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Solitons and periodic solutions for the fifth-order KdV equation

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Abstract

In this work we use the sine–cosine and the tanh methods for solving the fifth-order nonlinear KdV equation. The two methods reveal solitons and periodic solutions. The study confirms the power of the two schemes.

Keywords: KdV equation; Fifth-order KdV (fKdV) equation; Sine–cosine method; The tanh method; Solitons; Periodic solutions

1. Introduction

This work is concerned with the fifth-order KdV equation of the form [1–6]

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{1}$$

where $u = u(x, t)$ is a sufficiently often differentiable function. Eq. (1) is a special case of the standard fifth-order KdV equation (fKdV)

$$u_t + \alpha u^2u_x + \beta u_xu_{xx} + \gamma uu_{3x} + u_{5x} = 0. \tag{2}$$

The specific case (1) is called the Lax case, that is characterized by $\beta = 2\gamma$ and $\alpha = \frac{3}{10}\gamma^2$. We shall assume that the solution $u(x, t)$, along with its derivatives, tends to zero as $|x| \to \infty$. The fKdV equation (1) describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [1–6]. The nonlinear fKdV equation (1) is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory.

A great deal of research work has been invested during the past decades in the study of the fKdV equation. The main goal of these studies was its analytical and numerical solution. Several different approaches, such as Backland transformation, a bilinear form, and a Lax pair, have been used independently, by which soliton and multi-soliton solutions are obtained. Ablowitz et al. [4] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance where soliton solutions and rational solutions were developed.

Solitons are nonlinear waves that are characterized by:

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(i) localized waves that propagate without change of identity and whose character resembles particle-like behavior, and
(ii) stability against mutual collisions and width dependent on amplitude.

The objectives of this work are twofold. Firstly, we seek to establish exact solutions for the fKdV equation. Secondly, we aim to implement two strategies to achieve our goal, namely, the tanh method [7–10] and the sine–cosine method [11–14], and to emphasize the applicability of these methods in handling nonlinear problems.

The sine–cosine method [11–14] and the tanh method [7–10] have the advantage of reducing the nonlinear problem to a system of algebraic equations that can be easily solved by using a symbolic computation system such as *Mathematica* or *Maple*. The power of the two methods that will be used derives from the ease of use for determining shock or solitary types of solution. In what follows, the sine–cosine ansatz and the tanh method will be reviewed briefly.

2. The sine–cosine method

The features of this method can be summarized as follows. A PDE

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{3}$$

can be converted to an ODE

$$Q(u, u', u'', u''', \ldots) = 0, \tag{4}$$

upon using a wave variable $\xi = (x - ct)$. Eq. (4) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. The solutions of the reduced ODE equation can be expressed in the form

$$u(x,t) = \begin{cases} \{\lambda \cos^{\beta}(\mu\xi)\}, & |\xi| \le \dfrac{\pi}{2\mu}, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

or in the form

$$u(x,t) = \begin{cases} \{\lambda \sin^{\beta}(\mu\xi)\}, & |\xi| \le \dfrac{\pi}{\mu}, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $\lambda, \mu,$ and β are parameters that will be determined, μ and c are the wavenumber and the wave speed respectively. These assumptions give

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta - 1)\cos^{n\beta-2}(\mu\xi), \tag{7}$$

and

$$\begin{aligned} u^{(iv)} = {} & \mu^4\beta^4\lambda\cos^{\beta}(\mu\xi) - 2\mu^4\lambda\beta(\beta-1)(\beta^2 - 2\beta + 2)\cos^{\beta-2}(\mu\xi) \\ & + \mu^4\lambda\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(\mu\xi), \end{aligned} \tag{8}$$

where similar equations can be obtained for the sine assumption. Using the sine–cosine assumptions and its derivatives in the reduced ODE gives a trigonometric equation in $\cos^R(\mu\xi)$ or $\sin^R(\mu\xi)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosines or sines to determine R. We next collect all coefficients of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$, where these coefficients have to vanish. This gives a system of algebraic equations in the unknowns β, λ and μ that will be determined. The solutions proposed in (5) and (6) follow immediately.

3. The tanh method

The tanh method is developed by Malfliet [7–9] where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh also.

Introducing a new independent variable

$$Y = \tanh(\mu\xi), \tag{9}$$

leads to the change of derivatives

$$\begin{aligned}
\frac{d}{d\xi} &= \mu(1-Y^2)\frac{d}{dY},\\
\frac{d^2}{d\xi^2} &= -2\mu^2 Y(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2},\\
\frac{d^3}{d\xi^3} &= 2\mu^3(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6\mu^3 Y(1-Y^2)^2\frac{d^2}{dY^2} + \mu^3(1-Y^2)^3\frac{d^3}{dY^3},\\
\frac{d^4}{d\xi^4} &= -8\mu^4 Y(1-Y^2)(3Y^2-2)\frac{d}{dY} + 4\mu^4(1-Y^2)^2(9Y^2-2)\frac{d^2}{dY^2}\\
&\quad - 12\mu^4 Y(1-Y^2)^3\frac{d^3}{dY^3} + \mu^4(1-Y^2)^4\frac{d^4}{dY^4}.
\end{aligned} \tag{10}$$

We then apply the following finite series expansion:

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{11}$$

where M is a positive integer, in most cases, that will be determined. However, if M is not an integer, a transformation formula is usually used to overcome this difficulty. Substituting (10) and (11) into the ODE results in an algebraic equation in powers of Y.

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k(k = 0\cdots M)$, μ, and c. Having determined these parameters, knowing that M is a positive integer in most cases, and using (11), we obtain an analytic solution $u(x,t)$ in a closed form.

4. The fKdV equation approached by the sine–cosine method

The fifth-order KdV equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{12}$$

can be written as

$$u_t + 10(u^3)_x + 10(uu_{xx})_x + 5((u_x)^2)_x + u_{5x} = 0, \tag{13}$$

that can be converted to the ODE

$$-cu + 10u^3 + 10uu'' + 5(u')^2 + u^{(iv)} = 0, \tag{14}$$

upon using the wave variable $\xi = x - ct$ and integrating once. Using the assumptions (5)–(8) in (14) gives

$$\begin{aligned}
&-c\lambda\cos^{\beta}(\mu\xi) + 10\lambda^3\cos^{3\beta}(\mu\xi)\\
&\quad + 10\lambda\cos^{\beta}(\mu\xi)\left[-\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi)\right]\\
&\quad + 5\left[\lambda^2\mu^2\beta^2\cos^{2\beta-2}(\mu\xi) - \lambda^2\mu^2\beta^2\cos^{2\beta}(\mu\xi)\right]\\
&\quad + \lambda\mu^4\beta^4\cos^{\beta}(\mu\xi) - 2\lambda\mu^4\beta(\beta-1)(\beta^2-2\beta+2)\cos^{\beta-2}(\mu\xi)\\
&\quad + \lambda\mu^4\beta(\beta-1)(\beta-2)(\beta-3)\cos^{\beta-4}(\mu\xi) = 0.
\end{aligned} \tag{15}$$

Balancing $\cos^{3\beta}(\mu\xi)$ with $\cos^{\beta-4}(\mu\xi)$ gives

$$3\beta = \beta - 4, \tag{16}$$

so that $\beta = -2$. Using this value for β, Eq. (15) becomes

$$\left(-c\lambda + 16\lambda\mu^4\right)\cos^{-2}(\mu\xi) - \left(15\lambda^2\mu^2\beta^2 + 2\lambda\mu^4\beta(\beta-1)(\beta^2-2\beta+2)\right)\cos^{-4}(\mu\xi)$$
$$\left(10\lambda^3 + 10\lambda^2\mu^2\beta(\beta-1) + 5\lambda^2\mu^2\beta^2 + \lambda\mu^4\beta(\beta-1)(\beta-2)(\beta-3)\right)\cos^{-6}(\mu\xi) = 0. \tag{17}$$

Setting the coefficients of each $\cos^j(\mu\xi)$ to zero gives the system

$$\begin{aligned}
&(\beta-1)(\beta-2)(\beta-3) \neq 0,\\
&16\mu^4 = c,\\
&-15\lambda^2\mu^2\beta^2 = 2\lambda\mu^4\beta(\beta-1)(\beta^2-2\beta+2),\\
&10\lambda^3 + 10\lambda^2\mu^2\beta(\beta-1) + 5\lambda^2\mu^2\beta^2 = -\lambda\mu^4\beta(\beta-1)(\beta-2)(\beta-3).
\end{aligned} \tag{18}$$

Solving the system (18) leads to the results

$$\begin{aligned}
&\beta = -2,\\
&\mu = \frac{1}{2}\sqrt[4]{c},\\
&\lambda = -\frac{1}{2}\sqrt{c}.
\end{aligned} \tag{19}$$

This gives the periodic solutions for $u(x,t)$:

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\sec^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right), \tag{20}$$

and

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\csc^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right). \tag{21}$$

This also gives the soliton solutions for $u(x,t)$:

$$u(x,t) = \frac{1}{2}\sqrt{c}\,\operatorname{sech}^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right), \tag{22}$$

and

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\operatorname{csch}^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right). \tag{23}$$

5. The fKdV equation approached by the tanh method

In this section, we will use the tanh method to handle Eq. (14) given by

$$-cu + 10u^3 + 10uu'' + 5(u')^2 + u^{(iv)} = 0. \tag{24}$$

Balancing $u^{(iv)}$ with u^3 in (24) by using (10) we find

$$M + 4 = 3M, \tag{25}$$

so that $M = 2$. This means that

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2. \tag{26}$$

Substituting (26) into (24), collecting the coefficients of Y, and solving the resulting system, we find the following sets of solutions:

$$\begin{aligned} a_0 &= \frac{1}{2}\sqrt{c}, \\ a_1 &= 0, \\ a_2 &= -\frac{1}{2}\sqrt{c}, \\ \mu &= \frac{1}{2}\sqrt[4]{c}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} a_0 &= \frac{(\sqrt{10}+\sqrt{2})(11+5\sqrt{5})\sqrt{c}}{4(15+7\sqrt{5})}, \\ a_1 &= 0, \\ a_2 &= -\frac{\sqrt{3c+2\sqrt{5}}}{2}, \\ \mu &= \frac{1}{2}\sqrt[4]{3c+\sqrt{5}c}, \end{aligned} \tag{28}$$

Using the results (27) will give the periodic solutions

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\sec^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right), \tag{29}$$

and

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\csc^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right), \tag{30}$$

and the periodic solutions

$$u(x,t) = \frac{1}{2}\sqrt{c}\,\operatorname{sech}^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right), \tag{31}$$

and

$$u(x,t) = -\frac{1}{2}\sqrt{c}\,\operatorname{csch}^2\left(\frac{1}{2}\sqrt[4]{c}\,(x-ct)\right). \tag{32}$$

for the first set. However for the second set we find the soliton solutions

$$u(x,t) = \frac{(\sqrt{10}+\sqrt{2})(11+5\sqrt{5})\sqrt{c}}{4(15+7\sqrt{5})} - \frac{\sqrt{3c+c\sqrt{5}}}{2}\tanh^2\left[\frac{1}{2}\sqrt[4]{3c+\sqrt{5}c}\,(x-ct)\right], \tag{33}$$

and

$$u(x,t) = \frac{(\sqrt{10}+\sqrt{2})(11+5\sqrt{5})\sqrt{c}}{4(15+7\sqrt{5})} - \frac{\sqrt{3c+c\sqrt{5}}}{2}\coth^2\left[\frac{1}{2}\sqrt[4]{3c+\sqrt{5}c}\,(x-ct)\right], \tag{34}$$

and the periodic solutions

$$u(x,t) = \frac{(\sqrt{10}+\sqrt{2})(11+5\sqrt{5})\sqrt{c}}{4(15+7\sqrt{5})} + \frac{\sqrt{3c+c\sqrt{5}}}{2}\tan^2\left[\frac{1}{2}\sqrt{-\sqrt{3c+\sqrt{5}c}}\,(x-ct)\right], \tag{35}$$

and

$$u(x,t) = \frac{(\sqrt{10}+\sqrt{2})(11+5\sqrt{5})\sqrt{c}}{4(15+7\sqrt{5})} + \frac{\sqrt{3c+c\sqrt{5}}}{2}\cot^2\left[\frac{1}{2}\sqrt{-\sqrt{3c+\sqrt{5}c}}\,(x-ct)\right]. \tag{36}$$

6. Discussion

In this work the sine–cosine and the tanh methods were used to present an analytic study of the fifth-order KdV equation. Exact periodic and solitons solutions were obtained. The performances of the two schemes show that the two methods are powerful and reliable. However, the tanh method provided two sets of solutions, whereas the sine–cosine method gave only one set of solutions.

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