Bounds for Determinants

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Let $A = (a_{ij})$ be an $n \times n$ complex matrix, and let $P_i = \sum_{t=1,t\neq i}^{n} |a_{it}|$. The well-known Lévy-Desplanques Theorem [9, p. 146] states that if $|a_{ii}| > P_i$ for $i = 1, \ldots, n$, then $A$ is nonsingular. This theorem has been improved in many ways. For example, each of the following is known to be a sufficient condition for the nonsingularity of $A$:

(i) $|a_{ii}| |a_{jj}| > P_i P_j$ ($i, j = 1, \ldots, n; i \neq j$) [9, p. 149].

(ii) $|a_{ii}| > P_i$ ($i = 1, \ldots, n$), provided that at least one inequality is strict and $A$ is indecomposable [9, p. 147].

(iii) $|a_{ii}| > k_i m_i$, ($i = 1, \ldots, n$), where $k_1, \ldots, k_n$ are positive numbers satisfying $\sum_{i=1}^{n} (1 + k_i)^{-1} \leq 1$, and $m_i = \max_{t \neq i} |a_{ti}|$, [13; see also 3].

(iv) $|a_{ii}| > P_i Q_i^{1-\varepsilon}$, ($i = 1, \ldots, n$), where $0 \leq \varepsilon \leq 1$, and $Q_i = \sum_{t=1,t\neq i}^{n} |a_{ti}|$ [9, p. 150].

In a recent paper by Gudkov [5], the following improvement of the Lévy-Desplanques Theorem appears:

**Theorem A.** Let $R_1 = P_1$, and, for $i = 2, \ldots, n$, let

$$R_i = \sum_{t=1}^{i-1} \frac{|a_{it}|}{|a_{tt}|} R_t + \sum_{t=i+1}^{n} |a_{it}|.$$

If $|a_{ii}| > R_i$ for $i = 1, \ldots, n$, then $A$ is nonsingular.

The same theorem is implied by results of Ostrowski [15], who credits Nekrasov [10] with the discovery of these conditions. In what follows, we call a square matrix a Nekrasov matrix if it satisfies the hypothesis of Theorem A.
If \( A \) is a Nekrasov matrix, it might be expected that conditions analogous to (i)-(iv) above would be sufficient for the nonsingularity of \( A \). Surprisingly, none of these analogs is true. The matrix

\[
A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}
\]

is a counterexample to (i), (ii), and (iii). (In (iii), take \( k_1 = 10/9 \) and \( k_2 = 9/10 \), and redefine \( m_i \) as the maximum of the numbers

\[
\left\{ \left| a_{tt} \right|, \frac{R_t}{\left| a_{tt} \right|}, \ t = 1, \ldots, i - 1; \ \left| a_{tt} \right|, \ t = i + 1, \ldots, n \right\}.
\]

In (iv), let \( C_i \) be the column analog of \( R_i \). Taking \( \varepsilon = \frac{1}{4} \), the matrix

\[
A = \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & 1 & 1 \\ 19/24 & 7/12 & 13/12 \end{bmatrix}
\]

satisfies even the stronger conditions

\[
\left| a_{tt} \right| > \frac{R_t + C_i}{2} > (R_i C_i)^{1/2},
\]

yet \( A \) is singular.

A square matrix which satisfies the hypothesis of the Lévy-Desplanques Theorem is called \textit{diagonally dominant}. Positive lower bounds for the absolute value of the determinant of a diagonally dominant matrix have been discovered by many mathematicians, including Brenner [1, 2], Haynsworth [6–8], Ostrowski [11, 12, 14], Price [16], and Schneider [17].

In Theorems 1 and 2 of this paper, we generalize a result of Brenner [2] to obtain bounds for the determinant of a Nekrasov matrix. Our method yields an alternative proof of Theorem A which may be of interest in itself, since [5] and [10] are relatively inaccessible. Real Nekrasov matrices are considered in Theorems 3 and 4.

**Theorem 1.** Let \( A \) be a Nekrasov matrix, let

\[
l_i = \sum_{t=1}^{i-1} \frac{R_t}{\left| a_{tt} \right|} \quad \text{and} \quad L_i = \left| a_{ii} a_{ii}^{-1} \right| \sum_{t=i+1}^{n} \left| a_{tt} \right|.
\]

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Then

$$|\det A| \geq |a_{11}| \prod_{i=2}^{n} (|a_{ii}| - i + L_i).$$

The proof depends on a Nekrasov matrix analog of the lemma in [2]. Let $A = (a_{ij})$ be an $n \times n$ matrix, and let $B = (b_{ij})$ be a matrix defined by

$$b_{ij} = a_{ij} - a_{ii}^{-1}a_{1j}, \quad i, j = 2, \ldots, n.$$ 

Let $R_i', l_i', \text{ and } L_i'$ denote the numbers $R_i, l_i, L_i$ computed for the matrix $B$. Sylvester's determinant identity [4, p. 31] implies that $\det A = a_{11} \det B$.

**Lemma.** If $A$ is a Nekrasov matrix, then so is $B$. In fact,

$$\frac{R_i'}{|b_{ii}|} \leq \frac{R_i}{|a_{ii}|} \quad i = 2, \ldots, n.$$

**Proof.** The proof is by induction on $i$. For $i = 2$ we have

$$R_2' = \sum_{i=3}^{n} |b_{2i}| \leq \sum_{i=3}^{n} |a_{2i}| + |a_{21}a_{11}^{-1}| \sum_{i=3}^{n} |a_{1i}|$$

$$= \sum_{i=3}^{n} |a_{2i}| + |a_{21}a_{11}^{-1}|(R_1 - |a_{12}|)$$

$$= R_2 - |a_{21}a_{11}^{-1}a_{12}|.$$

Since $|b_{22} - |a_{22} - |a_{21}a_{11}^{-1}a_{12}|$ and $R_2/|a_{22}| < 1$, it follows that

$$\frac{R_2'}{|b_{22}|} \leq \frac{R_2 - |a_{21}a_{11}^{-1}a_{12}|}{|a_{22} - |a_{21}a_{11}^{-1}a_{12}|} \leq \frac{R_2}{|a_{22}|}.$$

Suppose now that $i > 2$, and that

$$\frac{R_i'}{|b_{ii}|} \leq \frac{R_t}{|a_{tt}|}; \quad t = 2, \ldots, i - 1.$$

We have

$$R_i' = \sum_{i=2}^{i-1} |b_{ii}| \frac{R_t'}{|b_{tt}|} + \sum_{i=t+1}^{n} |b_{ii}|$$
Since $\sum_{t=1}^{i-1} |a_{it}| + |a_{i1}a_{11}^{-1}a_{1i}| \frac{R_i}{|a_{tt}|} + \sum_{t=i+1}^{n} |a_{it}| + |a_{i1}a_{11}^{-1}| \sum_{t=i+1}^{n} |a_{1t}|$

\[ \leq \sum_{t=2}^{i-1} |a_{it}| \frac{R_i}{|a_{tt}|} + |a_{i1}a_{11}^{-1}| \sum_{t=2}^{i-1} |a_{1t}| + \sum_{t=i+1}^{n} |a_{it}| + |a_{i1}a_{11}^{-1}| \sum_{t=i+1}^{n} |a_{1t}| \]

\[ = \sum_{t=2}^{i-1} |a_{it}| \frac{R_i}{|a_{tt}|} + |a_{i1}a_{11}^{-1}|(R_1 - |a_{1i}|) + \sum_{t=i+1}^{n} |a_{it}| \]

\[ = R_i - |a_{i1}a_{11}^{-1}a_{1i}|. \]

Since $|b_{ii}| \geq |a_{ii} - |a_{i1}a_{11}^{-1}a_{1i}|$ and $R_i/|a_{ii}| < 1$, it follows that

\[ \frac{R_i'}{|b_{ii}|} \leq \frac{R_i - |a_{i1}a_{11}^{-1}a_{1i}|}{|a_{ii} - |a_{i1}a_{11}^{-1}a_{1i}|} \leq \frac{R_i}{|a_{ii}|}, \quad i = 2, \ldots, n. \]

It follows immediately that $B$ is a Nekrasov matrix, so the Lemma is proved. We remark that, since $\det A = a_{11} \det B$, the Lemma yields a new proof of Theorem A, by induction on the order of $A$.

**Proof of Theorem 1.** We proceed by induction on $n$, the theorem being obvious in case $n = 1$. Suppose that $n > 1$, and that the theorem is true for Nekrasov matrices of order smaller than $n$. By the Lemma, $B$ is a Nekrasov matrix. Thus

\[ |\det A| = |a_{11}| \det B \geq |a_{11}| |b_{22}| \prod_{t=3}^{n} (|b_{tt}| - l_t' + L_t'). \]

Now

\[ |b_{22}| \geq |a_{22} - |a_{21}a_{11}^{-1}a_{12}| \]

\[ = |a_{22} - |a_{21}a_{11}^{-1}|(R_1 - \sum_{t=3}^{n} |a_{1t}|) \]

\[ = |a_{22}| - l_2 + L_2. \]

Thus the theorem will be proved if we show, for $i > 2$, that

\[ |b_{ii}| - l_i' + L_i' \geq |a_{ii}| - l_i + L_i. \]

Using the Lemma, we proceed as follows:

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Some remarks are in order. The numbers $L_i$ in Theorem 1 are identical with those of Brenner [2, Theorem 1]. (There is a minor misprint in the definition of $L_i$ in [2].) However, in defining $l_i$, we have used the factors $R_i/a_{ii}$, which are generally smaller than Brenner's factors $P_i/a_{ii}$ and thus our lower bound for $|\det A|$ is generally larger than his. Moreover, our theorem applies to all Nekrasov matrices, a larger class than the class of diagonally dominant matrices.

The following theorem gives, in the notation of Theorem 1, an upper bound for the absolute value of the determinant of a Nekrasov matrix. The proof is entirely analogous to the proof of Theorem 1.

**Theorem 2.**

$$|\det A| \leq |a_{ii}| \prod_{i=2}^{n} (|a_{ii}| + l_i - L_i).$$

In case $A$ is a real Nekrasov matrix, the following result holds (cf. [18, Theorem IV]):

**Theorem 3.** Let $A$ be a real Nekrasov matrix with positive main diagonal elements. Then $\det A > 0$.

**Proof.** The Nekrasov matrix $B$ defined in the Lemma has main diagonal elements

$$b_{ii} = a_{ii} - a_{ii}a_{ii}^{-1}a_{ii}$$

$$\geq a_{ii} - |a_{ii}a_{ii}^{-1}a_{ii}|$$

$$\geq a_{ii} - |a_{ii}a_{ii}^{-1}|R_i$$

$$\geq a_{ii} - R_i > 0.$$
Since $\det A = a_{11} \det B$, the theorem follows by induction on the order of $A$.

Applying Theorem 3, we can restate the bounds of Theorems 1 and 2.

**Theorem 4.** Let $A$ be a real Nekrasov matrix with positive main diagonal elements. Then

$$a_{11} \prod_{i=2}^{n} (a_{ii} - l_i + L_i) \leq \det A \leq a_{11} \prod_{i=2}^{n} (a_{ii} + l_i - L_i).$$

We remark in closing that in each of the above theorems (including Theorem A), the hypothesis $|a_{11}| > R_1$ need not be satisfied, provided only that $a_{11} \neq 0$. The reason is that the similarity transformation effected by dividing the first row of $A$ by a positive number $p$ and then multiplying the first column by $p$ leaves $\det A, a_{ii}, l_i, L_i$, and, for $i > 1$, $R_i$ all unchanged. We are indebted to J. L. Brenner for pointing out that this phenomenon is related to condition (i): $|a_{ii}| |a_{jj}| > P_i P_j$, stated earlier. In fact, condition (i) implies that one of the inequalities $|a_{ii}| > P_i$ is permitted to fail, provided that the remaining diagonal elements dominate strongly enough to compensate for this single failure. (See [1, Section 6].) In our setting, a failure can in general occur only in the first row, and it need not be compensated for.

**REFERENCES**


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