On convolutions of B-splines

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Abstract

A smooth approximation to a function $f$ is achieved by convolving $f$ with a smooth function $\phi$. When $\phi$ is nonnegative, of unit mean value, compactly supported and has certain symmetry properties, convolving with $\phi$ respects the shape properties of the data $f$ such as local positivity, monotonicity and convexity. We study the convolution of $f$ and $\phi$ when $\phi$ is a univariate B-spline, tensor product B-spline, box spline or simplex spline, and $f$ is a linear combination of the same kind of splines as $\phi$. In terms of divided differences and blossoms, we express the convolution of univariate splines over nonuniform knots as linear combinations of B-splines. This conversion can be carried out by a stable recurrence.

Keywords: Convolution; Smoothing; Shape-preserving approximation; B-spline; Simplex spline; Box spline; Divided differences; Blossom; Polar form; Conversion

1. Introduction

Convolution is a widely used technique in mathematical analysis, statistics and approximation theory, and has many applications, e.g., image and signal processing. Convolution is also used in connection with univariate splines over uniform knots and box splines. However, convolution of piecewise polynomials over more general grids seems to have received little attention, but see, e.g., [7]. The technique is interesting in this more general setting since convolution offers a positivity-, monotonicity- and convexity-preserving method for smoothing data. To illustrate what we have in mind, suppose we are given some data we want to regularize, e.g., we might want to smooth the corners of some geometric object. The data to be regularized can for instance be given as a piecewise linear spline $f$. Convolving $f$ with a B-spline will give a smooth spline approximating the data. The ratio between the frequency of the oscillations in $f$ and the size of the B-spline’s support is of great importance to how close the resulting convolved function approximates the original function $f$. If the

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goal is to just smooth the corners and otherwise retain as much as possible of \( f \), the support of the B-spline should be chosen small compared to the frequency of the oscillations in \( f \).

In this paper we will mainly study convolutions of univariate splines over nonuniform knots, but we will also treat convolutions of multivariate B-splines. In the univariate case we study in particular how to express the convolution of two splines as linear combinations of higher-degree B-splines. By expressing these coefficients in terms of divided differences, we show that they can be evaluated stably by a recurrence relation. The method is easily extended to tensor product B-spline surfaces. Our argument is based on the polar-form representation of the dual functionals of univariate B-splines. We also make some remarks on convolutions of simplex splines and of box splines. Simplex splines generalize Bernstein polynomials on triangles, and hence results obtained for convolutions of simplex splines are useful when convolving piecewise polynomials on triangulations.

The spline obtained by convolving two lower-order splines will usually have more knots than any of the two lower-order splines convolved. As an example, consider the univariate case, and let \( \hat{f} \) be the spline resulting from convolving a spline \( f \) with a \( k \)th-order B-spline of small support relative to the spacing of the knots of \( f \). Then \( \hat{f} \) is a spline on a knot vector consisting of clouds of \( k + 1 \) knots located near every knot of the spline \( f \).

An outline of this paper goes as follows. In Section 2 we review some well-known properties of convolution, and examine the approximation \( \phi \ast f \) to a general \( f \) when \( \phi \) satisfies certain conditions. In Section 3 we study \( \phi \ast f \) when \( \phi \) is a univariate B-spline and \( f \) a B-spline curve. A brief comparison is done in the case of uniform knots between \( \phi \ast f \) and the Schoenberg variation diminishing spline approximation to \( f \) of suitable order (Section 3.2). Section 3 treats more generally convolution of univariate splines over nonuniform knots. Proposition 4 expresses the convolution of two B-splines in terms of divided differences of truncated powers. From this representation we derive continuity properties, and for \( f \in S_{k,t} \) and \( g \in S_{q,t} \) we may construct a knot vector \( \rho \) just large enough such that \( f \ast g \in S_{k+q,\rho} \) (Theorem 8). Proposition 10 describes the B-spline expansion of \( f \ast g \) in \( S_{k+q,\rho} \) in terms of blossoms. The conversion of \( f \ast g \) into a linear combination of B-splines can be implemented by a recurrence relation for blossoming the convolution of two B-splines given in Theorem 12. The method for convolving B-spline curves is extended to tensor product B-spline surfaces in Section 3.3. In Section 4 we examine \( \phi \ast f \) when \( \phi \) is a box spline or a simplex spline and \( f \) is a box spline surface respectively a simplex spline surface.

## 2. General properties of convolution

Let us recapture some properties of the convolution product which will be useful to us. See, e.g., [17] for discussions of the properties stated without proofs in this section. For functions \( \phi, f \) on \( \mathbb{R}^s \), the convolution of \( \phi \) and \( f \) is the function \( \phi \ast f \) defined by

\[
\phi \ast f (x) = \int_{\mathbb{R}^s} \phi(y) f(x - y) \, dy,
\]

for every \( x \) where the integral exists.

Roughly speaking, the convolution of two functions is at least as smooth as the one with greatest smoothness. The convolution product is commutative. Moreover it is bilinear, so for any integrable \( \phi \) we obtain a linear operator \( \phi \ast (\cdot) \) defined by \( \phi \ast (f) = \phi \ast f \). This operator commutes with
translations. The following lemma lists some shape-preserving properties of $\phi \ast (\cdot)$ under certain assumptions on $\phi$. In order to express the local nature of these properties, we will make use of the notation

$$\Omega - E = \{x - y: x \in \Omega, y \in E\},$$

for the set in $\mathbb{R}^d$ of all possible differences between points in $\Omega$ and $E$.

**Lemma 1.** Assume $\phi \geq 0$, $\int \phi = 1$ and $\text{supp} \phi \subseteq E \subseteq \mathbb{R}^d$. For any function $f$ defined on $\mathbb{R}^d$ and set $\Omega \subseteq \mathbb{R}^d$ such that $\phi \ast f(x)$ exists for all $x \in \Omega$, the following statements are valid.

(i) If $f|_{\Omega - E}$ is nonnegative, then so is $\phi \ast f|_{\Omega}$.

(ii) If $f|_{\Omega - E}$ is monotone in a direction $z$, then so is $(\phi \ast f)|_{\Omega}$.

(iii) For convex sets $\Omega$ and $E$, if $f|_{\Omega - E}$ is convex, then so is $(\phi \ast f)|_{\Omega}$.

(iv) If $f|_{\Omega - E}$ is a polynomial of exact degree $q$, then so is $(\phi \ast f)|_{\Omega}$ and the polynomials have the same highest-degree coefficients.

For the next statements we assume $\phi$ has the symmetry property $\phi(x) = \phi(-x)$.

(v) If $f|_{\Omega - E}$ is a polynomial of degree $\leq 1$, then $(\phi \ast f)|_{\Omega} = f|_{\Omega}$.

(vi) For convex sets $\Omega$ and $E$,

$$\phi \ast f(x) \geq \text{gcm}(f|_{\Omega - E})(x), \quad x \in \Omega,$$

where $\text{gcm}(f|_{\Omega - E})$ is the greatest convex minorant of the function $f|_{\Omega - E}$.

The greatest convex minorant of a real-valued function $f$ defined on a convex set $S \subseteq \mathbb{R}^d$ is the greatest convex function on $S$ majorized by $f$. Formally, for $x \in S$,

$$\text{gcm}(f)(x) = \inf\{\alpha \in \mathbb{R}: (x, \alpha) \in \text{co}(\text{epi}(f))\},$$

where $\text{co}(E)$ is the convex hull of the set $E$, and the epigraph of $f$ is a subset of $\mathbb{R}^{d+1}$ given by

$$\text{epi}(f) = \{(x, \alpha): x \in S, \alpha \in \mathbb{R}, \alpha \geq f(x)\}.$$

Note that statement (vi) assures that subject to the given conditions the graph of $(\phi \ast f)|_{\Omega}$ is contained in the convex hull of the graph of $f|_{\Omega - E}$.

The properties (i)-(v) of Lemma 1 are immediate from the definition of convolution (to see (iv), apply the binomial theorem). We include a discussion to justify (vi). By (i) the operator $\phi \ast (\cdot)$ is positive, and hence $\phi \ast f(x) \geq \phi \ast \text{gcm}(f|_{\Omega - E})(x)$ for $x \in \Omega$. Therefore, it suffices to show

$$g(x) \leq \phi \ast g(x), \quad x \in \Omega,$$

whenever $g$ is a convex real-valued function defined on $\Omega - E$. For such a $g$ the epigraph $\text{epi}(g)$ is a convex subset of $\mathbb{R}^{d+1}$ and hence for given $x \in \Omega$, there exists a supporting hyperplane of $\text{epi}(g)$ containing the point $(x, g(x))$ [23]. That is, there exist $a \in \mathbb{R}^d$, $\beta \in \mathbb{R}$ not both vanishing, such that

$$a \cdot x + \beta g(x) \leq a \cdot y + \beta \eta, \quad (y, \eta) \in \text{epi}(g).$$

By the symmetry property of $\phi$, the point $x$ is interior to $\Omega - E$, from which we may conclude that $\beta > 0$. Therefore the affine function $\theta$ defined by $\theta(y) = (a/\beta) \cdot (x - y) + g(x)$ satisfies

$$\theta(x) = g(x) \quad \text{and} \quad \theta(y) \leq g(y), \quad \text{for} \ y \in \Omega - E.$$
Using (v) and (i), we have \( g(x) = \theta(x) = \phi \ast \theta(x) \leq \phi \ast g(x) \), which shows (vi).

The more the mass of \( \phi \) is concentrated at the origin, the closer \( \phi \ast f \) approximates \( f \). We illustrate this with a result found, e.g., in [17], and use the following notation. For \( \phi \) an integrable function on \( \mathbb{R}^d \) and \( t > 0 \), define the function \( \phi_t \), by

\[
\phi_t(x) = t^{-d} \phi(t^{-1} x), \quad x \in \mathbb{R}^d.
\]

We see that \( \int \phi_t = \int \phi \).

**Proposition 2.** Assume \( \phi \in L^1 \) and \( \int \phi = 1 \). If \( f \in L^\infty \) and \( f \) is continuous on an open set \( U \), then \( \phi_t \ast f \rightarrow f \) uniformly on compact subsets of \( U \) as \( t \to 0 \).

For smooth \( f \) and certain compactly supported \( \phi \) an upper bound on the distance between \( f \) and \( \phi \ast f \) can be given that depends quadratically on the diameter of \( \text{supp} \phi \).

**Lemma 3.** Assume \( \int \phi = 1 \), \( \phi \geq 0 \), \( \phi(x) = \phi(-x) \) for all \( x \), and \( \text{supp} \phi \subseteq E \subseteq \mathbb{R}^d \), where \( E \) is star-shaped with respect to an open neighbourhood of the origin and contained in a ball with radius \( r \) centered at the origin. Then, for \( \Omega \subseteq \mathbb{R}^d \) and \( f \in C^2(\Omega - E) \),

\[
\| \phi \ast f - f \|_{\infty, \Omega} \leq r^2 \sup_{|z|=1} \| d_z^2 f \|_{\infty, \Omega - E},
\]

where \( d_z f \) is the directional derivative of \( f \) in direction \( z \), \( \| f \|_{\infty, \Omega} \) is the \( L^\infty \)-norm of \( f \) on the set \( \Omega \), and \( |z| \) is the Euclidean norm of \( z \).

**Proof.** Pick \( x \in \Omega \) and let \( p \) be the first-degree Taylor polynomial of \( f \) at \( x \). The operator \( \phi \ast (\cdot) \) reproduces polynomials of degree less than or equal to one; hence,

\[
|\phi \ast f(x) - f(x)| = |\phi \ast (f - p)(x) - (f - p)(x)| \leq 2 \| f - p \|_{\infty, \{x\} - E}.
\]

The assertion now follows by Taylor's theorem. \( \square \)

To summarize this section, the convolution operator \( \phi \ast (\cdot) \) is a linear operator producing smooth, shape-preserving approximations when \( \phi \) is

\[
\begin{aligned}
&\text{of unit integral: } \int \phi = 1, \\
&\text{positive: } \phi(x) \geq 0, \\
&\text{symmetric: } \phi(x) = \phi(-x), \\
&\text{compactly supported,} \\
&\text{smooth.}
\end{aligned}
\]

Moreover, the size of \( \text{supp} \phi \) is important to how closely \( \phi \ast f \) approximates \( f \).
3. Convolving with univariate B-splines

If $\phi$ is chosen equal to a univariate B-spline in the univariate case and equal to a tensor product B-spline or a box spline or a simplex spline in higher space dimensions, properties of $\phi$ like symmetry, size of support and smoothness are conveniently expressed in terms of the knots of the B-spline and are easily achieved. In this section we focus on the univariate case.

Our treatise on convolution with B-splines is based on the representation of B-splines as divided differences of truncated powers. We start by reviewing some basic properties of divided differences [20].

A divided difference $[x_0, \ldots, x_k]f$ of a function $f$ may be written as a linear combination of derivatives of $f$ at the points $x_0, \ldots, x_k$. More specifically, assume

$$
\{x_0, \ldots, x_k\} = \{\xi_1, \ldots, \xi_i, \ldots, \xi_m\}, \quad \text{where} \quad \xi_i < \xi_{i+1}, \quad i = 1, \ldots, m - 1.
$$

Then,

$$
[x_0, \ldots, x_k]f = \sum_{i=1}^{m} \sum_{j=0}^{n-1} a_{i,j} f^{(j)}(\xi_i), \quad (3)
$$

where the coefficients $a_{i,j}$ are rational functions in $\xi_1, \ldots, \xi_m$. If $f$ is piecewise smooth, we will use the convention that $[x_0, \ldots, x_k]f$ is defined by using left-sided limits of the derivatives of $f$ in the above sum.

The $k$th-order (degree +1) B-spline is defined as the divided difference

$$
M(x|x_0, \ldots, x_k) = k[x_0, \ldots, x_k](\cdot - x)^{k-1},
$$

where

$$
x_k^i = x^{i-1} \chi_{(0,\infty)}(x),
$$

and $\chi_E$ is the characteristic function of the set $E$. Even though it is common in CAGD (computer-aided geometric design) to use a normalization of the B-splines such that they make a partition of unity, we choose in this paper to work with the normalization which makes a B-spline have unit integral.

For sections to come we note that the truncated power $x^i_+$ is continuous from the left at the origin, so that $(t - x)_+^0$ is continuous from the left with respect to $t$ and from the right with respect to $x$. Hence, by our convention of divided differences, we may apply $[x_0, \ldots, x_k]$ to $(\cdot - x)_+^q$ for arbitrary $x$ and $k$.

A useful formula when examining convolutions of B-splines with polynomials and piecewise polynomials is the Peano representation for divided differences. Assume $x_0, \ldots, x_k$ is contained in the interval $[a, b]$. Peano’s formula says that for any $f \in C^{k-1}([a, b])$ with $f^{(k-1)}$ absolutely continuous on $[a, b]$ we have

$$
[x_0, \ldots, x_k]f = \frac{1}{k!} \int_{\mathbb{R}} M(t|x_0, \ldots, x_k)f^{(k)}(t) \, dt,
$$

as long as not all $x_0, \ldots, x_k$ coincide. This formula follows by applying the divided difference operator to the Taylor expansion of $f$, see, e.g., [25]. For a definition of absolute continuity see, e.g., [25].
For us, however, it will be sufficient to know that a continuous function which is piecewise \( C^1 \) with bounded derivative is absolutely continuous. Let us also mention the Hermite–Genocchi formula for divided differences. Under the same hypothesis as for Peano’s formula we have

\[
[x_0, \ldots, x_k] f = \int_{S^k} f^{(k)}(x_0 t_0 + \cdots + x_k t_k) \, dt_1 \cdots dt_k,
\]

where

\[
S^k = \left\{ (t_1, \ldots, t_k) : \sum_{j=1}^k t_j = 1 - t_0, t_j \geq 0, j = 0, \ldots, k \right\}.
\]

To show Hermite–Genocchi’s formula, one usually evaluates the right-hand side of (5) by iterated integration which yields a recurrence relation identical to that of the divided difference on the left-hand side, see, e.g., [18].

Combining Hermite–Genocchi’s formula and Peano’s formula, it will follow that \( M(\cdot|x_0, \ldots, x_k) \) is an excellent candidate for \( \phi \) in the convolution operator \( \phi \ast (\cdot) \) from the previous section. In addition to being nonnegative and supported on \( \text{co}(x_0, \ldots, x_k) \), we recall the following properties:

\[
\int M(t|x_0, \ldots, x_k) \, dt = 1,
\]

\[
M(t|x_0, \ldots, x_k) = |t^{-1}| M(t^{-1} x_0, \ldots, x_k), \quad t \neq 0,
\]

\[
M(x - y|x_0, \ldots, x_k) = M(x|x_0 + y, \ldots, x_k + y).
\]

Choosing \( t = -1 \) in the second of these equations, we see that the B-spline is symmetric if its knots are located symmetrically around the origin:

\[
M(-x|x_0, \ldots, x_k) = M(x|x_0, \ldots, x_k), \quad (x_0, \ldots, x_k) = (-x_0, \ldots, -x_k).
\]

The two last \((k+1)\)-tuples are considered to be without ordering. We see that it is easy to choose knots \( x_0, \ldots, x_k \) such that taking

\[
\phi = M(x|x_0, \ldots, x_k)
\]

makes \( \phi \) have all the properties (2) desired for the operator \( \phi \ast (\cdot) \) of the previous section. Moreover, the function \( \phi_t \) used in Proposition 2 is obtained by just scaling the knots:

\[
\phi_t = M(x|tx_0, \ldots, tx_k).
\]

The convolution of a function with a B-spline is at least as smooth as the B-spline. For example, convolving a broken line with a B-spline will have the effect of smoothing out the corner. In Fig. 1 we have convolved a translate of the absolute value function with two different B-splines which are both symmetric with respect to the origin. We see that for \( x \) such that the support of \( M(x - |x_0, \ldots, x_k) \) does not overlap the critical point of the broken line the convolution curve coincides with the broken line. This illustrates property (v) in Lemma 1.

By making appropriate choices for \( f \) in Peano’s formula (4), we may express in terms of divided differences the convolution of \( M(\cdot|x_0, \ldots, x_k) \) with a monomial, a truncated power and another B-spline.
Proposition 4. Assume \( x_0 \neq x_k \) and \( q \in \mathbb{Z}_+ \). Then,

\[
M(\cdot | x_0, \ldots, x_k) \ast \theta_q(z) = \binom{k + q}{q}^{-1} [x_0, \ldots, x_k]_t \theta_{k+q}(z-x),
\]

for \( \theta_j \) the power \( \theta_j(x) = (-x)^j \) or \( \theta_j \) the truncated power \( \theta_j(x) = (-x)^{j_+} \), \( j \in \mathbb{Z}_+ \). Moreover, if \( y_0 \neq y_q \), then

\[
M(\cdot | x_0, \ldots, x_k) \ast M(\cdot | y_0, \ldots, y_q)(z) = \frac{k! \cdot q!}{(k+q-1)!} [x_0, \ldots, x_k]_t [y_0, \ldots, y_q]_y (x+y-z)^{k+q-1}.
\]

Proof. To see the first assertion, let \( f \) be defined by

\[
f(x) = ((k+q)!)^{-1} q! \theta_{k+q}(z-x).
\]

Assume first \( \theta_j \) is the power \( \theta_j(x) = (-x)^j \), \( j \in \mathbb{Z}_+ \). Then \( f^{(j)}(x) = \theta_q(z-x) \) which, inserted into Peano’s formula (4), yields (7) in this case. Assume then \( \theta_j \) is the truncated power \( \theta_j(x) = (-x)^{j_+} \), \( j \in \mathbb{Z}_+ \). Then \( f \in C^{k-1} \) and \( f^{(j)}(x) = \theta_q(z-x) \), which is continuous except at the point \( x = z \) when \( k = q \). Hence we may also in this case apply Peano’s formula (4) to obtain (7). To see (8), let \( f \) be given by

\[
f(x) = [y_0, \ldots, y_q]_y (x+y-z)^{k+q-1}.
\]

From (3) we see that \( f \in C^{k-1} \) and \( f^{(k)} \) is continuous except at a possible \( q \)-tuple knot among \( \{y_0, \ldots, y_q\} \). We find \( f^{(k)}(x) \) by differentiating \( (x+y-z)^{k+q-1} \) with respect to \( x \) and then apply \( [y_0, \ldots, y_q]_y \) to obtain

\[
f^{(k)}(x) = (q+k-1)! \cdot (q!)^{-1} M(z-x|y_0, \ldots, y_q).
\]

Inserting this into Peano’s formula, we obtain

\[
[x_0, \ldots, x_k] \cdot f = (q+k-1)! \cdot (q! k!)^{-1} \int M(t|x_0, \ldots, x_k) M(z-t|y_0, \ldots, y_q) \, dt.
\]

\[\square\]
If the B-spline is defined over simple knots, the convolution with a monomial has a simple closed formula. Given $x_0 < \cdots < x_k$, replacing $\theta_q(x)$ in (7) with $\tilde{\theta}_q(x) = x^q$, a standard identity for divided differences implies (7) then takes the form

$$M(|x_0, \ldots, x_k) \ast \tilde{\theta}_q(x) = \binom{k + q}{k}^{-1} \sum_{i=0}^{k} \frac{(x - x_i)^{k+q}}{\prod_{j=0, j 
eq i}^{k}(x_j - x_i)}.$$ 

The function in (7) obtained by convolving a B-spline with a truncated power also appears in connection with natural splines. The Greville basis for the natural splines of order $2q$ on a knot vector $\tau = \{\tau_j\}_{j=1}^n$ consists of the ordinary B-splines on $\tau$ together with the following splines considered as functions of $z$:

$$[\tau_1, \ldots, \tau_{k+1}]x(x - z)^{2q-1}, \quad [\tau_{n-k}, \ldots, \tau_n]x((x - z)^{2q-1} - (x - z)^{2q-1}),$$

for $k \in \{q, \ldots, 2q - 1\}$ [25].

The identity (8) for the convolution of two B-splines is a generalization of the inner product formula for B-splines given in [14]. In Section 3.1 we will examine convolution of splines given as linear combination of B-splines. This study is largely based on the identity (8).

From Peano’s formula and Hermite-Genocchi’s formula for divided differences we obtain the following proposition.

**Proposition 5.** Let $a, b$ be numbers such that $a \leq x_i + y_j \leq b$ for $i = 0, \ldots, k, j = 0, \ldots, q$. For any $f \in C^{k+q-1}(a, b)$ such that $f^{(k+q-1)}$ is absolutely continuous on $[a, b]$ we have

$$[x_0, \ldots, x_k]_{x} [y_0, \ldots, y_q]_{y} f(x + y) = \int_{S_x \times S_y} f^{(k+q)}(\hat{X} \mu + \hat{Y} \nu) \, d\mu \, d\nu$$

$$= (k! \, q!)^{-1} \int_{\mathbb{R}} M(\cdot | X) \ast M(\cdot | Y)(t) \, f^{(k+q)}(t) \, dt,$$

whenever $x_0 \neq x_k$ and $y_0 \neq y_q$. Here $X = \{x_0, \ldots, x_k\}$, $Y = \{y_0, \ldots, y_q\}$ and

$$\hat{X} \mu = x_0 + \sum_{i=1}^{k} (x_i - x_0) \mu_i, \quad \hat{Y} \nu = y_0 + \sum_{i=1}^{q} (y_i - y_0) \nu_i.$$

According to Hermite-Genocchi’s formula, the first of the two equations remains valid even if $y_0 = \cdots = y_q$, while we have to replace $M(\cdot | X) \ast M(\cdot | Y)(t)$ by $M(t - y_0 | X)$ for the last equation to remain valid in this case.

**Proof.** Let the function $g$ be defined by

$$g(x) = [y_0, \ldots, y_q]_{y} f(x + y).$$

Then $g \in C^k$, and application of Peano’s formula (4) as well as Hermite-Genocchi’s formula (5) yield

$$[x_0, \ldots, x_k]_{x} g = \int_{S_x} g^{(k)}(\hat{X} \mu) \, d\mu = (k!)^{-1} \int_{\mathbb{R}} M(t | X) g^{(k)}(t) \, dt.$$
The substitution
\[ g^{(k)}(x) = [y_0, \ldots, y_q], f^{(k)}(x + y) \]
and yet another application of Hermite–Genocchi and Peano but now to \( f^{(k)}(\hat{X}\mu + \cdot) \) respectively \( f^{(k)}(t + \cdot) \) yields
\[
\langle x_0, \ldots, x_k \rangle g = \int_{s_0}^{s_1} \int_{s_0}^{s_1} f^{(k+q)}(\hat{X}\mu + \hat{Y}\nu) \, d\mu \, d\nu \\
= (k!q!)^{-1} \int_{\mathbb{R}} M(t|X) \int_{\mathbb{R}} M(s|Y) f^{(k+q)}(t + s) \, ds \, dt.
\]
For the last of these expressions to take on the desirable form we do the change of variables \( s \to s - t \) in the inner integral and observe that we may change the order of integration. □

The latter of the equations in Proposition 5 is a special case of an identity for more general polyhedral splines given in [6].

Several properties of the B-spline appear when comparing Peano’s formula and Hermite–Genocchi’s formula for divided differences. Both of these formulas may be extended to repeated application of divided differences and convolutions of B-splines. Such formulas are stated in Proposition 5 for the convolution of two B-splines, but the extension to \( n \)-fold convolution of B-splines is obvious.

As an illustration of Proposition 5, let us verify the well-known convolution formula for B-splines over uniform knots. In this case Hermite–Genocchi’s formula looks like
\[
[p, p + 1, \ldots, p + k] f = \int_{s_0}^{s_1} f^{(k)} \left( p + \sum_{j=1}^{k} j\mu_j \right) d\mu = (k!)^{-1} \int_{[0,1]^k} f^{(k)} \left( p + \sum_{j=1}^{k} z_j \right) dz,
\]
for sufficiently smooth \( f \). The last equality is seen when evaluating the last integral by iterated integration, which produces a recurrence relation identical to that of the divided difference. For the moment we will only be interested in the latter equality, so there is no need to talk about the \( k \)th derivative of \( f \). We therefore proceed with \( f^{(k)} \) replaced by any locally integrable \( f \). Twice applying this formula with \( p = 0 \) yields
\[
k!q! \int_{s_0}^{s_1} \int_{s_0}^{s_1} f \left( \sum_{j=1}^{k} j\mu_j + \sum_{t=1}^{q} t\nu_t \right) d\mu \, d\nu = \int_{[0,1]^k} \int_{[0,1]^q} f \left( \sum_{j=1}^{k} z_j + \sum_{t=1}^{q} u_t \right) dz \, du
\]
\[
= (k!q!) \int_{s_0}^{s_1} f \left( \sum_{j=1}^{k+q} j\mu_j \right) d\mu.
\]
The convolution identity for B-splines over uniform knots is now a consequence of invoking the above proposition:
\[
\int_{\mathbb{R}} M(\cdot|0, 1, \ldots, k) * M(\cdot|0, 1, \ldots, q) (x) f(x) \, dx = \int_{\mathbb{R}} M(x|0, 1, \ldots, k + q) f(x) \, dx.
\]
Suppose we want to write the convolution of two splines \( f \) and \( g \) as a linear combination of higher-degree B-splines. In case \( f \) and \( g \) both have knots at the integers, this is easy. Using the translation property of the convolution product, finding the new coefficients reduces to discrete convolution of the
coefficient vectors of $f$ and $g$. If $f$ and $g$ are splines over uniform knot vectors but which are different refinements of $Z$, say $2^{-m}Z$ respectively $2^{-n}Z$, this is still easy. We just refine the representations of the two splines to a common knot vector $2^{-\eta}Z$ where $\eta = \max(m, n)$. The coefficients of $f \ast g$ are then the discrete convolution of the coefficient vectors of $f$ and $g$ relative to $2^{-\eta}Z$. The topic of the next section is the nonuniform case. In this situation things are not so straightforward.

3.1. Convolutions of B-splines as linear combinations of B-splines

Convolution of two B-splines defined over the same uniform knot vector produces a B-spline defined over the same knot vector. In the more general situation with nonuniform knot vectors this is not the case because the knots of the two B-splines need not coincide, even if one of the B-splines is translated. As a consequence, convolving a spline with a B-spline produces a new curve with more knots, but which has more continuous derivatives than the original curve. If the distances between the knots of the original spline are large compared to the size of the B-spline’s support, the knots of the new spline will be clustered around the knots of the original one. This effect of smoothing the corners is particularly apparent if the original spline is piecewise linear. Some estimates on the global smoothness of the convolution of two B-splines are given by the general smoothness criterion for convolving two functions. But with the divided difference representation (8) at hand, we can give more precise estimates.

For the B-splines defined by a nondecreasing knot vector $\tau$ we will also make use of the common notation

$$M_{i,k,\tau} = M(\cdot | \tau_i, \ldots, \tau_{i+k}).$$

Hereafter we understand by knot vector a nondecreasing sequence. To such a knot vector $\tau$ we assign a strictly increasing sequence $\pi_\tau$ consisting of the distinct knots of $\tau$. We also assign a sequence $m_\tau$ indicating the multiplicities in $\tau$ of the elements in $\pi_\tau$. That is, if $n = \#\pi_\tau$ is the number of distinct elements in $\tau$, $\tau$ may be written as

$$\tau = \{\pi_{1,\tau}, \ldots, \pi_{n,\tau}\}.$$

To express the convolution of two B-splines as a linear combination of higher-order B-splines, we need some notation.

**Definition 6.** Given knot vectors $\tau$ and $t$. Let $\tau \uplus t$ be the nondecreasing sequence with distinct elements satisfying

$$\pi_{\tau \uplus t} = \pi_\tau + \pi_t = \{x: x = \pi_{i,\tau} + \pi_{j,t}, i = 1, \ldots, \#\pi_\tau, j = 1, \ldots, \#\pi_t\}$$

and where the multiplicity vector $m_{\tau \uplus t}$ is given by

$$m_{\ell,\tau \uplus t} = \max_{i,j} \{m_{i,\tau} + m_{j,t} - 1: \pi_{i,\tau} + \pi_{j,t} = \pi_{\ell,\tau \uplus t}\},$$

for $\ell = 1, \ldots, \#\pi_{\tau \uplus t}$.

We may now state a result on the smoothness for the convolution of two B-splines.
Lemma 7. Let $\tau = \{\tau_j\}_{j=1}^{k+1}$ and $t = \{t_i\}_{i=1}^{q+1}$ be two knot vectors such that $\tau_1 < \tau_{1+k}$ and $t_1 < t_{1+q}$ and let the knot vector $\tau \uplus t$ be as in Definition 6. Then, the spline $M_{1,k,\tau} * M_{1,q,t}$ is infinitely many times differentiable except at the points contained in $\pi_{r_{\text{ref}}}$. Moreover,

$$(M_{1,k,\tau} * M_{1,q,t})^{(r)}(x-) = (M_{1,k,\tau} * M_{1,q,t})^{(r)}(x+), \quad x = \pi_{r_{\text{ref}}},$$

for $r = 0, \ldots, k + q - m_{\ell,r_{\text{ref}}} - 1, \ell = 1, \ldots, \#\pi_{r_{\text{ref}}}$. 

Proof. The convolution $M_{1,k,\tau} * M_{1,q,t}$ may be written as a linear combination of truncated powers by applying Proposition 4 and Eq. (3) twice. Let $\mu = \#\pi_{r_{\text{ref}}}$ and $\nu = \#\pi_t$. Disregarding the scaling factor $(k + q)^{-1}$, the spline $M_{1,k,\tau} * M_{1,q,t}$ may be written

$$\begin{align*}
&[\tau_1, \ldots, \tau_{k+1}][t_1, \ldots, t_{q+1}]z(y + z - x)^{k+q-1} \\
&= \sum_{j=1}^{\mu} \sum_{\beta=1}^{m_{\tau,j}} \sum_{\alpha=1}^{m_{t,j}} \sum_{\gamma=1}^{\nu} c_{j,\beta}(\pi_{i,\tau} + \pi_{j,t} - x)^{k+q-\beta-\gamma-1},
\end{align*}$$

(10)

where $a_{i,\alpha}$ and $c_{j,\beta}$ only depend on $\tau_1, \ldots, \tau_{k+1}$ and $t_1, \ldots, t_{q+1}$, respectively. From this sum we see that the convolution is arbitrarily smooth on $\mathbb{R} \setminus \pi_{r_{\text{ref}}}$. For the continuity of $M_{1,k,\tau} * M_{1,q,t}$ at the point $\pi_{r_{\text{ref}}}$ assume $\pi_{i,\tau} + \pi_{j,t} = \pi_{r_{\text{ref}}}$ and consider the truncated power $(\pi_{i,\tau} + \pi_{j,t} - x)^{k+q+1-\alpha-\beta}$. Since $m_{\ell,r_{\text{ref}}} \geq m_{i,\tau} + m_{j,t} - 1$, clearly the truncated power is $k + q - m_{\ell,r_{\text{ref}}} - 1$ times continuously differentiable at $\pi_{r_{\text{ref}}}$. \hfill \Box

Differentiating $M_{1,k,\tau} * M_{1,q,t}$ more times than indicated in the lemma usually produces a discontinuity:

$$(M_{1,k,\tau} * M_{1,q,t})^{(r)}(x-) \neq (M_{1,k,\tau} * M_{1,q,t})^{(r)}(x+), \quad x = \pi_{r_{\text{ref}}},$$

when $r = k + q - m_{\ell,r_{\text{ref}}}$, unless

$$\sum_{(i,j) \in J_{\ell}} a_{i,m_{\tau}} c_{j,m_{\tau}} = 0, \quad J_{\ell} = \{(i, j) : \pi_{i,\tau} + \pi_{j,t} = \pi_{r_{\text{ref}}},\}.$$

where $a_{i,m_{\tau}}$ and $c_{j,m_{\tau}}$ are the coefficients in the sum (10). It is known from the theory on divided differences that $a_{i,m_{\tau}}$ is a rational, nowhere vanishing function in the elements of $\pi_{r_{\text{ref}}}$. and $c_{j,m_{\tau}}$ is nonvanishing and rational in the members of $\pi_t$. In particular this means that the above sum is nonzero when the set $J_{\ell}$ contains only one element.

With the continuity result of Lemma 7 at hand, it follows that the convolution of a spline on a knot vector $\tau$ with a spline on another knot vector $t$ is a spline on $\tau \uplus t$. Note that the construction of $\tau \uplus t$ is independent of the degrees of the splines involved.

Theorem 8. Let $\tau$ and $t$ be given knot vectors and define the knot vector $\tau \uplus t$ as in Definition 6. Then,

$$S_{k,t} * S_{q,t} \subseteq S_{k+q,r_{\text{ref}}}.$$
where
\[ S_{k,\tau} \ast S_{q,t} = \text{span}\{f \ast g : f \in S_{k,\tau}, g \in S_{q,t}\}. \]

**Proof.** Assume \( \pi_{\text{int}} \) is contained in an open interval \((a, b)\). We extend \( \tau \cup t \) with \((k + q)\)-tuple knots at \( a \) and \( b \) obtaining a knot vector \( \tau \cup t \). By the Curry–Schoenberg Theorem the spline space \( S_{k+q,\tau \cup t} \) contains all piecewise polynomials of degree less than \( k + q \) with \( k + q - m_{\ell,\tau \cup t} - 1 \) continuous derivatives at \( \pi_{\ell,\tau \cup t} \) for \( \ell = 1, \ldots, \#\pi_{\text{int}} \). Thus, Lemma 7 assures that the convolution of a B-spline from \( S_{k,\tau} \) with a B-spline from \( S_{q,t} \) is contained in \( S_{k+q,\tau \cup t} \), and hence
\[ f \ast g \in S_{k+q,\tau \cup t}, \quad \text{for } f \in S_{k,\tau}, \ g \in S_{q,t}. \]

Moreover, for such \( f \) and \( g \),
\[ \text{supp } f \ast g \subseteq \text{supp } f + \text{supp } g \subseteq \text{co}(\pi_{\tau}) + \text{co}(\pi_{t}) = \text{co}(\pi_{\text{int}}). \]

So, the spline \( f \ast g \) vanishes on open sets containing \( a \) and \( b \) and must therefore be contained in \( S_{k+q,\tau \cup t} \). \( \square \)

The comment following Lemma 7 shows we cannot in general expect to find a smaller B-spline space than \( S_{k+q,\tau \cup t} \) which contains \( S_{k,\tau} \ast S_{q,t} \).

From (9) we know that the convolution of two splines with knots at the integers is a new spline with uniform knots. In particular, the new spline will have simple knots only. By Theorem 8 and the definition of \( \tau \cup t \) this remains valid even in the nonuniform situation, convolving splines with simple knots produces a new spline with simple knots.

Theorem 8 is useful when dealing with the linear operator \( \phi \ast (\cdot) \) from Section 2 in the case when \( \phi \) is a B-spline. When the domain of this operator is a spline space, we may now construct a spline space just large enough to contain its range. Setting \( \tau = \{\tau_1, \ldots, \tau_{k+1}\} \), the space \( S_{k,\tau} \) is spanned by \( M(\cdot|\tau_1, \ldots, \tau_{k+1}) \) and
\[ M(\cdot|\tau_1, \ldots, \tau_{k+1}) \ast (\cdot) : S_{q,t} \rightarrow S_{k+q,\text{int}(\tau_1, \ldots, \tau_{k+1})}. \]

The knowledge that the convolution of two splines is contained in a given spline space is of more value if we are also able to expand the convolution spline in terms of the B-splines of the given space. There are several methods to find such an expansion. We will use blossoming techniques to express the unknown B-spline coefficients in terms of divided differences.

The following notation will be used. For \( f \) a polynomial of degree \( \leq n \) we let \( f^* \) be the \( n \)-th degree homogenization of \( f \),
\[ f^*(x, t) = t^n f(t^{-1}x), \quad x, t \in \mathbb{R}. \]

Furthermore we let \( B(f) \) denote the multilinear blossom of \( f \). This is the unique \( n \)-linear symmetric form from \((\mathbb{R}^2)^n \rightarrow \mathbb{R}\) which coincides with \( f^* \) on its diagonal. It can be shown that \( B(f) \) is a scaling of the \( n \)-th order total derivative of \( f^* \) and that the \( \ell \)-th order derivative of \( f^* \) for \( 0 \leq \ell \leq n \) is related to the blossom of \( f \) as follows:
\[ B(f)(u^1, \ldots, u^\ell, v, \ldots, v) = \frac{(n - \ell)!}{n!} \frac{d^{n-\ell}}{d u^{n-\ell}} f^*(v), \quad u^i, v \in \mathbb{R}^2, \] (11)
where $d_u$ is directional differentiation in direction $u \in \mathbb{R}^2$, see, e.g., [21]. We note that the blossoming operator $B(\cdot)$ is linear. If $f$ is a piecewise polynomial, we let $f_a$ be the polynomial representing $f$ at the point $a$. Similarly we will by the local blossom of $f$ at $a$ understand the blossom of $f_a$.

The B spline expansion of a piecewise polynomial $f$ contained in $S_{n+1,p}$ is found by evaluating the local blossom of $f$ in consecutive tuples from $\rho$. That is [21], if for real numbers $b_i$,

$$f = \sum_i b_i M_{i,n+1,p},$$

then for arbitrary $\xi_i \in [\rho_i, \rho_{i+n+1})$,

$$b_i = \frac{1}{n+1} (\rho_{i+n+1} - \rho_i) B(f_{\xi_i})(\tilde{\rho}_{i+1}, \ldots, \tilde{\rho}_{i+n}), \quad (12)$$

where $\tilde{\rho}_i = (\rho_i, 1) \in \mathbb{R}^2$. By means of Proposition 4 we will find below an expression for the local blossom of a piecewise polynomial $g$ when $g$ is the convolution of two B-splines or a linear combination of such convolutions. Then, if $g$ is contained in $S_{n+1,p}$, we will according to (12) also know the B-spline expansion of $g$ in $S_{n+1,p}$.

Before finding the local blossom of convolutions of B-splines, let us consider briefly what happens if we define a B-spline series $\sum_i b_i M_{i,k,p}$ where $b_i$ is determined by $f$ through (12) when $f$ is a piecewise polynomial of degree $\leq n$ not contained in $S_{n+1,p}$. The right-hand side of (12) is well-defined for all piecewise polynomials $f$ of degree $\leq n$. So, for a fixed sequence $\xi_i = \{\xi_i\}_i$ such that $\xi_i \in [\rho_i, \rho_{i+n+1})$, we obtain a linear projector $Q_{n+1,p}^f$ from the space of all piecewise polynomials of degree $\leq n$ onto $S_{n+1,p}$, defined by

$$Q_{n+1,p}^f(f) = \frac{1}{n+1} \sum_i (\rho_{i+n+1} - \rho_i) B(f_{\xi_i})(\tilde{\rho}_{i+1}, \ldots, \tilde{\rho}_{i+n}) M_{i,n+1,p}. \quad (12)$$

In [22] it is shown that $B(f_{\xi_i})(\tilde{\rho}_{i+1}, \ldots, \tilde{\rho}_{i+n})$ is the $i$th de Boor-Fix functional on $S_{n+1,p}$ applied to the polynomial $f_{\xi_i}$. Hence our B-spline series $\sum_i b_i M_{i,k,p} = Q_{n+1,p}^f(f)$ is the de Boor-Fix quasi-interpolant [11] in $S_{n+1,p}$ to $f$.

Let us consider the local blossom of the convolution of two B-splines, and see how we express this in terms of divided differences. For arbitrary numbers $x_0, \ldots, x_k, y_0, \ldots, y_q, z$ such that $k + q \geq 1$ let the function $C(z | X | Y) : (\mathbb{R}^2)^{k+q-1} \to \mathbb{R}$ be defined by

$$C(z | X | Y)(u^1, \ldots, u^{k+q-1}) \overset{\text{def}}{=} \frac{k! q!}{(k + q - 1)!} [x_0, \ldots, x_k, y_0, \ldots, y_q] (x + y - z)^0 \prod_{\ell=1}^{k+q-1} \det(x + y, u^\ell), \quad (13)$$

where $x + y = (x + y, 1) \in \mathbb{R}^2$ and $\det(u, v)$ is the determinant of the $2 \times 2$ matrix with first column $u$ and second $v$. We may note at once that if $x_m$ is the smallest and $x_M$ the largest element in $X$, and similarly for $y_m, y_M, Y$, then

$$C(z | X | Y) \equiv 0, \quad \text{whenever } z \not\in [x_m + y_m, x_M + y_M]. \quad (14)$$

The reason for introducing $C(z | X | Y)$ is the following lemma.
Lemma 9. Assume $\tau_i < \tau_{i+k}$ and $t_j < t_{j+q}$. Then $C(z | \tau_i, \ldots, \tau_{i+k} | t_j, \ldots, t_{j+q})$ is the local blossom of $M_{i,k,t} \ast M_{j,q,t}$ at $z$, i.e.,

$$B((M_{i,k,t} \ast M_{j,q,t})_z)(u^1, \ldots, u^{k+q-1}) = C(z | \tau_i, \ldots, \tau_{i+k} | t_j, \ldots, t_{j+q})(u^1, \ldots, u^{k+q-1}).$$

Proof. Let $f_z$ be the scaling

$$f_z = \frac{(k + q - 1)!}{k! q!} (M_{i,k,t} \ast M_{j,q,t})_z$$

of the polynomial coinciding with $M_{i,k,t} \ast M_{j,q,t}$ near $z$. Writing out the double divided difference (8) for $M_{i,k,t} \ast M_{j,q,t}$ as in the proof of Lemma 7, we see that

$$f_z(u) = [\tau_i, \ldots, \tau_{i+k}]_x [t_j, \ldots, t_{j+q}]_y (x + y - z)^0 (x + y - u)^{k+q-1}.$$

The multilinear blossom of this polynomial is given by

$$B(f_z)(u^1, \ldots, u^{k+q-1}) = [\tau_i, \ldots, \tau_{i+k}]_x [t_j, \ldots, t_{j+q}]_y (x + y - z)^0 \prod_{\ell=1}^{k+q-1} \det(x + y, u^\ell), \quad u^\ell \in \mathbb{R}^2.$$

To see this, we merely observe that the right-hand side is symmetric in $u^1, \ldots, u^{k+q-1}$, linear in each of these arguments and coincides with $f_z(u)$ if $u^\ell = (u, 1)$ all $\ell$. $\square$

Using this lemma together with the fact that the blossoming operator $B(\cdot)$ is linear, we may evaluate the local blossom of $f \ast g$ for any known B-spline expansions $f \in S_{k,t}$ and $g \in S_{q,r}$. If $f \ast g$ is contained in $S_{k+q,p}$, we may evaluate the local blossom of $f \ast g$ in order to convert the representation of $f \ast g$ to the B-spline representation in $S_{k+q,p}$.

Proposition 10. Given knot vectors $\tau$, $t$ and $p$. Assume coefficients $\{a_{i,j}\}_{i,j}$ and $\{d_\ell\}_\ell$ satisfy

$$\sum_{i,j} a_{i,j} M_{i,k,t} \ast M_{j,q,r} = \sum_\ell d_\ell M_{\ell,k+q,p}.$$

Then,

$$d_\ell = (k + q)^{-1} (\rho_{\ell+k+q} - \rho_\ell) \sum_{i,j} a_{i,j} c_{i,j},$$

where

$$c_{i,j} = C(x_\ell | \tau_i, \ldots, \tau_{i+k} | t_j, \ldots, t_{j+q}) (\tilde{\rho}_{\ell+1}, \ldots, \tilde{\rho}_{\ell+k+q-1}), \quad x_\ell \in [\rho_\ell, \rho_{\ell+k+q}),$$

if $\tau_i < \tau_{i+k}$ and $t_j < t_{j+q}$ and where $c_{i,j} = 0$ otherwise.

Proof. This follows from Lemma 9 and Eq. (12). $\square$

Evaluating the local blossom of a piecewise polynomial along the diagonal, we retrieve the piecewise polynomial. Moreover, from (11) it follows that the pointwise derivative is found by evaluating
the blossom at argument bags containing points in the plane with zero second component, i.e., assuming nonvanishing B-splines, we have

\[
\frac{(k-r)!}{k!} (M(\cdot|X)\ast M(\cdot|Y))^{(r)}(x) = C(x|X|Y)(e^1, \ldots, e^r, \bar{x}, \ldots, \bar{x}), \quad r \geq 0,
\]

(15)

where \( e^1 = (1,0) \). Thus, a recurrence relation for the evaluation of \( C(x|X|Y) \) at arbitrary argument bags will lead to an algorithm which might be used to evaluate and differentiate the convolution of two splines as well as to finding the B-spline coefficients relative to a suitable B-spline basis.

Recurrence relations for evaluating convolutions and inner products of polyhedral splines have been derived in [5,6]. These generalize formulas for inner product of univariate B-splines derived in [14]. Note that by (6) the relation between the inner product and the convolution of two nonvanishing univariate B-splines can be written

\[
\int M(x|X)M(x|Y) \, dx = C(0|X|Y)(\bar{\alpha}, \ldots, \bar{\alpha}),
\]

where \( \bar{\alpha} = (0,1) \). In order to generalize the recurrence relation for evaluating the convolution of B-splines to evaluating the local blossom of the convolution, we need some identities for divided differences.

Leibniz' formula for the divided difference of a product \( fg \) where \( g(x) = ax - b \) yields

\[
[x_0, \ldots, x_k](ax - b)f(x) = a[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]_xf(x)
\]

\[+ (ax_i - b)[x_0, \ldots, x_k]_xf(x).\]

Multiplying this equation by \( \mu_i/a \) and summing over all \( i \), we obtain

\[
[x_0, \ldots, x_k](ax - b)f(x) = \sum_{i=0}^{k} \mu_i[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]_xf(x)
\]

\[+ (u - b)[x_0, \ldots, x_k]_xf(x),
\]

(16)

whenever the numbers \( \mu_0, \ldots, \mu_k \) satisfy

\[
\sum_{i=0}^{k} \mu_i = a, \quad \sum_{i=0}^{k} \mu_i x_i = u.
\]

The last term in (16) disappears when choosing \( u \) equal to \( b \). The equation remains valid also in case \( \sum_i \mu_i = a = 0 \). This follows by scaling the formula

\[
(x_i - x_0)[x_0, \ldots, x_k]f = [x_1, \ldots, x_k]f - [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]f
\]

with \( \mu_i \) and summing over all \( i \).

We also need a result similar to (16) for iterated divided differences.

Lemma 11. For arbitrary numbers \( x_0, \ldots, x_k, y_0, \ldots, y_q, a, b \) and integers \( k, q \geq 1 \) we have
whenever the numbers $\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_q$ satisfy
\[
\sum_{i=0}^{k} \mu_i = a = \sum_{j=0}^{q} \nu_j, \quad \sum_{i=0}^{k} \mu_i x_i + \sum_{j=0}^{q} \nu_j y_j = b.
\]

Proof. Assume $\sum_{j=0}^{q} \nu_j y_j = v$. Eq. (16) yields
\[
[y_0, \ldots, y_q]_y (a(x + y) - b) f(x + y) = \sum_{j=0}^{q} \nu_j [y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_q]_y f(x + y) + (ax + v - b)[y_0, \ldots, y_q]_y f(x + y).
\]
Applying $[x_0, \ldots, x_k]$ to this equation and then using (16) once more, the assertion follows. \(\square\)

We are now ready to give a recurrence relation for the local blossom of the convolution of two B-splines.

Theorem 12. For arbitrary numbers $x_0, \ldots, x_k, y_0, \ldots, y_q, z$, points in the plane $u^\ell = (u_1^\ell, u_2^\ell)$, $\ell = 1, \ldots k + q - 1$, and integers $k, q \geq 0$ such that $k + q \geq 2$, we have
\[
C(z|X|Y)(u^1, \ldots, u^{k+q-1}) = \frac{k}{k + q - 1} \sum_{i=0}^{k} \mu_i C(z|X \setminus x_i|Y)(u^1, \ldots, u^{k+q-2}) + \frac{q}{k + q - 1} \sum_{j=0}^{q} \nu_j C(z|X|Y \setminus y_j)(u^1, \ldots, u^{k+q-2}),
\]
whenever the numbers $\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_q$ satisfy
\[
\sum_{i=0}^{k} \mu_i = u_2^{k+q-1} = \sum_{j=0}^{q} \nu_j, \quad \sum_{i=0}^{k} \mu_i x_i + \sum_{j=0}^{q} \nu_j y_j = u_1^{k+q-1}.
\]

Proof. For $k, q \geq 1$ this theorem is Lemma 11 applied to the function
\[
\det(x + y, u^{k+q-1}) f(x + y), \quad f(x) = \prod_{\ell=1}^{k+q-2} \det(x, u^\ell)(x - z)_+^0.
\]
If either $k$ or $q$ equals zero, we apply (16). \(\square\)

Note that as long as $X$ contains at least two nonequal elements $x_i \neq x_j$ or $Y$ contains two nonequal elements, there exist numbers $\mu_1, \ldots, \mu_k, \nu_0, \ldots, \nu_q$ satisfying the hypothesis of the theorem.
However, if \( x_0 = \cdots = x_k \) and \( y_0 = \cdots = y_q \), then \( C(z|X|Y) \) is identically zero for all \( z \). So, in this case (18) is true for arbitrary \( \mu_j \) and \( \nu_j \).

The recurrence in the theorem should be started with

\[
C(z|x_0, x_1, y_0, y_1) = \begin{cases} 
  (x_1 - x_0)^{-1}, & \text{if } x_0 + y_0 \leq z < x_1 + y_0, \\
  0, & \text{otherwise,}
\end{cases}
\]

\[
C(z|x_0, y_0, y_1) = \begin{cases} 
  (y_1 - y_0)^{-1}, & \text{if } x_0 + y_0 \leq z < x_0 + y_1, \\
  0, & \text{otherwise,}
\end{cases}
\]

(19)

in case \( x_0 \leq x_1 \) and \( y_0 \leq y_1 \). If \( x_1 > x_0 \), the roles of \( x_0 \) and \( x_1 \) should be interchanged, similarly for \( y_0 \) and \( y_1 \). Theorem 12 provides a great deal of freedom in choosing the parameters \( \mu_j \) and \( \nu_j \) in (18). It is always possible to choose these parameters in such a way that only three of them are different from zero. For example, if \( x_0 \neq x_k \), we may choose

\[
\mu_0 = \frac{u_2^{k+q-1}(x_k + y_q) - u_1^{k+q-1}}{x_k - x_0}, \quad \mu_k = u_2^{k+q-1} - \mu_0, \quad \nu_q = u_2^{k+q-1},
\]

and all other \( \mu_j \) and \( \nu_j \) equal to zero. This formula reduces to a formula with only two terms if \( \mu_2^{k+q-1} = 0 \). This is the case when evaluating derivatives of \( M(\cdot|X) \ast M(\cdot|Y) \). Also, in certain other situations the right-hand side of (18) reduces to only two nonzero terms:

\[
C(z|X|Y)(u^1, \ldots, u^{k+q-1})
\]

\[
= \left( k + q - 1 \right) C(z|x_1 \setminus x_i|Y)(u^i, \ldots, u^{i-1}, u^{i+1}, \ldots, u^{k+q-1})
\]

\[
+ \frac{q}{k + q - 1} C(z|X|Y \setminus y_j)(u^i, \ldots, u^{i-1}, u^{i+1}, \ldots, u^{k+q-1}),
\]

(20)

when

\[
x_i + y_j = \frac{u_i^j}{u_2^j}, \quad u_2^j \neq 0.
\]

Let us end the discussion on the recurrence relation (18) by showing that the B-spline expansion of \( M(\cdot|X) \ast M(\cdot|Y) \) may be evaluated stably for arbitrary \( X, Y \). We consider \( X = \{x_0, \ldots, x_k\} \), \( Y = \{y_0, \ldots, y_q\} \) to be two nondecreasing sequences. Define the piecewise polynomial \( C(\cdot|X|Y) \) by

\[
C(x|X|Y) \overset{\text{def}}{=} C(z|X|Y)(\bar{x}, \ldots, \bar{x}).
\]

(21)

Clearly \( C(z|X|Y) \) is the local blossom of \( C(\cdot|X|Y) \) at \( z \). Moreover,

\[
C(\cdot|X|Y) = \begin{cases} 
  M(\cdot|X) \ast M(\cdot|Y), & \text{if } x_0 < x_k, \quad y_0 < y_q, \\
  M(\cdot - x_0|X), & \text{if } x_0 = \cdots = x_k, \quad y_0 < y_q, \\
  M(\cdot - y_0|X), & \text{if } x_0 < x_i, \quad y_0 = \cdots = y_q.
\end{cases}
\]

Assume the knot vector \( X \cup Y \) is a subsequence of a knot vector \( \rho \). Then, according to Theorem 8 we have that \( M(\cdot|X) \ast M(\cdot|Y) \in S_{k+q,0} \). Actually, the proof of Lemma 7 yields somewhat more:

\[
C(\cdot|X|Y) \in S_{k+q,\rho}, \quad \text{when } X \cup Y \subseteq \rho.
\]
even when \( x_0 = \cdots = x_k \) or \( y_0 = \cdots = y_q \). Note that the B-spline expansion of \( C(\cdot|X|Y) \) in \( S_{k+q,p} \) is in fact a degree raising of \( M(\cdot|X) \) when all the elements in \( Y \) equal zero:

\[
M(z|X) = \sum_i \mathbb{C}(\xi_i|X|0,\ldots,0)(\rho_{i+1},\ldots,\rho_{i+k+q-1})M_{i,k+q,p}(z),
\]

(22)

where \( \xi_i \in [\rho_i, \rho_{i+k+q}] \). If we set \( q = 1 \), this is the formula in [3] used for raising the degree by 1 of a B-spline expansion.

The following corollary shows that the B-spline expansion of \( C(\cdot|X|Y) \) and hence of \( M(\cdot|X) \ast M(\cdot|Y) \) in \( S_{k+q,p} \) can be evaluated stably by the recurrence relation of Theorem 12 for arbitrary \( X, Y \) such that \( X \cup Y \) is a subsequence of \( \rho \).

**Corollary 13.** Assume \( X \cup Y \subseteq \rho \). Then,

\[
\mathbb{C}(z|X|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-1}) \geq 0, \quad z \in [\rho_i, \rho_{i+k+q}].
\]

If \( \rho_i = \cdots = \rho_{i+k+q} \), the inequality remains valid with \( z = \rho_i \).

**Proof.** The spline \( C(\cdot|X|Y) \) is contained in \( S_{k+q,p} \). Consequently, by uniqueness of B-spline expansion in \( S_{k+q,p} \), its \( i \)th B-spline coefficient \( \mathbb{C}(z|X|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-1}) \) is constant with respect to \( z \) for \( z \in [\rho_i, \rho_{i+k+q}] \). Assuming \( X \) and \( Y \) to be nondecreasing and recalling the support properties (14) of \( \mathbb{C}(\cdot|X|Y) \), we must have

\[
\mathbb{C}(z|X|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-1}) = 0,
\]

(23)

whenever

\[
z \in [\rho_i, \rho_{i+k+q}] \quad \text{and} \quad [\rho_i, \rho_{i+k+q}] \not\subseteq [x_0 + y_0, x_k + y_q].
\]

Here we may exchange \([\rho_i, \rho_{i+k+q}]\) with the point \( \{\rho_i\} \) if \( \rho_i = \cdots = \rho_{i+k+q} \). To prove the corollary we use induction on the order \( k + q \). By (19) the assertion holds for \( k + q = 1 \). Assume \( k + q > 1 \) and let \( \alpha \) be such that \( \rho_{i+k+q-1} = \alpha(x_0 + y_0) + (1 - \alpha)(x_k + y_q) \). By (23) we may assume \( i \) is such that \( \alpha \in [0, 1] \). According to Theorem 12 we have

\[
\mathbb{C}(z|X|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-1})
= \frac{1}{\alpha(k\mathbb{C}(z|X\setminus x_0|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-2}) + q\mathbb{C}(z|X\setminus y_0|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-2})}
+ \frac{1}{1 - \alpha}(k\mathbb{C}(z|X\setminus x_k|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-2})
+ q\mathbb{C}(z|X\setminus y_q|Y)(\bar{\rho}_{i+1},\ldots,\bar{\rho}_{i+k+q-2})).
\]

We may choose \( z \) freely in \([\rho_i, \rho_{i+k+q}]\), and setting \( z = \rho_i \) causes each of the terms in the sum to satisfy the hypothesis of the corollary for the order \( k + q - 1 \). With this choice of \( z \) the right-hand side is then by hypothesis a sum of four nonnegative terms. \( \square \)

While proving the corollary, we derived a stable four-term recurrence relation for the evaluation of the B-spline coefficients of \( C(\cdot|X|Y) \). One may also derive three-term recurrence relations for the
evaluation of these coefficients. Moreover, the evaluation can be performed stably by combining two or more of such three-term relations.

Considering our smoothing operator $\phi_t * (\cdot)$ where $\phi$ is a B-spline, we conclude that a B-spline expansion of the approximation $\phi_t * f$ of a spline $f$ may be calculated for arbitrarily small $t$ without running into serious numerical problems. Thus we are able to construct a smooth spline approximating $f$ as closely as we like.

In Fig. 2 we have plotted a piecewise linear spline and the result of convolving it with two second-order B-splines with different supports. Fig. 3 shows a piecewise constant spline and the result of convolving it with third-order B-splines. All the convolved curves are $C^2$ cubic splines. In order to plot these cubic splines we have first constructed knot vectors according to Theorem 8 and then we have found the linear B-spline expansions relative to the spline spaces given by these knot vectors using a four-term version of the blossoming algorithm in Theorem 12.

Fig. 2. Top: a piecewise linear spline (dashed) and its convolution with $M(\cdot - 0.15,0,0.15)$ (solid line). Bottom: the same spline and its convolution with $M(\cdot - 1,0,1)$. Both convolved curves are $C^2$ cubic.

Fig. 3. Top: a piecewise constant spline (dashed) and its convolution with $M(\cdot - 0.1,-0.03,0.03,0.1)$ (solid line). Bottom: the same spline and its convolution with $M(\cdot - 1,-0.3,0.3,1)$. Both convolved curves are $C^2$ cubic.
3.2. Connections to the Schoenberg variation diminishing spline approximation

The Schoenberg variation diminishing spline operator is widely used to construct smooth splines approximating given data. Other positive spline operators are studied, e.g., in [24]. The Schoenberg approximation from $S_{k,r}$ to given data $f$ is defined as

$$Vf = \sum_i f(\tau_i^*) \frac{\tau_{i+k} - \tau_i}{k} M(\cdot | \tau_i, \ldots, \tau_{i+k}),$$

where

$$\tau_i^* = \frac{1}{k-1} (\tau_{i+1} + \cdots + \tau_{i+k-1}).$$

The Schoenberg operator has excellent shape-preserving properties. For instance, it reproduces straight lines and it maps convex functions into convex functions. These are properties that are shared with the convolution operator $\phi * (\cdot)$ when $\phi$ satisfies (2). Moreover, $\phi * f \in S_{k+q,r|\tau_X}$ when $\phi = M(\cdot | X)$ is a spline of order $q$ and $f \in S_{k,r}$. In this case it is natural to compare $\phi * f$ with the Schoenberg approximation to $f$ in $S_{k+q,r|\tau_X}$. The comparison is complicated for general $\tau$ and $X$, so we concentrate on the uniform case. Also, this gives us the opportunity to demonstrate the simplicity of $\phi * (\cdot)$ in the uniform situation. Let $\phi$ be a B-spline with knots at the integers or at the integers translated with $\frac{1}{2}$:

$$\phi = M(\cdot + \frac{1}{2} k | 0, \ldots, k),$$

and assume $f$ is a B-spline series with knots at the integers:

$$f = \sum_{i \in \mathbb{Z}} c_i M(\cdot | i, \ldots, i + q).$$

The convolution of two B-splines with knots at the integers is a new B-spline with knots at the integers, so by the translation property of convolution we have

$$\phi * f = \sum_{\alpha \in \mathbb{Z} - \frac{1}{2} k} c_{\alpha+k/2} M(\cdot | \alpha, \alpha + 1, \ldots, \alpha + k + q). \quad (24)$$

Thus, in the uniform situation convolving a B-spline series with a B-spline amounts to keeping the coefficients and substituting the B-splines in the series with higher-order ones. The index set $\mathbb{Z} - \frac{1}{2} k$ over which the sum is taken is the set of integers if $k$ is even and the set of integers translated with $\frac{1}{2}$ if $k$ is odd. The corresponding Schoenberg approximation to $f$ is

$$Vf = \sum_{\alpha \in \mathbb{Z} - \frac{1}{2} (k+q)} f(\alpha + \frac{1}{2} (k+q)) M(\cdot | \alpha, \alpha + 1, \ldots, \alpha + k + q). \quad (25)$$

A first-order uniform B-spline is the characteristic function of an interval of unit length, while a second order B-spline with knots at the integers takes on the value 1 at one integer and zero at all other. Thus,

$$f(\alpha + \frac{1}{2} (k+q)) = c_{\alpha+k/2}, \quad q = 1, 2,$$
and we see that the Schoenberg approximation to $f$ is the convolution of $f$ with a B-spline when $f$ is piecewise constant or piecewise linear:

$$Vf = \phi \ast f, \quad \text{for } q = 1, 2. \quad (26)$$

For higher orders the two operators differ. This can be seen by applying them to the monomials. The $q$th-order uniform B-spline coefficients of $f$ are

$$c_i = \prod_{j=1}^{q-1} (i + j), \quad \text{whenever } f(x) = x^{q-1}.$$

So, for this $f$ we have

$$Vf - \phi \ast f = \sum_{\alpha \in \mathbb{Z} - k/2} \left( (\alpha + \frac{1}{2} (k + q))^{q-1} - \prod_{j=1}^{q-1} (\alpha + \frac{1}{2} k + j) \right) M(\cdot | \alpha, \alpha + 1, \ldots, \alpha + k + q).$$

In particular, we have $Vf - \phi \ast f = \frac{1}{4}$ when $f(x) = x^2$, independent of the difference $k$ between the order of $f$ and the order of $Vf$ and $\phi \ast f$. In Fig. 4 we have plotted $f$ and $\phi \ast f$ together with the corresponding Schoenberg approximation $Vf$ for $f = x^2, x^3$ and $k = 2$.

### 3.3. Tensor product surfaces

The method described for convolving spline curves is easily extended to handle convolution of tensor product spline surfaces. We believe this technique is of interest for geometric modeling. Convolving a tensor product surface with a tensor product B-spline provides us with an approximation method working entirely within the setting of tensor product B-spline surfaces. It smooths edges and corners, but preserves shape properties like local monotonicity and convexity. Moreover, on regions where the original spline is affine, the method reproduces the original spline except "near" the border of the region. What is meant by "near" is determined by the size of the B-spline’s support.

With the tensor product $f \otimes g$ of two univariate functions $f, g$ we understand the bivariate function

$$f \otimes g(x, y) = f(x) g(y).$$
If $V$ and $W$ are vector spaces of functions over $\mathbb{R}$, the tensor product $V \otimes W$ is the vector space
\[ V \otimes W = \text{span}\{f \otimes g: f \in V, g \in W\}. \]

It follows from the definition of convolution that convolving tensor product surfaces can be done by taking tensor products of convolved curves:
\[ (f_1 \otimes g_1) \ast (f_2 \otimes g_2) = (f_1 \ast f_2) \otimes (g_1 \ast g_2), \]
whenever $f_1, f_2, g_1, g_2$ are all integrable. As a consequence we may easily construct tensor product spline spaces just large enough to contain the convolution of two tensor product spline spaces. It follows from (27) that
\[ (S_{k_1,r_1} \otimes S_{k_2,r_2}) \ast (S_{q_1,r_1} \otimes S_{q_2,r_2}) = (S_{k_1,r_1} \ast S_{q_1,r_1}) \otimes (S_{k_2,r_2} \ast S_{q_2,r_2}) \subseteq S_{k_1+q_1,r_1,r_2} \otimes S_{k_2+q_2,r_2,r_2}. \]

The last inclusion follows from Theorem 8. We note for example that the convolution of two continuous piecewise bilinear splines is a $C^2$ bicubic spline.

Just like for curves, special attention should be given the case when one of the spline spaces spanned by a single B-spline. This is the case for $S_{k_1,r_1} \otimes S_{q_2,r_2}$ when the numbers of elements in $t^1$ and $t^2$ are $q_1 + 1$ respectively $q_2 + 1$. Hence, with
\[ \phi = M(\cdot | X) \otimes M(\cdot | Y), \]
we know a spline space just large enough to contain the image of $S_{k_1,r_1} \otimes S_{k_2,r_2}$ under the operator $\phi \ast (\cdot)$. Moreover, all the shape-preserving properties of $\phi \ast (\cdot)$ described in Section 1 are achieved just like in the univariate case. A tensor product B-spline $\phi$ is locally supported, smooth, nonnegative and of unit integral. The remaining property of those listed in (2) is symmetry. The desired symmetry is obtained if the knots are symmetric with respect to the origin:
\[ \phi(-x,-y) = \phi(x,y), \quad \text{if } X = -X, \ Y = -Y \]
when $X$ and $Y$ are considered as tuples without ordering. Finally we note that the function $\phi_t(x, y) = t^{-2}\phi(x/t, y/t)$, used in Proposition 2 describing approximation by convolution, is obtained by scaling the knots:
\[ \phi_t = M(\cdot | tx) \otimes M(\cdot | ty). \]

Linear B-spline expansions of convolved tensor product surfaces may be derived with the techniques developed for curves. First of all we recall the one to one correspondence between bivariate polynomials of degree \( \leq (m, n) \) and the bisymmetric multilinear forms mapping $(\mathbb{R}^2)^m \times (\mathbb{R}^2)^n$ into $\mathbb{R}$ [21]. If $b_0, \ldots, b_m$ and $\tilde{b}_0, \ldots, \tilde{b}_n$ are bases for univariate polynomials of degree \( \leq m \) respectively \( \leq n \). and $g$ is the tensor product polynomial
\[ g = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} b_i \otimes \tilde{b}_j, \]
then the bisymmetric multilinear blossom of $g$, $B_{\text{TP}}(g) : (\mathbb{R}^2)^m \times (\mathbb{R}^2)^n \rightarrow \mathbb{R}$ is given by
\[ B_{\text{TP}}(g) (u^1, \ldots, u^m; v^1, \ldots, v^n) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} B(b_i) (u^i, \ldots, u^m) B(\tilde{b}_j) (v^1, \ldots, v^n), \quad u^i, v^j \in \mathbb{R}^2, \]
where $B(b_i)$ is the ordinary blossom of $b_i$. For a bivariate piecewise polynomial $f$ of degree $\leq (m, n)$ we understand by $f_a$ the local polynomial of $f$ at the point $a$, and by $B_{TP}(f_a)$ we mean the local bisymmetric blossom.

The tensor product B-spline expansion of $f \in S_{m+1, p} \otimes S_{n+1, q}$ is written in terms of bisymmetric blossoms as

$$f = \frac{1}{(m+1)(n+1)} \sum_{i,j} (\rho_{i+m+1} - \rho_i) (\sigma_{j+n+1} - \sigma_j) \beta_{i,j} M_{i,m+1,p} \otimes M_{j,n+1,q},$$

where

$$\beta_{i,j} = B_{TP}(f_{x_{ij}})(\tilde{\rho}_{i+m}, \ldots, \tilde{\rho}_{i+m}; \tilde{\sigma}_{j+n}, \ldots, \tilde{\sigma}_{j+n}),$$

for arbitrary points $x_{ij} \in [\rho_i, \rho_{i+m}] \times [\sigma_j, \sigma_{j+n}]$ [21]. Hence we know the tensor product B-spline expansion of $f$ if we know the local bisymmetric blossom of $f$.

When $f$ is a convolution of two tensor product B-splines or a known linear combination of such, we can find the local blossoms of $f$. Suppose knot sets $X^i, Y^j$ consist of $k_i + 1$ respectively $q_j + 1$ elements $i = 1, 2$, and that $m = k_1 + q_1 - 1, n = k_2 + q_2 - 1$. The bisymmetric blossom of

$$f = (M(\cdot|X^1) \otimes M(\cdot|X^2)) \ast (M(\cdot|Y^1) \otimes M(\cdot|Y^2))$$

is found by invoking the recurrence relation for the local blossom of convolution of univariate B-splines. This is so because in this case, unless $f$ is completely vanishing, we have, by applying (27),

$$B_{TP}(f_a)(u^1, \ldots, u^m, v^1, \ldots, v^n) = C(a_1|X^1|Y^1)(u^1, \ldots, u^m)C(a_2|X^2|Y^2)(v^1, \ldots, v^n), \quad (28)$$

where $a = (a_1, a_2)$, and $C(a_i|X^i|Y^i)$ is defined in (13).

In Fig. 5 we have plotted a bilinear surface $f$ and the convolutions of $f$ with two bilinear tensor product B-splines. In the figure the convolution surfaces are scaled.

### 4. Box splines and simplex splines

In Section 2 we saw that the convolution product with a function $\phi$ satisfying (2) is a linear operator with good approximation and shape-preserving properties. In the multivariate case we have seen that we can easily achieve tensor product B-splines satisfying (2), and therefore we can treat tensor product surfaces. But we may also apply the convolution technique to more general polyhedral spline surfaces. Results for convolutions of polyhedral splines are obtained in [6]. In this section we make some remarks on the special cases of box splines and simplex splines, where we notice that several of these B-splines satisfy (2). At the end of the section we comment in particular on recurrence relations for evaluating and differentiating the convolution of two simplex splines.

A box spline $B(\cdot|X)$ with knot directions $\{x^1, \ldots, x^k\} = X \subseteq \mathbb{R}^k$ where $X$ spans $\mathbb{R}^k$ is defined by requiring that

$$\int_{\mathbb{R}^k} B(u|X) f(u) \, du = \int_{[0,1]^k} f(\nu_1 x^1 + \cdots + \nu_k x^k) \, d\nu$$

where $\nu_i$ are the components of $\nu$. These integrals form a basis for the space of all functions on $X$. The resulting box splines are piecewise polynomials that are linear in each variable and continuous up to the boundary of the box.

When $f$ is a convolution of two tensor product B-splines or a known linear combination of such, we can find the local blossoms of $f$. Suppose knot sets $X^i, Y^j$ consist of $k_i + 1$ respectively $q_j + 1$ elements $i = 1, 2$, and that $m = k_1 + q_1 - 1, n = k_2 + q_2 - 1$. The bisymmetric blossom of

$$f = (M(\cdot|X^1) \otimes M(\cdot|X^2)) \ast (M(\cdot|Y^1) \otimes M(\cdot|Y^2))$$

is found by invoking the recurrence relation for the local blossom of convolution of univariate B-splines. This is so because in this case, unless $f$ is completely vanishing, we have, by applying (27),

$$B_{TP}(f_a)(u^1, \ldots, u^m, v^1, \ldots, v^n) = C(a_1|X^1|Y^1)(u^1, \ldots, u^m)C(a_2|X^2|Y^2)(v^1, \ldots, v^n), \quad (28)$$

where $a = (a_1, a_2)$, and $C(a_i|X^i|Y^i)$ is defined in (13).

In Fig. 5 we have plotted a bilinear surface $f$ and the convolutions of $f$ with two bilinear tensor product B-splines. In the figure the convolution surfaces are scaled.
The two convolution surfaces are $C^2$ bicubic.

is valid for all locally integrable $f$ [13]. It can be shown that $B(\cdot | X)$ is a piecewise polynomial of degree $k - s$ supported on $\{\nu_1 x^1 + \cdots + \nu_k x^k : 0 \leq \nu_i \leq 1, i = 1, \ldots, k\}$ which has $m - 1$ continuous derivatives if every subset of $k - m$ elements from $X$ spans $\mathbb{R}^s$. From the definition it follows that

$$\int B(u|X) \, du = 1$$

and that

$$B(x^* + u|X) = B(x^* - u|X), \quad u \in \mathbb{R}^s,$$

where $x^* = \frac{1}{s} \sum_{i=1}^{k} x^i$. So, a good candidate for the approximation operator $\phi \ast (\cdot)$ is obtained by choosing

$$\phi = B(\cdot + x^*|X).$$

This $\phi$ has the properties listed in (2). In particular, $\phi$ has the symmetry property required for $\phi \ast (\cdot)$ to reproduce affine functions. From the definition of box spline we see that the function
\( \phi_t(x) = t^{-s} \phi(t^{-1} x), \ t > 0, \) used in Proposition 2 describing approximation by convolution, is given as

\[ \phi_t = B(\cdot + t x^s | t X). \]

Important for the evaluation of \( \phi \ast f \) when \( f \) is a box spline surface is the convolution identity

\[ B(\cdot | X) \ast B(\cdot | Y) = B(\cdot | X, Y), \]

where the knot directions of \( B(\cdot | X, Y) \) are \( \{x^0, \ldots, x^k, y^0, \ldots, y^q\} \). Using the translation property of the convolution product we may now evaluate the image of a box spline surface under \( \phi \ast (\cdot) \) as

\[ \phi \ast \sum_{y \in E} c_y B(\cdot - y | Y) = \sum_{y \in E} c_y B(\cdot - y + x^* | X, Y), \]

where \( E \) is a subset of \( \mathbb{R}^s \) finite on compacts. Usually one works with translates of box splines that are piecewise polynomials over the same mesh of \( \mathbb{R}^s \). For our applications \( B(\cdot | X) \) will typically be of smaller support than \( B(\cdot | Y) \), so it will be necessary to refine their representation to a common mesh before performing the convolution. Performing the convolution would then be done by convolving the coefficients, just like what was the case when convolving translates of univariate B-spline series with knots at the integers.

The final piecewise polynomial we will consider is the simplex spline. Recall that simplex splines generalize the Bernstein polynomials on triangles. Moreover, it has recently been shown [26] that multivariate piecewise polynomials over arbitrary triangulations may be represented as linear combinations of simplex splines. Convolution of two piecewise polynomials over triangulations can therefore be studied by examining the convolution product of two simplex splines.

The multivariate B-spline or simplex spline was introduced in [9]. For arbitrary knots \( X = \{x^0, \ldots, x^k\} \subseteq \mathbb{R}^s \) such that

\[ \text{vol}_s([X]) > 0, \]

where \([E]\) denotes the convex hull of the set \( E \), the s-variate simplex spline \( M(\cdot | X) \) is defined by

\[ \int_{\mathbb{R}^s} f(x) M(x | X) \, dx - k! \int_{s^k} f(\hat{x} \mu) \, d\mu, \]

for all locally integrable functions \( f \). Here \( \hat{x} \mu = x^0 + \sum_{j=1}^k (x^j - x^0) \mu_j \) and the right-hand side is integrated over the standard \( k \)-simplex \( S^k \). In this section we will use the shorter notation \([E]\) for the convex hull of the set \( E \) instead of \( \text{co}(E) \) as was used above. If \( \text{vol}_s([X]) \) equals zero, the simplex spline \( M(\cdot | X) \) is also set equal to zero. It can be shown, see, e.g., [18], that \( M(\cdot | X) \) is a piecewise polynomial of degree \( k - s \) supported on \([X]\) which is of class \( C^{k-s-m} \) if the convex hull of any subset of \( s + m \) elements from \( \{x^0, \ldots, x^k\} \) has positive volume in \( \mathbb{R}^s \). From the definition it follows that \( \int M(\cdot | X) = 1 \) and

\[ M(x | x^0, \ldots, x^k) = M(x | x^{\pi(0)}, \ldots, x^{\pi(k)}), \]

\[ M(x | Ax^0, \ldots, Ax^k) = \frac{1}{|\text{det} A|} M(A^{-1} x | x^0, \ldots, x^k), \]

\[ M(x - y | x^0, \ldots, x^k) = M(x | x^0 + y, \ldots, x^k + y), \]
where $\pi$ is any permutation on $\{0, \ldots, k\}$ and $A$ any $s \times s$ invertible matrix.

From these properties of a simplex spline we see that it is easy to choose knots $\{x_0, \ldots, x^k\}$ such that the convolution product with $M(\cdot | X)$ is an operator with good approximation and shape-preserving properties. If we set

$$\phi = M(\cdot | x_0, \ldots, x^k)$$

and let $\phi_t$ be defined by $\phi_t(x) = t^{-s} \phi(t^{-1} x)$, $t > 0$, then

$$\phi_t(x) = M(x|x_0, \ldots, x^k).$$

So the distance from $\phi \ast f$ to $f$ is in view of Proposition 2 to a large extent governed by the diameter of the knot set. For $\phi \ast (\cdot)$ to reproduce affine functions, we need some symmetry of $\phi$. Again, the symmetry of $\phi$ is governed by the symmetry of the knots:

$$\phi(x) = \phi(-x), \quad \text{if } \{x_0, \ldots, x^k\} = \{-x_0, \ldots, -x^k\}.$$

In the last equation the ordering within each of the two $(k + 1)$-tuples is disregarded. Clearly, $\phi$ a suitable simplex spline will have the properties listed in (2), and for such a $\phi$ the convolution operator $\phi \ast (\cdot)$ has the approximation and shape-preserving properties discussed in Section 2.

In order to investigate properties for the convolution of two simplex splines, we consider the following polyhedral spline. Let $X = \{x_0, \ldots, x^k\}$ and $Y = \{y_0, \ldots, y^q\}$ with $k + q \geq s$, $k, q \geq 0$, be two sets in $\mathbb{R}^k$ such that

$$\text{vol}_s([X] + [Y]) > 0. \quad (29)$$

Then there exist sets $U = \{u_0, \ldots, u^k\}, V = \{v_0, \ldots, v^q\} \subseteq \mathbb{R}^{k+q}$ such that

$$\text{vol}_{k+q}([U] + [V]) > 0, \quad P(U) = X, \quad P(V) = Y$$

where $P$ is the coordinate projection

$$Pz = (z_1, \ldots, z_t), \quad z \in \mathbb{R}^{k+q}.$$

Define the $s$-variate function $C(\cdot | X | Y)$ by

$$C(x|X|Y) = \frac{\text{vol}_{k+q-s}\{z \in [U] + [V]: Pz = x\}}{\text{vol}_{k+q}([U] + [V])}. \quad (30)$$

To see that we may lift $X, Y$ to knot sets $U, V$ as stated, note that $[X] + [Y]$ is the image of $S^k \times S^q$ under the affine map $T$ given by $T(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_q) = x^0 + y^0 + \sum_{i=1}^k (x^i - x^0) \mu_i + \sum_{i=1}^q (y^i - y^0) \nu_i$. Since $[X] + [Y]$ has positive $s$-dimensional volume, $T$ is onto $\mathbb{R}^s$. Consequently, the set $\{x^1 - x^0, \ldots, x^k - x^0, y^1 - y^0, \ldots, y^q - y^0\}$ contains a subset $J$ of $s$ linearly independent vectors.

Sets $U$ and $V$ may now be obtained by augmenting the elements in $X$ and $Y$ as follows. The vectors $x^0, y^0$ and those $x^i, y^j$ such that $x^i - x^0, y^j - y^0$ are contained in $J$ are augmented with the origin of $\mathbb{R}^{k+q-s}$. The other elements in $X$ and $Y$ are augmented with the unit vectors of $\mathbb{R}^{k+q-s}$.

The definition of $C(\cdot | X | Y)$ does not depend on the particular sets $U, V$ to which $X, Y$ are lifted.
Lemma 14. Assume $\text{vol}_s([X] + [Y]) > 0$. Then, for all $f \in C^\infty$ we have
\begin{equation}
\int_{\mathbb{R}^s} C(x|X|Y) f(x) \, dx = k! \cdot q! \int_{S^s \times S^q} f(\hat{X}\mu + \hat{Y}\nu) \, d\nu \, d\mu,
\end{equation}
where $\hat{X}\mu = x^0 + \sum_{j=1}^k (x^j - x^0)\mu_j$ and $\hat{Y}\nu = y^0 + \sum_{j=1}^q (y^j - y^0)\nu_j$.

Proof. Let $X, Y$ be lifted to sets $U, V \subseteq \mathbb{R}^{k+q}$ and define the affine map $W$ of $\mathbb{R}^{k+q}$ onto itself by $W(\mu, \nu) = \hat{U}\mu + \hat{Y}\nu$. We have $W(S^k \times S^q) = [U] + [V]$ and the Jacobian $\det DW$ has determinant $|\det DW| = (k!q!)\text{vol}_{k+q}([U] + [V])$. Hence,
\begin{align*}
k!q! \int_{S^k \times S^q} f \circ W(\mu, \nu) \, d\nu \, d\mu \\
= \text{vol}_{k+q}([U] + [V])^{-1} \int_{[U] + [V]} f \circ P(z) \, dz \\
= \text{vol}_{k+q}([U] + [V])^{-1} \int_{\mathbb{R}^s} f(x) \int_{\mathbb{R}^q} \chi_{[U] + [V]}(x, \eta) \, d\eta \, dx.
\end{align*}

When $\text{vol}_s([X] + [Y]) = 0$, we identify $C(\cdot|X|Y)$ with the functional given by the right-hand side of (31). By [12], $C(\cdot|X|Y)$ is a polynomial spline of degree $\leq k + q - s$ when $k + q \geq s$. The definition assures that $C(\cdot|X|Y)$ is nonnegative and supported on $[X] + [Y]$. The simplest one looks like
\begin{equation}
C(x|X|Y) = (\text{vol}_s([X] + [Y]))^{-1}\chi_{[X] + [Y]}(x),
\end{equation}
where
\begin{align*}
X &= \{x^0, \ldots, x^k\}, \\
Y &= \{y^0, \ldots, y^q\}, \\
k + q &= s,
\end{align*}
and $[X] + [Y]$ has positive $s$-dimensional volume. From Lemma 14 we obtain
\begin{align*}
\int C(\cdot|X|Y) &= 1, \\
C(x|X + z^1|Y + z^2) &= C(x - z^1 - z^2|X|Y), \\
C(x|AX|AY) &= \frac{1}{|\det A|} C(A^{-1}x|X|Y),
\end{align*}
where $A$ is any nonsingular $s \times s$ matrix, $AX = \{Ax^0, \ldots, Ax^k\}$, and $z^1, z^2$ are arbitrary in $\mathbb{R}^s$. When $\text{vol}_s([X]) > 0$ as well as $\text{vol}_s([Y]) > 0$, it follows from [6] and Lemma 14 that
\begin{equation}
M(\cdot|X) \ast M(\cdot|Y) = C(\cdot|X|Y).
\end{equation}
In this case a recursive evaluation relation for $C(\cdot|X|Y)$ is obtained as a special case of a formula given in [6], see also [5]. It turns out that the spline $C(\cdot|X|Y)$ satisfies similar recursive differentiation and evaluation relations even when the number of elements in $X$ or $Y$ is less than or equal to $s$. More precisely, for $k + q \geq s$, knot sets $X, Y$ satisfying (29), and points $x$ at which $C(\cdot|X|Y)$ is smooth we have
\begin{equation}
d_x C(x|X|Y) = k \sum_{i=0}^k \mu_i C(x|X \setminus x^i|Y) + q \sum_{j=0}^q \nu_j C(x|X|Y \setminus y^j).
\end{equation}
whenever
\[ \sum_{i=0}^{k} \mu_i x^i + \sum_{j=0}^{q} v_j y^j = z, \quad \sum_{i=0}^{k} \mu_i = 0 = \sum_{j=0}^{q} v_j. \]

Furthermore,
\begin{equation}
C(x|X|Y) = \frac{k}{k+q-s} \sum_{i=0}^{k} \mu_i C(x|x_1x_2...|Y) + \frac{q}{k+q-s} \sum_{j=0}^{q} v_j C(x|X|Y \setminus y^j),
\end{equation}
whenever \( \mu_0, \ldots, \mu_k, v_0, \ldots, v_q \) are numbers such that
\[ \sum_{i=0}^{k} \mu_i x^i + \sum_{j=0}^{q} y^j = x, \quad \sum_{i=0}^{k} \mu_i = \sum_{j=0}^{q} v_j = 1. \]

This can be seen by modifying appropriately the discussion in [6], or by tuning the proof of [13, Theorem 1] to the special case of \( C(\cdot|X|Y) \).

We may ask if there are any regions in \( \mathbb{R}^s \) where the local blossom of \( C(\cdot|X|Y) \) can be calculated by a multilinear version of (34) similar to the recurrence relation (18) in the univariate case. This question is of interest since spaces spanned by certain simplex splines have de Boor–Fix functionals quite similar to those existing in the univariate case [8].

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References