots \ I_q]L^{-1} - L^{-1}\begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}[D_0 \ D_1 \ \dots \ D_{n-1}]L^{-1}.$$

Using the definition of P_j, U_j, $\tilde{P}_i$ and $\tilde{U}_i$, we derive

$$L^{-1}\operatorname{diag}(y_i) - \operatorname{diag}(z_j)L^{-1}$$

$$= \begin{bmatrix} \tilde{U}_0 \\ \tilde{U}_1 \\ \vdots \\ \tilde{U}_{n-1} \end{bmatrix}[P_0 \ P_1 \ \dots \ P_{m-1}] - \begin{bmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \vdots \\ \tilde{P}_{n-1} \end{bmatrix}[U_0 \ U_1 \ \dots \ U_{m-1}].$$

Taking the $(i,j)(q \times p)$-block of the left- and right-hand side, it follows that

$$(L^{-1})_{i,j}y_j - z_i(L^{-1})_{i,j} = \tilde{U}_iP_j - \tilde{P}_iU_j.$$

Therefore, the (i,j) block of L^{-1} can be written as

$$(L^{-1})_{i,j} = \frac{\tilde{U}_iP_j - \tilde{P}_iU_j}{y_j - z_i}. \qquad \square$$

3. The connection with matrix rational interpolants

In this section, we give the relationship between P, U, $\tilde{P}$, $\tilde{U}$ and certain matrix rational interpolants.

Definition 3.1 (*Degree of polynomial matrices*). Given $P(x) \in \mathbb{F}^{p\times q}[x]$, we write $\deg P(x) < n$ iff each of the polynomial elements of $P(x)$ has degree smaller than n. We say $\deg P(x) = n$ iff $P(x) = P_nz^n + P_{n-1}z^{n-1} + \cdots + P_0$ with $P_n \in \mathbb{F}^{p\times q}$ having full rank.

Take the following basis for the vector space $\mathbb{F}_n[x]$ of all polynomials having degree $\leqslant n$:

$$\{b_0, b_1, \dots, b_{n-1}, b\}$$

with

$$b(x) = \prod_{j=0}^{n-1}(x - z_j)$$

and

$$b_j(x) = \frac{b(x)}{(x - z_j)}, \quad j = 0, 1, \ldots, n - 1.$$

Similarly we can write a basis for $\mathbb{F}_m[z]$: $\{a_0, a_1, \ldots, a_{m-1}, a\}$ based on the points $Y = \{y_0, y_1, \ldots, y_{m-1}\}$ instead of the points $Z = \{z_0, z_1, \ldots, z_{n-1}\}$.

Constructing the following matrix polynomials from the solutions of Eqs. (4) and (6),

$$\mathbb{F}^{q\times q}[x] \ni \tilde{U}(x) = b(x)I_q - \sum_{j=0}^{n-1} b_j(x)\tilde{U}_j, \tag{7}$$

$$\mathbb{F}^{p\times q}[x] \ni \tilde{V}(x) = -\sum_{j=0}^{n-1} b_j(x)D_j\tilde{U}_j, \tag{8}$$

$$\mathbb{F}^{p\times p}[x] \ni U(x) = a(x)I_p - \sum_{i=0}^{m-1} a_i(x)U_i \tag{9}$$

and

$$\mathbb{F}^{p\times q}[x] \ni V(x) = -\sum_{i=0}^{m-1} a_i(x)U_iC_i, \tag{10}$$

we get the following interpretation for $\tilde{U}$ and U.

Theorem 3.1 (Connection with matrix rational interpolation). *If the matrix polynomials $\tilde{V}(x)$ and $\tilde{U}(x)$ are given by* (7) *and* (8), *the matrix rational function*

$$R(x) = \tilde{V}(x)\tilde{U}(x)^{-1}$$

represents the unique matrix rational function having $\deg \tilde{V}(x) \leqslant n - 1$ *and* $\deg \tilde{U}(x) = n$ *such that $R(x)$ satisfies the linearized interpolation conditions, i.e.*

$$\tilde{V}(\bar{x}) = F_{\bar{x}}\tilde{U}(\bar{x}) \quad \forall\, \bar{x} \in Y \cup Z, \tag{11}$$

with

$$F_{\bar{x}} = C_i \quad \text{if } \bar{x} = y_i$$

and

$$F_{\bar{x}} = D_i \quad \text{if } \bar{x} = z_i.$$

A similar result is also true for $U(x)^{-1}V(x)$, given by (9) *and* (10). *Moreover, they represent the same matrix rational function, i.e.*

$$R(x) = \tilde{V}(x)\tilde{U}(x)^{-1} = U(x)^{-1}V(x).$$

Proof. Let us look at all matrix rational functions $R(x)$ of the form $(\deg \leqslant n - 1)(\deg = n)^{-1}$ interpolating the data. We can always normalize $R(x)$ such that the highest degree coefficient of the denominator is I_q.

We take the basis $\{b_0, b_1, \dots, b_{n-1}, b\}$ based on the interpolation points $Z = \{z_0, z_1, \dots, z_{n-1}\}$ and parametrize $R(x)$ such that

$$R(x) = A(x)B(x)^{-1},$$

with

$$B(x) = b(x)I_q - \sum_{j=0}^{n-1} b_j(x)B_j, \tag{12}$$

$$A(x) = -\sum_{j=0}^{n-1} b_j(x)A_j. \tag{13}$$

The matrix rational function has to satisfy the linearized interpolation conditions, i.e.

$$A(y_i) = C_i B(y_i), \quad i = 0, 1, \dots, m-1, \tag{14}$$

$$A(z_j) = D_j B(z_j), \quad j = 0, 1, \dots, n-1. \tag{15}$$

Using the parametrization (12) and (13), the interpolation conditions (15) can be rewritten as

$$b_j(z_j)A_j = b_j(z_j)D_jB_j \quad \text{or} \quad A_j = D_jB_j, \qquad j = 0, 1, \dots, n-1.$$

Once we have the B_j, we can compute the A_j. How do we compute the B_j? Rewriting (14) gives us

$$-\sum_{j=0}^{n-1} b_j(y_i)D_jB_j = C_i\left(b(y_i)I_q - \sum_{j=0}^{n-1} b_j(y_i)B_j\right)$$

or

$$\sum_{j=0}^{n-1} b_j(y_i)(C_i - D_j)B_j = C_i b(y_i).$$

Using the definition of $b_j(x)$, i.e.

$$b_j(x) = \frac{b(x)}{x - z_j},$$

we derive the equation

$$\sum_{j=0}^{n-1} \frac{b(y_i)}{y_i - z_j}(C_i - D_j)B_j = C_i b(y_i).$$

Because $b(y_i) \neq 0$,

$$\sum_{j=0}^{n-1} \left(\frac{C_i - D_j}{y_i - z_j}\right) B_j = C_i, \quad i = 0, 1, \dots, m-1.$$

Hence, the coefficients B_j are uniquely determined as the solution of the following set of linear equations

$$L\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{m-1} \end{bmatrix}.$$

Therefore, $\tilde{U}(x) = B(x)$ and $\tilde{V}(x) = A(x)$.

Similarly, we can prove that $U^{-1}(x)V(x)$ is the only matrix rational function satisfying the linearized interpolation conditions and having the form

$$(\deg = m)^{-1}(\deg < m).$$

It remains to show that

$$U^{-1}(x)V(x) = \tilde{V}(x)\tilde{U}(x)^{-1}$$

or

$$U(x)\tilde{V}(x) = V(x)\tilde{U}(x).$$

The matrix rational functions $U^{-1}(x)V(x)$ and $\tilde{V}(x)\tilde{U}(x)^{-1}$ both satisfy similar linearized interpolation conditions, i.e.

$$\tilde{V}(\bar{x}) - F_{\bar{x}}\tilde{U}(\bar{x}) = 0 \tag{16}$$

and

$$V(\bar{x}) - U(\bar{x})F_{\bar{x}} = 0 \tag{17}$$

$\forall\, \bar{x} \in Y \cup Z$. Multiplying (16) to the left by $U(\bar{x})$ and (17) to the right by $\tilde{U}(\bar{x})$ and subtracting, we get

$$U(\bar{x})\tilde{V}(\bar{x}) = V(\bar{x})\tilde{U}(\bar{x}).$$

Hence, there exists some polynomial matrix $M(x) \in \mathbb{F}^{p\times q}[x]$ such that

$$U(x)\tilde{V}(x) = V(x)\tilde{U}(x) + a(x)b(x)M(x).$$

Because $\deg(U(x)\tilde{V}(x))$ and $\deg(V(x)\tilde{U}(x)) < m + n$, $M(x) = 0$.

This completes the proof. □

Similarly, we can use P and $\tilde{P}$ to construct matrix polynomials and give an interpretation as a tangential interpolation problem.

Define $\tilde{P}(x)$, $\tilde{Q}(x)$, $P(x)$ and $Q(x)$ based on the solutions of Eqs. (3) and (5) as follows:

$$\mathbb{F}^{q\times p}[x] \ni \tilde{P}(x) = \sum_{j=0}^{n-1} b_j(x)\tilde{P}_j, \tag{18}$$

$$\mathbb{F}^{p\times p}[x] \ni \tilde{Q}(x) = b(x)I_p + \sum_{j=0}^{n-1} b_j(x)D_j\tilde{P}_j, \tag{19}$$

$$\mathbb{F}^{q\times p}[x] \ni P(x) = \sum_{i=0}^{m-1} a_i(x)P_i \tag{20}$$

and

$$\mathbb{F}^{q\times q}[x] \ni Q(x) = a(x)I_q + \sum_{i=0}^{m-1} a_i(x)P_iC_i. \tag{21}$$

The parameters P and $\tilde{P}$ of the inversion formula of a block Löwner matrix are related to an interpolation problem as follows.

Theorem 3.2 (Connection to a tangential interpolation problem). *If the polynomial matrices $\tilde{Q}(x)$ and $\tilde{P}(x)$ are given by* (18) *and* (19), *the matrix polynomial couple $(\tilde{Q}(x), \tilde{P}(x))$ is the only couple such that*

- $\tilde{Q}(x) \in \mathbb{F}^{p\times p}[x]$ *and* $\deg \tilde{Q}(x) = n$; $\tilde{P}(x) \in \mathbb{F}^{q\times p}[x]$ *and* $\deg \tilde{P}(x) < n$; *highest degree coefficient of* $\tilde{Q}(x)$ *is* I_p;
- $\tilde{Q}(\bar{x}) = F_{\bar{x}}\tilde{P}(\bar{x})\ \forall\, \bar{x} \in Y \cup Z$.

Similarly, if $Q(x)$ and $P(x)$ are given by (20) *and* (21), *the matrix polynomial couple $(Q(x), P(x))$ is the only couple such that*

- $Q(x) \in \mathbb{F}^{q\times q}[x]$ *and* $\deg Q(x) = m$; $P(x) \in \mathbb{F}^{q\times p}[x]$ *and* $\deg P(x) < m$; *highest degree coefficient of* $Q(x)$ *is* I_q;
- $Q(\bar{x}) = P(\bar{x})F_{\bar{x}}\ \forall\, \bar{x} \in Y \cup Z$.

Moreover,

$$P(x)\tilde{Q}(x) = Q(x)\tilde{P}(x).$$

Proof. The proof goes along the same lines as the previous proof. We give the proof for the couple $(Q(x), P(x))$.

We can parametrize $P(x)$ and $Q(x)$ as follows:

$$P(x) = \sum_{i=0}^{m-1} a_i(x)P_i$$

and

$$Q(x) = a(x)I_q + \sum_{i=0}^{m-1} a_i(x)Q_i.$$

Because P and Q have to satisfy the interpolation conditions

$$Q(y_i) = P(y_i)C_i, \quad i = 0, 1, \ldots, m-1,$$

we get that $Q_i = P_iC_i$.

The remaining interpolation conditions, $j = 0, 1, \ldots, n-1$, transform into

$$a(z_j)I_q + \sum_{i=0}^{m-1} a_i(z_j)P_iC_i = \sum_{i=0}^{m-1} a_i(z_j)P_iD_j$$

or

$$\sum_{i=0}^{m-1} a_i(z_j)P_i(C_i - D_j) = -a(z_j)I_q$$

or

$$\sum_{i=0}^{m-1} P_i\left(\frac{C_i - D_j}{y_i - z_j}\right) = I_q$$

or

$$[P_0 \; P_1 \; \ldots \; P_{m-1}]L = [I_q \; I_q \; \ldots \; I_q].$$

Because L is nonsingular, the solution is unique. As in the previous proof, it is easy to show that

$$P(x)\tilde{Q}(x) = Q(x)\tilde{P}(x). \qquad \square$$

4. Some properties of the parameters of the inversion formula

In this section, we indicate how to compute $P, \tilde{P}, U$ and $\tilde{U}$. The interpolation conditions of Theorems 3.1 and 3.2 on the corresponding polynomial matrices $P(x)$, $\tilde{P}(x)$, $U(x)$ and $\tilde{U}(x)$ can be summarized as follows:

Look for a polynomial matrix

$$\tilde{T}(x) \in \mathbb{F}^{(p+q)\times(p+q)}[x]$$

such that

- $[I_p \; -F_{\bar{x}}]\tilde{T}(\bar{x}) = 0$, $\forall \bar{x} \in Y \cup Z$,
- $\deg \tilde{T}(x) = n$,
- highest degree coefficient of $\tilde{T}(x)$ is I_{p+q} (hence, $\deg \det \tilde{T}(x) = n(p+q)$).

If we partition the matrix $\tilde{T}(x)$, we get

$$\tilde{T}(x) = \begin{bmatrix} \tilde{Q}(x) & \tilde{V}(x) \\ \tilde{P}(x) & \tilde{U}(x) \end{bmatrix}. \tag{22}$$

Hence, $\tilde{T}(x)$ is unique.

In the same way, $Q(x)$, $P(x)$, $V(x)$ and $U(x)$ can be found as the blocks of a square polynomial matrix $T(x) \in \mathbb{F}^{(p+q)\times(p+q)}[x]$,

$$T(x) = \begin{bmatrix} U(x) & -V(x) \\ -P(x) & Q(x) \end{bmatrix} \tag{23}$$

such that

- $T(\bar{x})\begin{bmatrix} F_{\bar{x}} \\ I_q \end{bmatrix} = 0, \ \forall \bar{x} \in Y \cup Z$,
- $\deg T(x) = m$ (hence, $\deg\det T(x) = m(p+q)$),
- highest degree coefficient of $T(x)$ is I_{p+q}.

Clearly, $T(x)$ is also unique.

The polynomial matrices $T(x)$ and $\tilde{T}(x)$ are related as follows.

Theorem 4.1. *The polynomial matrices $T(x)$ and $\tilde{T}(x)$ given by* (22) *and* (23) *satisfy*

$$T(x)\tilde{T}(x) = \tilde{T}(x)T(x) = a(x)b(x)I_{p+q}.$$

Moreover,

$$\det T(x) = (a(x)b(x))^q, \qquad \det \tilde{T}(x) = (a(x)b(x))^p.$$

Proof. Using the linearized interpolation conditions, we write

$$\begin{bmatrix} I_p & -F(x) \\ 0 & a(x)b(x)I_q \end{bmatrix}\tilde{T}(x) = a(x)b(x)R(x) \tag{24}$$

with $R(x) \in \mathbb{F}^{(p+q)\times(p+q)}[x]$ and $F(x) \in \mathbb{F}^{p\times q}[x]$ such that $F(\bar{x}) = F_{\bar{x}}$.

A possible choice for $F(x)$ is the interpolating matrix polynomial of degree $< m+n$. Taking the determinant of left- and right-hand side of Eq. (24) gives us

$$(a(x)b(x))^q \det \tilde{T}(x) = (a(x)b(x))^{(p+q)} \det R(x).$$

Because $\deg a(x) = m$, $\deg b(x) = n$ and $\deg\det \tilde{T}(x) = n(p+q)$, we derive

$$\deg\det R(x) = nq - mp = 0 \quad \text{or} \quad \det R(x) = c \neq 0.$$

Hence, $R(x)$ is a unimodular matrix and $\det \tilde{T}(x) = c(a(x)b(x))^p$. Because the highest degree coefficient of $\tilde{T}(x)$ is I_{p+q}, $\det \tilde{T}(x)$ is a monic polynomial. The polynomials $a(x)$ and $b(x)$ are monic. Therefore, $c = 1$ (similarly for $T(x)$).

Take the inverse of left- and right-hand side of Eq. (24)

$$\tilde{T}(x)^{-1}\begin{bmatrix} I_p & \dfrac{1}{a(x)b(x)}F(x) \\ 0 & \dfrac{1}{a(x)b(x)}I_q \end{bmatrix} = \frac{1}{a(x)b(x)}R^{-1}(x)$$

or

$$a(x)b(x)\tilde{T}(x)^{-1} = R^{-1}(x)\begin{bmatrix} I_p & -F(x) \\ 0 & a(x)b(x)I_q \end{bmatrix}. \tag{25}$$

The right-hand side of (25) is a polynomial matrix. Hence,

$$T^*(x) = a(x)b(x)\tilde{T}(x)^{-1}$$

is polynomial. Moreover, because

$$\tilde{T}(x)^{-1} = I_{p+q}x^{-n} + \mathrm{O}_-(x^{-n-1}),$$

we get that

$$T^*(x) = a(x)b(x)(I_{p+q}x^{-n} + \mathrm{O}_-(x^{-n-1}))$$
$$= I_{p+q}x^m + \mathrm{O}_-(x^{m-1}).$$

Multiplying (25) to the right by

$$\begin{bmatrix} a(x)b(x)I_p & F(x) \\ 0 & I_q \end{bmatrix}$$

we derive the following interpolation conditions on $T^*(x)$:

$$T^*(x)\begin{bmatrix} a(x)b(x)I_p & F(x) \\ 0 & I_q \end{bmatrix} = a(x)b(x)R^{-1}(x).$$

Hence, $T^*(x) = T(x)$. □

Note that the previous result is an extension of the classical duality between type I (Latin) and type II (German) polynomial systems in a normal point of the Padé–Hermite approximation problem [20, 18, 10].

The following theorem is based on the results of [24]. Column and row reducedness of a polynomial matrix is also defined in Definition 6.1.

Theorem 4.2 (Connection with module theory). *The polynomial matrix $\tilde{T}(x)$ given by* (22) *forms a column* (*and row*) *reduced basis matrix for the submodule $\tilde{S}$ of all polynomial* $(p+q)$*-tuples $\boldsymbol{p}(x) \in \mathbb{F}^{(p+q)\times 1}[x]$ satisfying*

$$[I_p \quad -F_{\bar{x}}]\boldsymbol{p}(\bar{x}) = 0 \quad \forall \bar{x} \in Y \cup Z.$$

Similarly, the polynomial matrix $T(x)$ given by (23) *forms a row* (*and column*) *reduced basis matrix for the submodule S of all polynomial* $(p+q)$*-tuples $\boldsymbol{p}(x) \in \mathbb{F}^{1\times(p+q)}[x]$ satisfying*

$$\boldsymbol{p}(\bar{x})\begin{bmatrix} F_{\bar{x}} \\ I_q \end{bmatrix} = 0 \quad \forall \bar{x} \in Y \cup Z.$$

Proof. We prove the theorem for the polynomial matrix $\tilde{T}(x)$. It is clear that the columns of $\tilde{T}(x)$ are $\mathbb{F}[x]$-linearly independent and satisfy the interpolation conditions. Clearly, $\tilde{T}(x)$ is column (and row) reduced. Moreover,

$$\deg\det\tilde{T}(x) = n(p+q)$$

which is equal to the number of independent interpolation conditions

$$p(m+n) = qn + pn = n(p+q).$$

This proves the theorem for $\tilde{T}(x)$. □

5. Efficient computation of $T(x)$ and $\tilde{T}(x)$

Using the results of [24], we get the following algorithm to compute $T(x)$ and $\tilde{T}(x)$ in a recursive way.

Algorithm 5.1
given
  $\bar{X} = Y \cup Z$ *with* $Y = \{y_0, y_1, \dots, y_{m-1}\}$ *and* $Z = \{z_0, z_1, \dots, z_{n-1}\}$
  $F_{\bar{x}} \in \mathbb{F}^{p \times q}$, $\forall \bar{x} \in \bar{X}$
  $pm = qn$
initialization
  $\tilde{T}(x) = I_{p+q}$ *with column degrees* $\vec{\delta} = [\delta_1 \dots \delta_{p+q}] = [0\,0 \dots 0] \in \mathbb{N}^{p+q}$
  $T(x) = I_{p+q}$
while $\bar{X} \neq$ **do**
  take an arbitrary $\bar{x} \in \bar{X}$; $\bar{X} = \bar{X} \backslash \{\bar{x}\}$
  for $i \in \{1, 2, \dots, p\}$ **do**
  - *compute the residuals, i.e.,*
    $\mathbb{F}^{1 \times (p+q)} \ni [r_1\, r_2 \dots r_{p+q}] = e_i [I_p \quad -F_{\bar{x}}] \tilde{T}(\bar{x})$
    with $\mathbb{F}^{1 \times p} \ni e_i = [0 \dots 0\,1\,0 \dots 0]$ (1 *on the ith place*)
  - *take* $j = \min\{k \mid \delta_k = \delta^*\}$ *with* $\delta^* = \min\{\delta_i \mid r_i \neq 0\}$
  - $\tilde{T}(x) = \tilde{T}(x) W(x)$ *with column degrees*
    $[\delta_1, \dots, \delta_{j-1}, \delta_j + 1, \delta_{j+1}, \dots, \delta_{p+q}]$
    $T(x) = (x - \bar{x}) W^{-1}(x) T(x)$

with

$$W(x) = \begin{bmatrix} r_j & & & & & & & \\ & r_j & & & & & & \\ & & \ddots & & & & & \\ & & & r_j & & & & \\ -r_1 & -r_2 & \cdots & -r_{j-1} & (x - \bar{x}) & -r_{j+1} & \cdots & -r_{p+q} \\ & & & & & r_j & & \\ & & & & & & \ddots & \\ & & & & & & & r_j \end{bmatrix}$$

finally
  $\tilde{T}(x) = \tilde{T}(x) H^{-1}$
  $T(x) = H T(x)$
  with $H =$ *highest degree coefficient of* $\tilde{T}(x)$.

Algorithm 5.1 requires $O((m+n)^2)$ operations in the field $\mathbb{F}$.

Once we have $T(x)$ and $\tilde{T}(x)$, we can immediately partition $\tilde{T}(x)$ and $T(x)$ to get the parameters of the inversion formula.

Note: Algorithm 5.1 is based on the updating procedure described in [24] which computes basis matrices connected to a general matrix rational interpolation problem. In each step of Algorithm 5.1 the set $\bar{X}$ decreases. The polynomial matrix $T(x)$ is a row reduced basis matrix for the submodule S of all polynomial $(p+q)$-tuples $\boldsymbol{p}(x) \in \mathbb{F}^{1\times(p+q)}[x]$ satisfying

$$\boldsymbol{p}(\bar{x})\begin{bmatrix} F_{\bar{x}} \\ I_q \end{bmatrix} = 0 \quad \forall \bar{x} \in (Y \cup Z)\backslash \bar{X}.$$

The polynomial matrix $\tilde{T}(x)$ is a column reduced basis matrix for the submodule $\tilde{S}$ of all polynomial $(p+q)$-tuples $\boldsymbol{p}(x) \in \mathbb{F}^{(p+q)\times 1}[x]$ satisfying

$$[I_p \quad -F_{\bar{x}}]\boldsymbol{p}(\bar{x}) = 0 \quad \forall \bar{x} \in (Y \cup Z)\backslash \bar{X}.$$

Finally, the basis matrices $T(x)$ and $\tilde{T}(x)$, the output of Algorithm 5.1, have the special form as described by Theorem 4.2 because the matrix rational interpolation problem is connected to a nonsingular square block Löwner matrix.

We want to make the link here with the behavioral approach to linear exact modelling [5].

Suppose we look for all matrix rational functions Z of size $p \times q$ given the first part of the Taylor series expansion of Z for a finite number of points $\bar{x}$, i.e.

$$Z(x) = Z(\bar{x}) + (x-\bar{x})Z^{(1)}(\bar{x}) + \cdots + (x-\bar{x})^{(\kappa_{\bar{x}}-1)}Z^{(\kappa_{\bar{x}}-1)}(\bar{x})/(\kappa_{\bar{x}}-1)! + \mathrm{O}((x-\bar{x})^{\kappa_{\bar{x}}}).$$

This is equivalent to looking for all linear systems having transfer function $Z(x)$ which have output $Y_{\bar{x}}(t)$ corresponding to input $U_{\bar{x}}(t)$ with

$$\begin{aligned}\begin{bmatrix} Y_{\bar{x}}(t) \\ U_{\bar{x}}(t) \end{bmatrix} &= W_{\bar{x}}(t) \\ &= \mathrm{e}^{\bar{x}t}\left\{\begin{bmatrix} Z(\bar{x}) \\ I_q \end{bmatrix} t^{\kappa_{\bar{x}}-1}/(\kappa_{\bar{x}}-1)! + \cdots + \begin{bmatrix} Z^{(j)}(\bar{x}) \\ 0 \end{bmatrix} t^{\kappa_{\bar{x}}-j-1}/(\kappa_{\bar{x}}-j-1)! \right. \\ &\quad \left. + \cdots + \begin{bmatrix} Z^{(\kappa_{\bar{x}}-1)}(\bar{x}) \\ 0 \end{bmatrix}\right\}.\end{aligned}$$

Hence, if we take only function values and no derivatives, we look for all $Z(x)$ such that $Z(\bar{x}) = F_{\bar{x}}$. The input/output data set is

$$W_{\bar{x}}(t) = [w_{1,\bar{x}}(t), \ldots, w_{q,\bar{x}}(t)] = \begin{bmatrix} Y_{\bar{x}}(t) \\ U_{\bar{x}}(t) \end{bmatrix} = \mathrm{e}^{\bar{x}t}\begin{bmatrix} \mathrm{F}_{\bar{x}} \\ \mathrm{I}_q \end{bmatrix}.$$

If we take the polynomial-exponential “time series” $w_{i,\bar{x}}(t)$, $\forall \bar{x} \in Y \cup Z$ as the data set, the rows of a row reduced autoregressive equation representation $\theta^*(x)$ described in [5] form what is called in [24] a **0**-reduced basis for the submodule connected to the matrix rational interpolation problem. Hence, the autoregressive equation representation $\theta^*(x)$ being row reduced with highest degree coefficient I_{p+q} is nothing else but our $T(x)$ matrix. Moreover, the recursive update described in [5, Section 8] is very similar to the update described in [24] and worked out in our algorithm of Section 5.

If we take the input/output data

$$W_{\bar{x}}(t) = e^{\bar{x}t}\begin{bmatrix} I_p \\ F_{\bar{x}}^T \end{bmatrix},$$

we get the row reduced autoregressive equation representation $\tilde{T}^{\mathrm{T}}(x)$.

Note that $\tilde{T}(x)$ and $T^{\mathrm{T}}(x)$ can be seen as $\theta(x)$-matrices playing a central role in [4].

If one also wants to consider pole information for the matrix rational interpolant, this problem can be solved as a no-pole problem when enough interpolation data are known at each pole [22].

In Algorithm 5.1 one new datum is added at each step. In [25] it is described how a new basis matrix can be computed from a previous one adding several data all at once, a so-called "look-ahead" step. Taking more data at each step could be used to enhance the numerical stability of the algorithm (for the scalar rational interpolation problem, see [8]) as was done for Hankel and Toeplitz matrices (see, e.g., [9, 14–16]).

6. Matrix continued fraction representations

If we denote the successive polynomial matrices $W(x)$ appearing in Algorithm 5.1 as $W_1(x), W_2(x), \ldots, W_l(x)$, with $l = p(m+n)$, denote H^{-1} as W_{l+1}, and partition $W_i(x)$ as

$$W_i(x) = \begin{bmatrix} \tilde{Q}_i(x) & \tilde{V}_i(x) \\ \tilde{P}_i(x) & \tilde{U}_i(x) \end{bmatrix},$$

with $\tilde{Q}_i(x)$ $p \times p$, we have the following connection with matrix continued fractions.

Theorem 6.1. *The matrix rational function $\tilde{V}(x)\tilde{U}(x)^{-1}$ of Theorem 3.1 is the $(l+1)$st convergent of the matrix continued fraction*

$$\frac{\tilde{V}_1(x) + \tilde{Q}_1(x)\dfrac{\tilde{V}_2(x) + \tilde{Q}_2(x)\div}{\tilde{U}_2(x) + \tilde{P}_2(x)\div}}{\tilde{U}_1(x) + \tilde{P}_1(x)\dfrac{\tilde{V}_2(x) + \tilde{Q}_2(x)\div}{\tilde{U}_2(x) + \tilde{P}_2(x)\div}}.$$

The matrix rational function $\tilde{P}(x)\tilde{Q}(x)^{-1}$ is the $(l+1)$st convergent of the matrix continued fraction

$$\frac{\tilde{P}_1(x) + \tilde{U}_1(x)\dfrac{\tilde{P}_2(x) + \tilde{U}_2(x)\div}{\tilde{Q}_2(x) + \tilde{V}_2(x)\div}}{\tilde{Q}_1(x) + \tilde{V}_1(x)\dfrac{\tilde{P}_2(x) + \tilde{U}_2(x)\div}{\tilde{Q}_2(x) + \tilde{V}_2(x)\div}}.$$

The notation A/B stands for AB^{-1}. There are similar results for $U(x)^{-1}V(x)$ and $Q(x)^{-1}P(x)$.

The matrix continued fraction is very similar to the one introduced in [2] to decompose matrix formal power series in x^{-1}, i.e., solving the rational interpolation problem around the multiple point ∞ (see also [23]). By changing the variable as described in, e.g., [7, 5], the so-called minimal partial realization problem around ∞ can be changed into a matrix rational interpolation problem around 0. The matrix continued fraction can be interpreted as a cascade interconnection of linear two-port systems (see, e.g., [5, Section 10]).

Before we can prove previous theorem, we need some additional results. We use the notation $W_{i,j}(x)$ for

$$W_{i,j}(x) = W_i(x) W_{i+1}(x) \dots W_j(x), \quad i \leqslant j.$$

$W_{i,j}(x)$ is partitioned similarly to $W_i(x)$ as

$$W_{i,j}(x) = \begin{bmatrix} \tilde{Q}_{i,j}(x) & \tilde{V}_{i,j}(x) \\ \tilde{P}_{i,j}(x) & \tilde{U}_{i,j}(x) \end{bmatrix}.$$

Definition 6.1 (*Column reduced polynomial matrix*). A polynomial matrix $P(x) \in \mathbb{F}[x]^{n \times n}$ is called *column reduced* iff

$$P(x) = (P^* + \mathrm{O}_-(x^{-1}))x^{\vec{\delta}}$$

with $P^* \in \mathbb{F}^{n \times n}$ nonsingular and $x^{\vec{\delta}} = \mathrm{diag}(x^{\delta_1}, x^{\delta_2}, \dots, x^{\delta_n})$, $\vec{\delta} \in \mathbb{N}^n$. The natural number δ_i is called the column degree of the ith column of $P(x)$ and P^* is called the highest degree coefficient (hdc) of $P(x)$. Row reducedness is defined in a similar way.

Lemma 6.2. *The polynomial matrices* $W_{1,i}(x)$, $i = 1, 2, \dots, l+1$, *are column reduced with an upper triangular and nonsingular hdc.*

Proof. This is true for $W_{1,1}(x)$. Suppose it is true for $W_{1,i-1}(x)$. The choice of j in Algorithm 5.1 guarantees that the hdc of the kth column of $W_{1,i}(x)$ is a nonzero multiple of the hdc of the kth column of $W_{1,i-1}(x)$ to which in some cases a nonzero multiple of the hdc of a *previous* column of $W_{1,i-1}(x)$ is added. This proves the lemma. □

Note that the previous lemma implies that H and H^{-1} are nonsingular and upper triangular matrices.

Lemma 6.3. *The polynomial matrices* $\tilde{U}_{i,j}(x)$ *and* $\tilde{Q}_{i,j}(x)$ *are nonsingular,* $i \leqslant j \leqslant l+1$.

Proof. We start from the following equality:

$$W_{1,j}(x) = W_{1,i-1}(x) W_{i,j}(x).$$

Because each $W_k(x)$ is invertible, also $W_{1,i-1}(x)$ is invertible. Hence,

$$W_{i,j}(x) = W_{1,i-1}(x)^{-1} W_{1,j}(x).$$

From

$$W_{1,j}(x) = (A + \mathrm{O}_-(x^{-1}))x^{\vec{\delta}_1}$$

and

$$W_{1,i-1}(x) = (B + O_-(x^{-1}))x^{\vec{\delta}_2}$$

with A and B nonsingular and upper triangular, we get

$$\begin{aligned} W_{i,j}(x) &= x^{-\vec{\delta}_2}(B^{-1} + O_-(x^{-1}))(A + O_-(x^{-1}))x^{\vec{\delta}_1} \\ &= x^{-\vec{\delta}_2}(B^{-1}A + O_-(x^{-1}))x^{\vec{\delta}_1} \end{aligned}$$

with $B^{-1}A$ nonsingular and upper triangular. Hence, the (1, 1) block $\tilde{Q}_{i,j}(x)$ and the (2, 2) block $\tilde{U}_{i,j}(x)$ of $W_{i,j}(x)$ are invertible. □

Now we have all the ingredients to give the proof of Theorem 6.1.

Proof of Theorem 6.1. We give the proof only for $\tilde{V}(x)\tilde{U}(x)^{-1}$. The proof for $\tilde{P}(x)\tilde{Q}(x)^{-1}$ is similar. We rewrite $\tilde{V}(x)\tilde{U}(x)^{-1}$ as follows:

$$\begin{aligned} \tilde{V}(x)\tilde{U}(x)^{-1} &= \tilde{V}_{1,l+1}(x)\tilde{U}_{1,l+1}(x)^{-1} \\ &= \frac{\tilde{Q}_1(x)\tilde{V}_{2,l+1}(x) + \tilde{V}_1(x)\tilde{U}_{2,l+1}(x)}{\tilde{P}_1(x)\tilde{V}_{2,l+1}(x) + \tilde{U}_1(x)\tilde{U}_{2,l+1}(x)}. \end{aligned}$$

Because $\tilde{U}_{2,l+1}(x)$ is invertible, we get

$$\tilde{V}(x)\tilde{U}(x)^{-1} = \frac{\tilde{V}_1(x) + \tilde{Q}_1(x)Z_{2,l+1}(x)}{\tilde{U}_1(x) + \tilde{P}_1(x)Z_{2,l+1}(x)}$$

with $Z_{2,l+1}(x) = \tilde{V}_{2,l+1}(x)\tilde{U}_{2,l+1}(x)^{-1}$. Following the same reasoning for $Z_{2,l+1}(x), Z_{3,l+1}(x), \ldots$ leads us to the matrix continued fraction representation for $\tilde{V}(x)\tilde{U}(x)^{-1}$. □

Note that also each convergent i, $i = 1, 2, \ldots, l$ is well-defined and connected to a matrix rational interpolation problem considering the first i interpolation conditions.

7. Unattainable points

Definition 7.1. Consider the linearized interpolation problem given by (11). Then the interpolation point y_i (or z_j) is called *attainable* iff the matrix $\tilde{U}(y_i)$ (or $\tilde{U}(z_j)$) is nonsingular so that the corresponding interpolation condition can be written as a proper rational interpolation condition

$$\tilde{V}(y_i)\tilde{U}(y_i)^{-1} = C_i.$$

We give a small example showing that the nonsingularity of the (block) Löwner matrix does not necessarily guarantee that for $\tilde{V}(x)\tilde{U}(x)^{-1} = U(x)^{-1}V(x)$ all the interpolation points are attainable.

Take $p = q = 1$, $m = n = 2$ and

$$y_0 = 0, \quad C_0 = 2, \quad y_1 = 1, \quad C_1 = 6,$$
$$z_0 = 2, \quad D_0 = 4, \quad z_1 = 3, \quad D_1 = 3.$$

The Löwner matrix

$$L = \begin{bmatrix} 1 & \frac{1}{3} \\ -2 & -\frac{3}{2} \end{bmatrix}$$

is nonsingular with determinant $-\frac{5}{6}$.

We get $\tilde{U}$ and U as the solution of

$$L\tilde{U} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

and

$$UL = [D_0 \;\; D_1].$$

We derive

$$\tilde{U} = \begin{bmatrix} 6 \\ -12 \end{bmatrix} \quad \text{and} \quad U = [0 \;\; -2].$$

Hence,

$$\tilde{U}(x) = x^2 + x, \qquad \tilde{V}(x) = 12x,$$
$$U(x) = x^2 + x, \qquad V(x) = 12x.$$

The rational function

$$\tilde{V}(x)\tilde{U}(x)^{-1} = U(x)^{-1}V(x) = \frac{12x}{x^2 + x}$$

has an unattainable point for $x = y_0 = 0$.

This is also mirrored in the fact that $U_0 = 0$.

Before we give equivalent conditions for the attainability of the interpolation points, we show the following equality.

Lemma 7.1. *It holds that*

$$\det U(x) = \det \tilde{U}(x).$$

Proof. By elementary block elimination operations, it is easy to show that

$$T(x)^{-1} = \begin{bmatrix} U(x)^{-1} + U(x)^{-1}V(x)\Delta(x)^{-1}P(x) & U(x)^{-1}V(x)\Delta(x)^{-1} \\ \Delta(x)^{-1}P(x) & \Delta(x)^{-1} \end{bmatrix}$$

with $\Delta(x) = Q(x) - P(x)U(x)^{-1}V(x)$. Moreover, $\det T(x) = \det \Delta(x) \det U(x)$. Because $a(x)b(x)T(x)^{-1} = \tilde{T}(x)$, we also have

$$\tilde{U}(x) = a(x)b(x)\Delta(x)^{-1}.$$

Therefore,

$$\begin{aligned}\det T(x) &= \det \Delta(x) \det U(x) \\ &= (a(x)b(x))^q \det U(x) / \det \tilde{U}(x).\end{aligned}$$

Because $\det T(x) = (a(x)b(x))^q$, $\det U(x) = \det \tilde{U}(x)$. □

Corollary 7.1. *The interpolation point y_i (or z_j) is attainable iff* $\det U(y_i) = \det \tilde{U}(y_i) \neq 0$ *(or* $\det U(z_j) = \det \tilde{U}(z_j) \neq 0$).

Theorem 7.2. *If the block Löwner matrix $L = [(C_i - D_j)/(y_i - z_j)]$ is nonsingular, the Smith canonical form of the polynomial matrices*

$$L_1(x) = \begin{bmatrix} & -a(x)I_p \\ & a_0(x)I_p \\ L_{+\text{row}} & \vdots \\ & a_{m-1}(x)I_p \end{bmatrix},$$

where

$$L_{+\text{row}} = \begin{bmatrix} D_0 \ldots D_{n-1} \\ \left[\dfrac{C_i - D_j}{y_i - z_j}\right] \end{bmatrix},$$

resp.,

$$L_2 = \begin{bmatrix} L_{+\text{col}} \\ -b(x)I_q, b_0(x)I_q, \ldots, b_{n-1}(x)I_q \end{bmatrix},$$

where

$$L_{+\text{col}} = \begin{bmatrix} C_0 & \\ \vdots & \left[\dfrac{C_i - D_j}{y_i - z_j}\right] \\ C_{m-1} & \end{bmatrix}$$

is

$$\begin{bmatrix} I_N & 0 \\ 0 & S_U(x) \end{bmatrix},$$

resp.,

$$\begin{bmatrix} I_N & 0 \\ 0 & S_{\tilde{U}}(x) \end{bmatrix},$$

where $S_U(x) \in \mathbb{F}^{p \times p}[x]$, *resp.* $S_{\tilde{U}}(x) \in \mathbb{F}^{q \times q}[x]$ *are the Smith canonical forms of the polynomial matrices* $U(x)$, *resp.* $\tilde{U}(x)$, *and* $N = mp = nq$.

(This result can be compared with a result for Hankel matrices [13, Theorem 2.12]. In a future paper, we shall elaborate on this.)

Proof. Let us prove the assertion for $L_1(x)$. Because L is nonsingular, there is a matrix M_1 of dimension $N \times (N + p)$ such that

$$M_1 L_{+\text{row}} = I_N.$$

Then

$$\begin{bmatrix} M_1 \\ -I_p, U_0, \dots, U_{m-1} \end{bmatrix} \begin{bmatrix} & -\ a(x)I_p \\ & a_0(x)I_p \\ L_{+\text{row}} & \vdots \\ & a_{m-1}(x)I_p \end{bmatrix} = \begin{bmatrix} I_N & P(x) \\ 0 & U(x) \end{bmatrix}$$

for a polynomial matrix $P(x)$. Because $U(x)$ is nonsingular ($\det U(x) = x^N + \cdots \neq 0$), the last matrix on the right is nonsingular. Hence, the (constant) transformation matrix on the left is unimodular. The matrix on the right can be evidently multiplied by a unimodular matrix $R(x)$ to the right to get

$$\begin{bmatrix} I_N & 0 \\ 0 & U(x) \end{bmatrix}.$$

Then, it is easy to come over to

$$\begin{bmatrix} I_N & 0 \\ 0 & S_U(x) \end{bmatrix}$$

by unimodular transformations again. □

In the sequel, we shall need the following corollary.

Corollary 7.2.

$$\det U(x) = \kappa_1 \det \begin{bmatrix} & -\ a(x)I_p \\ & a_0(x)I_p \\ L_{+\text{row}} & \vdots \\ & a_{m-1}(x)I_p \end{bmatrix},$$

$$\det \tilde{U}(x) = \kappa_2 \det \begin{bmatrix} L_{+\mathrm{col}} \\ -b(x)I_q, b_0(x)I_q, \ldots, b_{n-1}(x)I_q \end{bmatrix},$$

where κ_1 and κ_2 are nonzero constants.

The characterization of solvability of the rational interpolation problem is the following.

Theorem 7.3. *Let the block Löwner matrix $L = [(C_i - D_j)/(y_i - z_j)]$ be nonsingular. Then all the interpolation points are attainable iff both matrices $L_{+\mathrm{row}}$ and $L_{+\mathrm{col}}$ (defined in the previous theorem) have all block minors formed of $m \times n$ blocks of dimension $p \times q$ different from zero.*

Proof. Deleting the ith block row from $[(C_i - D_j)/(y_i - z_j)]$, denote the resulting matrix by L_i. By Theorem 7.2,

$$\begin{bmatrix} D_0 \ldots D_{n-1} \\ L_i \end{bmatrix}$$

is nonsingular iff $U(y_i) \neq 0$. An analogous assertion holds for $\tilde{U}(z_j)$. With this fact and with Lemma 7.1, the proof becomes evident. □

Now we show that if $\bar{x}$ is an unattainable point that

$$\lim_{x \to \bar{x}} U(x)^{-1} V(x) = \lim_{x \to \bar{x}} \tilde{V}(x)\tilde{U}(x)^{-1} \neq F_{\bar{x}}.$$

We need the following lemma.

Lemma 7.4. *If $U(\bar{x})$ is singular, then $U(x)$ and $V(x)$ have a left common divisor the determinant of which is a constant multiple of $(x - \bar{x})$. Similarly for $\tilde{V}(x)$ and $\tilde{U}(x)$.*

Proof. If $U(\bar{x})$ is singular, there exists a vector $c \in \mathbb{F}^p$ such that

$$c^{\mathrm{T}} U(x) = (x - \bar{x})u(x), \quad \text{with } u(x) \in \mathbb{F}^{p \times 1}[x].$$

Multiplying

$$[U(x) \quad -V(x)] \begin{bmatrix} F(x) \\ I_q \end{bmatrix} = a(x)b(x)R'(x)$$

to the left by c^{T}, we also get that $c^{\mathrm{T}} V(x) = (x - \bar{x})v(x)$ with $v(x)$ polynomial. If $C \in \mathbb{F}^{p \times p}$ is any nonsingular matrix with its first row equal to c^{T}, then

$$G(x) = C^{-1} \begin{bmatrix} x - \bar{x} & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

is a common left divisor of $U(x)$ and $V(x)$ and $\det G(x) = (x - \bar{x})(\det C)^{-1} \in \mathbb{F}[x]$. □

Theorem 7.5. *If $U(\bar{x})$ is singular, then*

$$\lim_{x \to \bar{x}} U(x)^{-1} V(x) \neq F_{\bar{x}}.$$

Moreover, for any common left divisor $G(x)$ of $U(x)$ and $V(x)$ with $\det G(\bar{x}) = 0$, after deleting this divisor even the linearized interpolation condition in $\bar{x}$ is not satisfied, i.e.

$$U'(\bar{x}) F_{\bar{x}} - V'(\bar{x}) \neq 0$$

with $U(x) = G(x)U'(x)$ and $V(x) = G(x)V'(x)$. Similarly, for $\tilde{V}(x)$ and $\tilde{U}(x)$.

Proof. If $U(\bar{x})$ is singular, we know from the previous lemma that there is at least one common left divisor $G(x)$ of $U(x)$ and $V(x)$ such that $\bar{x}$ is a zero of $\det G(x)$. Take such a $G(x)$ with

$$\det G(x) = (x - \bar{x})^{\delta} p(x), \quad \text{with } p(\bar{x}) \neq 0 \text{ and } \delta > 0.$$

Defining the polynomial matrices $U'(x)$ and $V'(x)$ by

$$U(x) = G(x)U'(x), \qquad V(x) = G(x)V'(x),$$

we can write

$$\begin{bmatrix} U'(x) & -V'(x) \\ -P(x) & Q(x) \end{bmatrix} \begin{bmatrix} a(x)b(x)I_p & F(x) \\ 0 & I_q \end{bmatrix} = a(x)b(x) \begin{bmatrix} G(x)^{-1} & 0 \\ 0 & I_q \end{bmatrix}. \tag{26}$$

We assume now that $U'(\bar{x})F_{\bar{x}} - V'(\bar{x}) = 0$ or

$$\begin{bmatrix} U'(x) & -V'(x) \\ -P(x) & Q(x) \end{bmatrix} \begin{bmatrix} a(x)b(x)I_p & F(x) \\ 0 & I_q \end{bmatrix} = (x - \bar{x})R'(x) \tag{27}$$

with $R'(x) \in \mathbb{F}[x]^{(p+q)\times(p+q)}$. Looking at the factor $(x - \bar{x})$ in the determinant of the right-hand sides of (26) and (27), we get

$$(x - \bar{x})^{(p+q)-\delta} = (x - \bar{x})^{(p+q)+\kappa}$$

with $\det R'(x) = (x - \bar{x})^{\kappa} p'(x)$ where $\kappa \geqslant 0$ is the multiplicity of the root $\bar{x}$ in $\det R'(x)$. Hence, $\delta = -\kappa \leqslant 0$. Therefore, our assumption cannot be true or

$$U'(\bar{x})F_{\bar{x}} - V'(\bar{x}) \neq 0.$$

In the sequel, take for $G(x)$ a greatest common left divisor of $U(x)$ and $V(x)$. Hence, $U'(x)$ and $V'(x)$ are left coprime. Also, $\bar{x}$ is a zero of $\det G(x)$. There are two possibilities.

- $U'(\bar{x})$ is nonsingular. Hence,

$$\lim_{x \to \bar{x}} U(x)^{-1} V(x) = U'(\bar{x})^{-1} V'(\bar{x}) \neq F_{\bar{x}}.$$

- $U'(\bar{x})$ is singular. Hence, the matrix rational function $U'(x)^{-1}V'(x)$ has a pole in $\bar{x}$. Therefore,

$$\lim_{x \to \bar{x}} U(x)^{-1} V(x) = \lim_{x \to \bar{x}} U'(\bar{x})^{-1} V'(\bar{x}) \neq F_{\bar{x}}.$$

This proves the theorem. □

The next theorem shows that the zeros of the determinant of a common left divisor of $U(x)$ and $V(x)$ can only be interpolation points.

Theorem 7.6. *The determinant of a common left divisor of $U(x)$ and $V(x)$ divides $(a(x)b(x))^q$. Similarly, the determinant of a common right divisor of $\tilde{V}(x)$ and $\tilde{U}(x)$ divides $(a(x)b(x))^p$.*

Proof. If $G(x)$ is a common left divisor of $U(x)$ and $V(x)$, we can rewrite $T(x)$ as

$$T(x) = \begin{bmatrix} G(x) & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} U'(x) & -V'(x) \\ -P(x) & Q(x) \end{bmatrix}$$

with $U(x) = G(x)U'(x)$ and $V(x) = G(x)V'(x)$. Because $\det T(x) = (a(x)b(x))^q$, $\det G(x)$ is a divisor of $(a(x)b(x))^q$. □

Using the last two theorems, we get the following corollary.

Corollary 7.3. *If $U(x)$ and $V(x)$ are left coprime, there are no unattainable points. If $U(x)$ and $V(x)$ are not left coprime, with $G(x)$ a greatest common left divisor, the zeros of the determinant of $G(x)$ are the unattainable points.*

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