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Weak Property (Y_0) and Regularity of Inductive Limits

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An inductive limit $(E, t) = \text{ind}(E_n, t_n)$ is regular if and only if it satisfies the weak property (Y_0); i.e., each weakly unconditionally Cauchy series in (E, t) is contained and is a weakly unconditionally Cauchy series in some (E_n, t_n). In particular, an (LF)-space $(E, t) = \text{ind}(E_n, t_n)$ is regular if and only if every weakly unconditionally Cauchy series $\sum_k x_k$ is a C-series; i.e., for any scalar sequence $(\xi_k) \in c_0$, the series $\sum_k \xi_k x_k$ is convergent. Furthermore, for inductive limits of Fréchet spaces containing no copy of c_0, a number of characteristic conditions of regularity are given.

Key Words: inductive limit; (LF)-space; regularity; weak property (Y_0); C-series.

1. INTRODUCTION

Let $(E_n, t_n)_{n \in \mathrm{N}}$ be an inductive sequence of locally convex spaces; i.e., (E_n, t_n) is continuously included in (E_{n+1}, t_{n+1}) for all $n \in \mathrm{N}$. Let $E := \bigcup_{n \in \mathrm{N}} E_n$ and let (E, t) be endowed with the finest locally convex topology such that the injections from (E_n, t_n) into E are continuous. We call (E, t) the inductive limit of $(E_n, t_n)_{n \in \mathrm{N}}$ and denote (E, t) by $\text{ind}(E_n, t_n)$ (e.g., see [3, 8, 10, 15, 23]). We always assume that each (E_n, t_n) is Hausdorff and that the inductive limit $(E, t) = \text{ind}(E_n, t_n)$ is also Hausdorff. If each (E_n, t_n) is a Fréchet space (i.e., a complete metrizable locally convex space), then $(E, t) = \text{ind}(E_n, t_n)$ is called an (LF)-space. An inductive limit $(E, t) = \text{ind}(E_n, t_n)$ is said to be regular if each bounded set in (E, t) is contained and bounded in some (E_n, t_n). The Dieudonné–Schwartz Theorem (e.g., see [8, Chapter 2, Section 12]) states that if $(E, t) = \text{ind}(E_n, t_n)$ is a strict inductive limit and each E_n is closed in (E_{n+1}, t_{n+1}), then (E, t) is regular. Korobeinik [9] introduced the following property (Y_0), which is related to regularity. An inductive sequence $(E_n, t_n)_{n \in \mathrm{N}}$ or its inductive limit $(E, t) = \text{ind}(E_n, t_n)$ is said to satisfy property (Y_0) if each absolutely

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summable sequence $(x_k)_{k \in \mathrm{N}}$ in (E, t) is contained and absolutely summable in some (E_n, t_n). Here, a sequence $(x_k)_{k \in \mathrm{N}}$ in (E, t) is said to be absolutely summable if for each continuous seminorm ρ on (E, t), $\sum_{k=1}^{\infty} \rho(x_k) < \infty$. Fernandez and Melikhov investigated the relationship between property (Y_0) and regularity. If all (E_n, t_n) are normed spaces, Melikhov [13] proved that $(E, t) = \mathrm{ind}(E_n, t_n)$ has property (Y_0) if and only if it is regular. In general, for inductive limits $(E, t) = \mathrm{ind}(E_n, t_n)$ of locally convex spaces, they proved that "(E, t) has property (Y_0)" implies "(E, t) is regular," but the converse is not true (see [7, 13, 14]). In this paper we introduce weak property (Y_0) and prove that weak property (Y_0) is equivalent to regularity. If all (E_n, t_n) are sequentially complete strictly webbed spaces (in particular, if all (E_n, t_n) are Fréchet spaces), we show that $(E, t) = \mathrm{ind}(E_n, t_n)$ is regular if and only if (E, t) satisfies the following completeness property: every weakly unconditionally Cauchy (briefly denoted by w.u.c.) series $\sum_{k=1}^{\infty} x_k$ is a C-series; i.e., for any scalar sequence $(\xi_k) \in c_0$, $\sum_{k=1}^{\infty} \xi_k x_k$ is convergent (for w.u.c. series and C-series, refer to [1, p. 92–94] or [20, Chapter 3]). Moreover, if all (E_n, t_n) are Fréchet spaces containing no copy of c_0, then $(E, t) = \mathrm{ind}(E_n, t_n)$ is regular if and only if every w.u.c. series in (E, t) is unconditionally convergent.

2. WEAK PROPERTY (Y_0) AND REGULARITY

Let X be a locally convex space and let X' be its topological dual. A series $\sum_{k=1}^{\infty} x_k$ in X is said to be weakly unconditionally Cauchy (briefly, w.u.c.) if $\sum_{k=1}^{\infty} |\langle f, x_k \rangle| < \infty$ for every $f \in X'$ (see [1, p. 94; 2; 6, p. 44; 12, p. 98; 20] and others).

Definition 1. An inductive limit $(E, t) = \mathrm{ind}(E_n, t_n)$ is said to satisfy weak property (Y_0) if for each w.u.c. series $\sum_{k=1}^{\infty} x_k$ in (E, t), there exists $m \in \mathrm{N}$ such that $(x_k)_{k \in \mathrm{N}} \subset E_m$ and $\sum_{k=1}^{\infty} x_k$ is w.u.c. in (E_m, t_m).

Theorem 1. *Let $(E, t) = \mathrm{ind}(E_n, t_n)$ be an inductive limit of locally convex spaces. Then the following statements are equivalent*:

(i) *(E, t) is regular*.

(ii) *(E, t) satisfies weak property (Y_0)*.

Proof. (i) ⇒ (ii). Let $\sum_{k=1}^{\infty} x_k$ be a w.u.c. series in (E, t); i.e., $\sum_{k=1}^{\infty} |\langle f, x_k \rangle| < \infty$ for every $f \in E' := (E, t)'$. It is easy to see that the set

$$\left\{ \sum_{k \in \sigma} \theta_k x_k : \sigma \subset \mathrm{N} \text{ is finite}, |\theta_k| = 1 \text{ for every } k \in \sigma \right\}$$

is bounded in (E, t). (For the case of Banach spaces, refer to [6, p. 44] and [1, pp. 94–95]. Here the proof is similar.) For the sake of completeness, the proof follows. Since bounded sets and weakly bounded sets are the same in (E, t), assume to the contrary that there is $f \in E'$ such that the scalar set

$$\left\{\left\langle f, \sum_{k \in \sigma} \theta_k x_k \right\rangle : \sigma \subset \mathrm{N} \text{ is finite}, |\theta_k| = 1 \text{ for every } k \in \sigma \right\}$$

is unbounded. Thus, for each $n \in \mathrm{N}$, there is a finite subset σ_n of N and $|\theta_k| = 1$ for every $k \in \sigma_n$ such that $|\langle f, \sum_{k \in \sigma_n} \theta_k x_k \rangle| \geq n$, which implies that $\sum_{k \in \sigma_n} |\langle f, x_k \rangle| \geq n$. This contradicts the assumption that $\sum_{k=1}^{\infty} |\langle f, x_k \rangle| < \infty$.

By (i), there is $m \in \mathrm{N}$ such that the set

$$\left\{ \sum_{k \in \sigma} \theta_k x_k : \sigma \subset \mathrm{N} \text{ is finite}, |\theta_k| = 1 \text{ for every } k \in \sigma \right\}$$

is contained and bounded in (E_m, t_m). From this we easily see that $(x_k) \subset E_m$, and we assert that $\sum_{k=1}^{\infty} |\langle f_m, x_k \rangle| < \infty$ for every $f_m \in (E_m, t_m)'$. If not, there exists $f_m \in (E_m, t_m)'$ such that $\sum_{k=1}^{\infty} |\langle f_m, x_k \rangle| = \infty$. For each $k \in \mathrm{N}$, take a scalar θ_k such that $|\theta_k| = 1$ and $\theta_k \langle f_m, x_k \rangle = |\langle f_m, x_k \rangle|$. Thus, $\langle f_m, \sum_{k=1}^{i} \theta_k x_k \rangle = \sum_{k=1}^{i} \langle f_m, \theta_k x_k \rangle = \sum_{k=1}^{i} |\langle f_m, x_k \rangle| \to_i \infty$, which contradicts the above conclusion that the set

$$\left\{ \sum_{k \in \sigma} \theta_k x_k : \sigma \subset \mathrm{N} \text{ is finite and } |\theta_k| = 1 \text{ for every } k \in \sigma \right\}$$

is bounded in (E_m, t_m). Thus, we have proved that $(x_k) \subset E_m$ and $\sum_{k=1}^{\infty} x_k$ is w.u.c. in (E_m, t_m); that is to say, (E, t) satisfies weak property (Y_0).

(ii) $\Rightarrow$ (i). It suffices to prove that each bounded sequence $(x_k)_{k \in \mathrm{N}}$ in (E, t) is contained and bounded in some (E_m, t_m). Take a fixed scalar sequence $(\lambda_k) \in l^1$ such that every $\lambda_k \neq 0$. Since $(x_k)_{k \in \mathrm{N}}$ is bounded in (E, t), $\sum_{k=1}^{\infty} |\langle f, \lambda_k x_k \rangle| < \infty$ for every $f \in E'$; i.e., the series $\sum_{k=1}^{\infty} \lambda_k x_k$ is w.u.c. in (E, t). By (ii), there exists $m \in \mathrm{N}$ such that $(\lambda_k x_k)_{k \in \mathrm{N}} \subset E_m$ and $\sum_{k=1}^{\infty} \lambda_k x_k$ is w.u.c. in (E_m, t_m). First, we remark that $(x_k)_{k \in \mathrm{N}} \subset E_m$ since $(\lambda_k x_k)_{k \in \mathrm{N}} \subset E_m$ and every $\lambda_k \neq 0$. Without loss of generality, we may assume that $(x_k)_{k \in \mathrm{N}} \subset E_1$. We are going to prove that there exists $m \in \mathrm{N}$ such that $(x_k)_{k \in \mathrm{N}}$ is bounded in (E_m, t_m). Assume to the contrary that for any $n \in \mathrm{N}$, there is an $f_n \in (E_n, t_n)'$ such that the scalar set $\{\langle f_n, x_k \rangle : k \in \mathrm{N}\}$ is unbounded. Thus, we obtain a sequence of subsequences of $(x_k)_{k \in \mathrm{N}}$ as follows,

$$x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots \text{ such that } \left|\langle f_1, x_k^{(1)} \rangle\right| \geq (1 + k)^4;$$

$$x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots \text{ such that } \left|\langle f_2, x_k^{(2)} \rangle\right| \geq (2 + k)^4;$$

$$x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \dots \text{ such that } \left|\langle f_3, x_k^{(3)} \rangle\right| \geq (3 + k)^4;$$

$$\cdots.$$

In general, $|\langle f_n, x_k^{(n)}\rangle| \geq (n+k)^4$, for $k = 1, 2, \dots$ and $n = 1, 2, \dots$. By the diagonal procedure, we put $z_1 = x_1^{(1)}$, $z_2 = x_2^{(1)}$, $z_3 = x_1^{(2)}$, $z_4 = x_3^{(1)}$, $z_5 = x_2^{(2)}$, $z_6 = x_1^{(3)}, \dots$. For $z_j = x_k^{(n)}$, let $\lambda_j := n + k$; then $j \to \infty$ if and only if $\lambda_j \to_j \infty$. Indeed, it is easy to verify that $j \geq \lambda_j = n + k$ for any $j \geq 3$. And for $1 + 2 + \cdots + l = (1/2)l(l+1) < j \leq 1 + 2 + \cdots + l + (l+1) = (1/2)(l+1)(l+2)$, one has $\lambda_j = l + 2$, hence $(1/2)\lambda_j(\lambda_j - 1) = (1/2)(l+1)(l+2) \geq j$. Thus, $\lambda_j^4 \geq (1/4)\lambda_j^2(\lambda_j - 1)^2 \geq j^2$ for any $j \in \mathrm{N}$. On the one hand, since $0 < 1/\lambda_j^4 \leq 1/j^2$ and $(z_j)_{j \in \mathrm{N}}$ is a bounded sequence in (E, t), we have

$$\sum_{j=1}^{\infty} \left| \left\langle f, \frac{1}{\lambda_j^4} z_j \right\rangle \right| \leq \left(\sup_{j \in \mathrm{N}} |\langle f, z_j \rangle| \right) \left(\sum_{j=1}^{\infty} \frac{1}{\lambda_j^4} \right)$$

$$\leq \left(\sup_{j \in \mathrm{N}} |\langle f, z_j \rangle| \right) \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) < \infty,$$

for every $f \in E'$. This means that the series $\sum_{j=1}^{\infty}(1/\lambda_j^4) z_j$ is w.u.c. in (E, t). On the other hand, for each $n \in \mathrm{N}$, there is $f_n \in (E_n, t_n)'$ such that $|\langle f_n, x_k^{(n)}\rangle| \geq (n+k)^4$, $k = 1, 2, \dots$; or

$$\left| \left\langle f_n, \frac{1}{\lambda_j^4} z_j \right\rangle \right| = \left| \left\langle f_n, \frac{1}{(n+k)^4} x_k^{(n)} \right\rangle \right| \geq 1,$$

which implies that the series $\sum_{j=1}^{\infty}(1/\lambda_j^4) z_j$ is not w.u.c. in any (E_n, t_n). This contradicts the assumption that (ii) holds. ∎

Following Rolewicz, a series $\sum_{k=1}^{\infty} x_k$ in a topological vector space X is said to be a C-series (or c_0-multiplier convergent) if for any scalar sequence $(\xi_k)_{k \in \mathrm{N}} \in c_0$, the series $\sum_{k=1}^{\infty} \xi_k x_k$ converges in X (see [1, p. 92] or [20, Chapter 3]). If all (E_n, t_n) are sequentially complete, we have the following.

THEOREM 2. *Let $(E, t) := \mathrm{ind}(E_n, t_n)$ and let every (E_n, t_n) be sequentially complete. Then the following statements are equivalent*:

(i) *(E, t) is regular.*

(ii) *(E, t) satisfies weak property (Y_0).*

(iii) *For any w.u.c. series $\sum_{k=1}^{\infty} x_k$ in (E, t), there exists $m \in \mathrm{N}$ such that $(x_k)_{k \in \mathrm{N}} \subset E_m$ and $\sum_{k=1}^{\infty} x_k$ is a C-series in (E_m, t_m).*

Proof. (i) ⇔ (ii). See Theorem 1.

(ii) ⇒ (iii). Let $\sum_{k=1}^{\infty} x_k$ be a w.u.c. series in (E, t) and $(\xi_k)_{k \in \mathrm{N}} \in c_0$. By (ii), there exists $m \in \mathrm{N}$ such that $(x_k)_{k \in \mathrm{N}} \subset E_m$ and $\sum_{k=1}^{\infty} x_k$ is a w.u.c. series in (E_m, t_m). From this, it follows that the set

$$\left\{ \sum_{k \in \sigma} \theta_k x_k,\ \sigma \subset \mathrm{N} \text{ is finite and } |\theta_k\| = 1 \text{ for every } k \in \sigma \right\}$$

is bounded in (E_m, t_m), as mentioned in the proof of Theorem 1 (cf. [1, pp. 94–95] and [6, p. 44]). Hence, for any closed absolutely convex 0-neighborhood U_m in (E_m, t_m), there is $\lambda > 0$ such that

$$\left\{ \sum_{k \in \sigma} \theta_k x_k : \sigma \subset \mathrm{N} \text{ is finite and } |\theta_k| = 1 \text{ for every } k \in \sigma \right\} \subset \lambda U_m.$$

For each $f_m \in U_m^0 \subset E_m'$,

$$\begin{aligned} \left| \left\langle f_m, \sum_{k=i}^{j} \xi_k x_k \right\rangle \right| &= \left| \sum_{k=i}^{j} \xi_k \langle f_m, x_k \rangle \right| \\ &\le \left(\sup_{i \le k \le j} |\xi_k| \right) \left(\sum_{k=i}^{j} |\langle f_m, x_k \rangle| \right) \\ &= \left(\sup_{i \le k \le j} |\xi_k| \right) \left(\sum_{k=i}^{j} \theta_k \langle f_m, x_k \rangle \right) \\ &= \left(\sup_{i \le k \le j} |\xi_k| \right) \left\langle f_m, \sum_{k=i}^{j} \theta_k x_k \right\rangle \\ &\le \left(\sup_{i \le k \le j} |\xi_k| \right) \lambda \le 1, \end{aligned}$$

provided that i is large enough, since $(\xi_k) \in c_0$. Here, θ_k is a scalar such that $|\theta_k| = 1$ and $\theta_k \langle f_m, x_k \rangle = |\langle f_m, x_k \rangle|$. Thus, $\sum_{k=i}^{j} \xi_k x_k \in U_m^{00} = U_m$ for large enough i. That is, $(\sum_{k=1}^{i} \xi_k x_k)_{i \in \mathrm{N}}$ is a Cauchy sequence in (E_m, t_m). Since (E_m, t_m) is sequentially complete, $\sum_{k=1}^{\infty} \xi_k x_k$ is convergent in (E_m, t_m), or $\sum_{k=1}^{\infty} x_k$ is a C-series in (E_m, t_m).

(iii) ⇒ (ii). Let $\sum_{k=1}^{\infty} x_k$ be a w.u.c. series in (E, t). By (iii), there exists $m \in \mathrm{N}$ such that $(x_k)_{k \in \mathrm{N}} \subset E_m$ and $\sum_{k=1}^{\infty} x_k$ is a C-series in (E_m, t_m). Hence, for any $(\xi_k) \in c_0$, the series $\sum_{k=1}^{\infty} \xi_k x_k$ is convergent in (E_m, t_m). Thus, for each $f_m \in E_m'$, the scalar series $\langle f_m, \sum_{k=1}^{\infty} \xi_k x_k \rangle = \sum_{k=1}^{\infty} \xi_k \langle f_m, x_k \rangle$ is convergent for any $(\xi_k) \in c_0$. By the uniform boundedness principle, $(\langle f_m, x_k \rangle)_{k \in \mathrm{N}} \in l^1$; i.e., $\sum_{k=1}^{\infty} |\langle f_m, x_k \rangle|$ is convergent. This means that $\sum_{k=1}^{\infty} x_k$ is w.u.c. in (E_m, t_m) and hence (E, t) satisfies weak property (Y_0).

Recall that a series $\sum_{k=1}^{\infty} x_k$ in a topological vector space is said to be bounded multiplier convergent if for each bounded sequence $(\xi_k)_{k \in \mathrm{N}}$ of scalars, the series $\sum_{k=1}^{\infty} \xi_k x_k$ is convergent (for example, see [1, p. 77]). Refs. [1, 2, 6, 11, 12, 20, 22] contain entertaining discussions on bounded multiplier convergent series. In a Banach space, a series $\sum_{k=1}^{\infty} x_k$ is bounded multiplier convergent if and only if it is subseries convergent if and only if it is unconditionally convergent (see [12, p. 15; 6, p. 29]). It is easy to see that this equivalence still holds for sequentially complete locally convex spaces (see [11; 22; 23, Problem 15-1-113]). In general, a C-series need not be convergent. But for sequentially complete locally convex spaces, the property that every C-series is convergent (or bounded multiplier convergent) characterizes those spaces which contain no copy of c_0. Many authors investigated the characterization of Banach spaces (and locally convex spaces) containing no copy of c_0; for example, Antosik and Swartz [1], Bessaga and Pelczynski [2], Diestel [6], Li and Bu [11], Lindenstrauss and Tzafriri [12], Rolewicz [20], and others. Combining Theorem 2 and [11, Theorem 4], we immediately obtain the following.

COROLLARY 1. *Let every (E_n, t_n) be sequentially complete and contain no copy of $(c_0, \| \|_\infty)$. The following statements are equivalent*:

(i) $(E, t) = \mathrm{ind}(E_n, t_n)$ *is regular*.

(ii) (E, t) *satisfies weak property* (Y_0).

(iii) *For any w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t), *there exists* $m \in \mathrm{N}$ *such that* $(x_k) \subset E_m$ *and* $\sum_{k=1}^{\infty} x_k$ *is a* C-*series in* (E_m, t_m).

(iv) *For any w.u.c. in series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t), *there exists* $m \in \mathrm{N}$ *such that* $(x_k) \subset E_m$ *and* $\sum_{k=1}^{\infty} x_k$ *is unconditionally convergent* (*equivalently, bounded multiplier convergent, or subseries convergent*) *in* (E_m, t_m).

In [11, Theorem 5], it is stated that a weakly sequentially complete locally convex space contains no copy of $(c_0, \| \|_\infty)$. Besides, every weakly sequentially complete locally convex space is sequentially complete. By Corollary 1, we have:

COROLLARY 2. *Let every* (E_n, t_n) *be weakly sequentially complete. Then for* $(E, t) = \mathrm{ind}(E_n, t_n)$, *the relationships in Corollary* 1 *remain equivalent*.

3. ON REGULARITY OF INDUCTIVE LIMITS OF STRICTLY WEBBED SPACES

In this section, we investigate the regularity of inductive limits of strictly webbed spaces. For the notion of strictly webbed spaces, refer to [10, Section 35]. We begin this section with the following basic observation.

LEMMA 1. *Let $(E, t) = \operatorname{ind}(E_n, t_n)$ be an inductive limit of strictly webbed spaces. Then (E, t) is regular if it is locally complete.*

Proof. Let A be a bounded set in (E, t). Since (E, t) is locally complete, there exists a Banach disk B such that $A \subset B$ (see [15, Proposition 5.1.6]). Here a subset B of (E, t) is called a Banach disk if B is absolutely convex bounded and the span$[B]$ with the gauge ρ_B of B is a Banach space. We denote the Banach space (span$[B], \rho_B$) by E_B. Since each (E_p, t_p) is a strictly webbed space, we may assume that (E_p, t_p) has a strict web $\{C^{(p)}_{n_1, \dots, n_k}\}$ for any $p \in \mathrm{N}$. Put $D_{n_1} = E_{n_1}$ and $D_{n_1, \dots, n_k} = C^{(n_1)}_{n_2, \dots, n_k}$ for all natural numbers $n_1, n_2, \dots$. Then $\{D_{n_1, \dots, n_k}\}$ is a strict web in (E, t); see [10, II, Section 35, 4(8)]. Consider the inclusion map: $E_B \to (E, t)$. Obviously, it is continuous. By the localization theorem for strictly webbed spaces (see [10, II, Section 35, 6(1)(2)]), there exists a sequence n_k and a sequence $\alpha_k > 0$ such that $B \subset \alpha_k D_{n_1, \dots, n_k} = \alpha_k C^{(n_1)}_{n_2, \dots, n_k} \subset E_{n_1}$ for every $k \in \mathrm{N}$. In the strictly webbed space (E_{n_1}, t_{n_1}), if $\rho_{k-1} > 0$ are the numbers corresponding to the sequence $C^{(n_1)}_{n_2, \dots, n_k}$, $k = 2, 3, \dots$, and if U_{n_1} is a neighborhood of 0 in (E_{n_1}, t_{n_1}), then there exists k_0 such that $\rho_{k_0 - 1} C^{(n_1)}_{n_2, \dots, n_{k_0}} \subset U_{n_1}$ (see [10, II, Section 35, 1(3)]). Thus, $B \subset \alpha_{k_0} C^{(n_1)}_{n_2, \dots, n_{k_0}} \subset (\alpha_{k_0}/\rho_{k_0 - 1}) U_{n_1}$. That is to say, B is contained and bounded in (E_{n_1}, t_{n_1}) and so is A. ∎

Concerning local completeness, we would like to mention some recent progress. Saxon and Sánchez Ruiz [21] proved that a space E is locally complete if and only if it is l^1-complete. Here, a space E is said to be l^1-complete if for each bounded sequence $(x_k) \subset E$ and each $(\lambda_k) \in l^1$, the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in E. This concept can be extended to l^q-completeness $(1 \le q \le \infty)$. If $1 \le p \le \infty$, let $l^p(E)$ denote the set of all sequences (x_k) in E such that $(\rho(x_k)) \in l^p$ for each continuous seminorm ρ on E (refer to [15, Definitions 4.8.1 and 4.8.2]). Thus, for example, $l^\infty(E)$ denotes all bounded sequences in E. A space E is said to be l^q-complete $(1 \le q \le \infty)$, if for each $(\lambda_k) \in l^q$ and each $(x_k) \in l^p(E)$, the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in E, where $(1/p) + (1/q) = 1$. We [18] proved further that a space E is locally complete if and only if it is l^q-complete $(1 < q < \infty)$. Combining these results and Lemma 1, we can obtain the following.

THEOREM 3. *Let* $(E, t) = \operatorname{ind}(E_n, t_n)$ *and let every* (E_n, t_n) *be a sequentially complete strictly webbed space. Then the following statements are equivalent*:

(i) $(E, t) = \operatorname{ind}(E_n, t_n)$ *is regular.*

(ii) (E, t) *satisfies weak property* (Y_0).

(iii) *For each w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t), *there exists* $m \in \mathrm{N}$ *such that* $(x_k) \subset E_m$ *and* $\sum_{k=1}^{\infty} x_k$ *is a* C*-series in* (E_m, t_m).

(iv) *Every w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t) *is a* C*-series in* (E, t).

(v) (E, t) *is locally complete.*

(vi) (E, t) *is* l^q*-complete* $(1 \leq q < \infty)$.

Proof. By Theorem 2, we have: (i) ⇔ (ii) ⇔ (iii). Since the topology t_m is finer than the topology on E_m induced by t, the implication (iii) ⇒ (iv) is obvious.

(iv) ⇒ (v). Let (x_k) be a locally null sequence in (E, t). There is an increasing unbounded sequence (μ_k) of positive real numbers such that $(\mu_k x_k)_{k \in \mathrm{N}}$ converges to 0 in (E, t); see [15, Proposition 5.1.3]. For any $(\lambda_k) \in l^1$ with $\sum_{k=1}^{\infty} |\lambda_k| \leq 1$, clearly the series $\sum_{k=1}^{\infty} \lambda_k \mu_k x_k$ is w.u.c. in (E, t). By (iv), the series $\sum_{k=1}^{\infty} \lambda_k \mu_k x_k$ is a C-series in (E, t); hence, $\sum_{k=1}^{\infty} (1/\mu_k)(\lambda_k \mu_k x_k) = \sum_{k=1}^{\infty} \lambda_k x_k$ converges in (E, t). By [4, Proposition III.1.4], the closed absolutely convex hull of the set $\{x_k \colon k \in \mathrm{N}\}$ is exactly the set $\{\sum_{k=1}^{\infty} \lambda_k x_k \colon \sum_{k=1}^{\infty} |\lambda_k| \leq 1\}$ and the latter is compact. Thus we have proved that the closed absolutely convex hull of every locally null sequence in (E, t) is compact. By Dierolf's characterization of local completeness (see [5] or [15, Theorem 5.1.11]), (E, t) is locally complete.

(v) ⇔ (vi). See [18] and [21].

(v) ⇒ (i). It follows from Lemma 1. ∎

Since Fréchet spaces are sequentially complete strictly webbed spaces, in particular we have:

THEOREM 4. *For an* (LF)*-space* $(E, t) = \operatorname{ind}(E_n, t_n)$, *all the statements in Theorem* 3 *are equivalent.*

Moreover, if all (E_n, t_n) are Fréchet spaces containing no copy of c_0, we have an even stronger result.

THEOREM 5. *Let* $(E, t) := \operatorname{ind}(E_n, t_n)$ *and let every* (E_n, t_n) *be a Fréchet space containing no copy of* $(c_0, \| \, \|_\infty)$. *Then the following statements are equivalent*:

(i) (E, t) *is regular.*

(ii) (E, t) *satisfies weak property* (Y_0).

(iii) *For any w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t), *there exists* $m \in \mathrm{N}$ *such that* $(x_k) \subset E_m$ *and* $\sum_{k=1}^{\infty} x_k$ *is a* C-*series in* (E_m, t_m).

(iv) *For any w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t), *there exists* $m \in \mathrm{N}$ *such that* $(x_k) \subset E_m$ *and* $\sum_{k=1}^{\infty} x_k$ *is unconditionally convergent* (*equivalently, bounded multiplier convergent, or subseries convergent*) *in* (E_m, t_m).

(v) *Every w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *in* (E, t) *is unconditionally convergent* (*or bounded multiplier convergent, or subseries convergent*) *in* (E, t).

(vi) $(E, \sigma(E, E'))$ *is* l^{∞}-*complete*; *i.e., every w.u.c. series* $\sum_{k=1}^{\infty} x_k$ *is convergent in* $(E, \sigma(E, E'))$.

(vii) (E, t) *is* Σ-*complete* (*see* [15, *p.* 164]); *i.e., every unconditionally Cauchy series* $\sum_{k=1}^{\infty}$ *is convergent*.

(viii) (E, t) *is* l^{∞}-*complete*; *i.e., every series* $\sum_{k=1}^{\infty} x_k$ *satisfying* $\sum_{k=1}^{\infty} \rho(x_k) < \infty$ *for each continuous seminorm* ρ *on* E *is convergent in* (E, t).

(ix) (E, t) *is* l^q-*complete* $(1 \leq q < \infty)$.

(x) (E, t) *is locally complete*.

Proof. By Corollary 1, we know that (i) ⇔ (ii) ⇔ (iii) ⇔ (iv). The implications (iv) ⇒ (v) ⇒ (vi) are obvious.

(v) ⇒ (vii). Let $\sum_{k=1}^{\infty} x_k$ be an unconditionally Cauchy series in (E, t). Certainly, $\sum_{k=1}^{\infty} x_k$ is an unconditionally Cauchy series in $(E, \sigma(E, E'))$; i.e., $\sum_{k=1}^{\infty} x_k$ is w.u.c. in (E, t). By (v), the series $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent.

(vii) ⇒ (viii). Let the series $\sum_{k=1}^{\infty} x_k$ satisfy $\sum_{k=1}^{\infty} \rho(x_k) < \infty$ for each continuous seminorm ρ on (E, t). Then clearly, $\sum_{k=1}^{\infty} x_k$ is an unconditionally Cauchy series. By (vii), $\sum_{k=1}^{\infty} x_k$ is convergent in (E, t).

(viii) ⇒ (ix). For each $(\lambda_k) \in l^q$ $(1 \leq q < \infty)$ and each $(x_k) \in l^p(E)$,

$$\sum_{k=1}^{\infty} \rho(\lambda_k x_k) = \sum_{k=1}^{\infty} |\lambda_k| \rho(x_k) \leq \left(\sum_{k=1}^{\infty} |\lambda_k|^q \right)^{1/q} \left(\sum_{k=1}^{\infty} (\rho(x_k))^p \right)^{1/p} < \infty,$$

where ρ is any continuous seminorm on (E, t). By (viii), $\sum_{k=1}^{\infty} \lambda_k x_k$ is convergent in (E, t). That is to say, (E, t) is l^q-complete $(1 \leq q < \infty)$.

Similarly, we can deduce that "$(E, \sigma(E, E'))$ is l^{∞}-complete" implies "$(E, \sigma(E, E'))$ is l^q-complete $(1 \leq q < \infty)$." As mentioned above, l^q-completeness $(1 \leq q < \infty)$ is equivalent to local completeness. And these two properties are duality invariant. Thus we have (vi) ⇒ (ix) ⇔ (x). Finally, by Lemma 1, we have (x) ⇒ (i). ∎

COROLLARY 3. *If all* (E_n, t_n) *are weakly sequentially complete Fréchet spaces, then for* $(E, t) = \mathrm{ind}(E_n, t_n)$, *all the statements in Theorem* 5 *are still equivalent*.

Theorems 4 and 5 and Corollary 3 have more or less presented the relationship between regularity and various completeness properties for (LF)-spaces. Using the results in [16, 17], we [19] gave a sufficient condition for (LF)-spaces $(E, t) = \mathrm{ind}(E_n, t_n)$ to be complete as follows:

If for each $n \in \mathrm{N}$, there is $m = m(n) \geq n$ such that $\overline{E}_n^{E_p} \subset E_m$ for all $p \geq n$ (or, such that $\overline{E}_n^{E} \subset E_m$), then (E, t) is regular and complete, where $\overline{E}_n^{E_p}$ and $\overline{E}_n^{E}$ denote the closures of E_n in (E_p, t_p) and in (E, t), respectively. However, the longstanding problem of Grothendieck "whether regular (LF)-spaces are complete (or sequentially complete)" is still open. At the end of this paper, we raise the following problems, which are related to Grothendieck's above famous problem.

Open Problems. If all (E_n, t_n) are Fréchet spaces containing a copy of c_0, are all the equivalent relationships in Theorem 5 still true? Is there a regular (LF)-space $(E, t) = \mathrm{ind}(E_n, t_n)$, where every (E_n, t_n) contains a copy of c_0, such that (E, t) is not Σ-complete (or, is not l^∞-complete)?

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