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Generalized Euler Integrals and A-Hypergeometric Functions

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INTRODUCTION

Let $A \subset \mathbb{Z}^n$ be a finite set. In [GGZ, GZK 1, GZK 2] we have associated to A a holonomic system of linear differential equations on a function $\Phi(v)$, $v \in \mathbb{C}^A$. We call it the *A-hypergeometric system*, and its solutions the *A-hypergeometric functions*. Their main properties will be reviewed in the next section. In [GGZ, GZK 1, GZK 2] we have constructed a basis in the space of A-hypergeometric functions consisting of series of hypergeometric type. Here we study integral representations of these functions. The corresponding integrals are of the form

$$\int_\sigma \prod P_i(x_1, ..., x_k)^{\alpha_i} x_1^{\beta_1} \cdots x_k^{\beta_k} \, dx_1 \cdots dx_k \tag{1}$$

for some Laurent polynomials P_i; the integrals are considered as functions of the coefficients of P_i. Here σ is some k-cycle; the precise meaning of the integral will be explained in Section 2. It is natural to call the integrals of type (1) *generalized Euler integrals*. They generalize the classical Euler integral

$$\int t^\alpha (1-t)^\beta \, (1-zt)^\gamma \, dt$$

representing the Gauss hypergeometric function. More generally, these integrals include as special cases the integrals of products of powers of linear functions studied in [A, VGZ].

We shall prove that any integral of type (1) satisfies a certain A-hypergeometric system. Conversely, we show that for any A the integrals of type (1) form the complete system of solutions of the A-hypergeometric equations (see Theorem 2.10 below). This is a generalization of the corresponding results in [VGZ] concerned with hypergeometric functions on

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Grassmannians. As in [VGZ] the completeness is proved under certain non-resonance conditions on the exponents α_i, β_j.

Our proof of Theorem 2.10 is based on the theory of $\mathscr{D}$-modules. In fact a large part of the theory of hypergeometric functions of several variables can be placed into the context of microlocal study of holonomic regular $\mathscr{D}$-modules on a toric variety which are smooth with respect to stratification by torus orbits.

The microlocal approach to $\mathscr{D}$-modules [MV, Gi] associates to such a module a collection of local systems of "vanishing cycles" on some open parts of conormal bundles to the strata. The local system formed by hypergeometric functions is a special case of a vanishing cycles local system corresponding to a one-point stratum in the toric variety.

In contrast with [VGZ] we do not construct explicitly the cycles σ giving the required number of Euler integrals. It is a very interesting problem to give such a construction. As in [VGZ] our proof is based on irreducibility of the monodromy representation. To prove it we use the classification of irreducible perverse sheaves by means of Deligne–Goresky–MacPherson extensions of local systems [BBD].

The paper is organized as follows. In Section 1 we review the main properties of A-hypergeometric systems. In Section 2 we define Euler integrals and state our main results: Theorems 2.7, 2.10, 2.11. Section 3 is devoted to topological considerations. Here we study the sheaf on $\mathbb{C}^A$ whose stalk at a given point consists of Euler cycles. Finally, in Section 4 we study the hypergeometric $\mathscr{D}$-module and complete the proof of the main results of Section 2.

1. The A-Hypergeometric System

1.1. *Notation and Assumptions.* Let A be a finite subset of an integral lattice $\mathbb{Z}^n$. We shall assume that A satisfies the following two conditions:

(a) A generates $\mathbb{Z}^n$ as an Abelian group.

(b) There is a group homomorphism $h: \mathbb{Z}^n \to \mathbb{Z}$ such that $h(\omega) = 1$ for any $\omega \in A$.

Let $\mathbb{C}^A$ be the space of vectors (v_ω), $\omega \in A$, where $v_\omega \in \mathbb{C}$. Denote by $L = L(A) \subset \mathbb{Z}^A$ the lattice of relations among elements of A, i.e., the set of integer vectors (a_ω), $\omega \in A$, such that $\sum_{\omega \in A} a_\omega \omega = 0$. For any $a \in L$ we define the differential operator $\Box_a$ on $\mathbb{C}^A$ by

$$\Box_a = \prod_{\omega:\, a_\omega > 0} \left(\frac{\partial}{\partial v_\omega}\right)^{a_\omega} - \prod_{\omega:\, a_\omega < 0} \left(\frac{\partial}{\partial v_\omega}\right)^{-a_\omega}.$$

Note that (b) implies $\sum_{\omega \in A} a_\omega = 0$ for any $a \in L$, so $\Box_a$ is homogeneous.

Define also differential operators

$$Z_i = \sum_{\omega \in A} \omega_i v_\omega (\partial/\partial v_\omega), \qquad i = 1, ..., n$$

on $\mathbb{C}^A$, where ω_i is the ith coordinate of $\omega \in A \subset \mathbb{Z}^n$.

1.2. DEFINITION. Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a complex vector. The A-hypergeometric system with parameters γ is the following system of linear differential equations on a function $\Phi(v)$, $v \in \mathbb{C}^A$:

$$\Box_a \Phi = 0 \qquad (a \in L), \qquad Z_i \Phi = \gamma_i \Phi \qquad (\mathrm{i} = 1, ..., n). \tag{2}$$

This system was introduced and studied in [GGZ, GZK 1, GZK 2]. It is holonomic, and so the number of linearly independent solutions at a generic point is finite.

We shall denote by Hyp(γ) or simply Hyp the sheaf on $\mathbb{C}^A$ whose sections are hypergeometric functions, i.e., holomorphic solutions of (2). By the general theory of holonomic systems, Hyp(γ) is a constructible sheaf. By $\mathrm{Hyp}(\gamma)_v$ we denote the space of local holomorphic solutions near v, i.e., the stalk of Hyp(γ) at v.

1.3. *Generic Stratum and the Number of Solutions.* Let $Q \subset \mathbb{R}^n$ be the convex hull of A. It is a convex polytope of dimension $n-1$ lying in the hyperplane $h(u) = 1$. We introduce the volume form Vol on this hyperplane by setting the volume of an elementary simplex on the lattice $\{u \in \mathbb{Z}^n : h(u) = 1\}$ to be equal to 1.

In [GZK 3] we have associated to each face $\Gamma \subset Q$ an irreducible polynomial $\Delta_{A \cap \Gamma}$ on $\mathbb{C}^A$ called $A \cap \Gamma$-discriminant; it depends only on the variables v_ω for $\omega \in A \cap \Gamma$. Define the *generic stratum* $\mathbb{C}^A_{\mathrm{gen}} \subset \mathbb{C}^A$ by conditions $\Delta_{A \cap \Gamma} \neq 0$ for all Γ (for a geometric interpretation see proposition 1.6(b) below).

1.4. THEOREM [GZK 1, 2]. *The number of linearly independent holomorphic solutions of the hypergeometric system* (2) *at each point of* $\mathbb{C}^A_{\mathrm{gen}}$ *is equal to* Vol(Q).

In other words, the restriction of the (*constructible*) *sheaf* Hyp *to* $\mathbb{C}^A_{\mathrm{gen}}$ *is a local system of rank* Vol(Q).

More generally, we have described the whole characteristic cycle of (2) (i.e., the characteristic cycle of the corresponding $\mathscr{D}$-module, see [K, Bj]). The answer is given in terms of volumes of certain polytopes similar to Q. According to the general theory of holonomic systems the multiplicities arising in the characteristic cycle coincide with the numbers of some "microlocal solutions" of (2) which are (multivalued) microfunctions; see [K, Gi]. These "hypergeometric microfunctions" deserve further study.

1.5. *Interpretation via Fourier Transform.* Put $V = \mathbb{C}^A$. Let V^* be the dual vector space to V, and (ξ_ω), $\omega \in A$ be the coordinates dual to (v_ω). For each $a \in L$ consider a polynomial $\hat{\Box}_a(\xi) = \prod_{\omega: a_\omega > 0} \xi_\omega^{a_\omega} - \prod_{\omega: a_\omega < 0} \xi_\omega^{-a_\omega}$ which is the symbol of the differential operator $\Box_a$. Let S be the subvariety in V^* defined by equations $\hat{\Box}_a(\xi) = 0$ for all $a \in L$. As shown in [GZK 2] S is a toric variety (not necessarily normal). More precisely, consider the action of the torus $(\mathbb{C}^*)^n$ on V^* given by $(\lambda\xi)_\omega = \lambda^{-\omega}\xi_\omega$, $\omega \in A$, where $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda^\omega = \prod \lambda_i^{\omega_i}$. For any face $\Gamma \subset Q$, including Q and $\varnothing$ (where Q is the convex hull of A) consider the subvariety $S(\Gamma) \subset S$ defined by equations $\xi_\omega = 0$ for $\omega \notin \Gamma$.

1.6. Proposition [GZK 1, 2]. (a) *S is the closure of the orbit of the point* $(1, ..., 1)$. *There is a 1–1 correspondence $\Gamma \to S_0(\Gamma)$ between faces of Q and orbits of $(\mathbb{C}^*)^n$ on S such that the closure of $S_0(\Gamma)$ is $S(\Gamma)$.*

(b) *The generic stratum $\mathbb{C}^A_{\mathrm{gen}} \subset V$ is the complement of the union of the subvarieties in V projectively dual to all $S(\Gamma)$.*

Consider the Fourier transform of the system (2),

$$\hat{\Box}_a \Xi = 0 \ (a \in L), \qquad \hat{Z}_i \Xi = \gamma_i \Xi \ (i = 1, ..., n), \tag{2'}$$

where $\hat{Z}_i = -\sum \omega_i(\xi_\omega(\partial/\partial\xi_\omega) + 1)$. Since the first equations are not differential but algebraic, this system has no analytic solutions. So we can consider solutions of (2′) in the sheaf of hyperfunctions [K]. Then the first group of equations means that Ξ is supported on S and the second that Ξ is homogeneous with respect to the action of the torus.

1.7. *Functoriality.* Let $g: \mathbb{Z}^n \to \mathbb{Z}^n$ be an automorphism, i.e., an integral unimodular matrix, and let $A' = g(A)$. Then g gives a natural identification $\mathbb{C}^A \cong \mathbb{C}^{A'}$. It is straightforward to prove

1.8. Proposition. *The identification $\mathbb{C}^A \cong \mathbb{C}^{A'}$ given by g takes the solutions of the A-hypergeometric system with parameters γ to solutions of the A'-hypergeometric system with parameters $g(\gamma)$.*

2. Main Results

2.1. *Notation.* Let $A_1, ..., A_m \subset \mathbb{Z}^k$ be arbitrary finite subsets. To each element $\omega = (\omega_1, ..., \omega_k) \in \mathbb{Z}^k$ we associate the Laurent monomial $x^\omega = x_1^{\omega_1} \cdots x_k^{\omega_k}$ in k variables $x_1, ..., x_k$. We shall regard the vector space

$\mathbb{C}^{A_i}$ as the space of Laurent polynomials of the form $P_i(x)=\sum v_\omega x^\omega$, $\omega\in A_i$.

Let $\alpha=(\alpha_1, ..., \alpha_m)\in\mathbb{C}^m$, $\beta=(\beta_1, ..., \beta_k)\in\mathbb{C}^k$ be complex vectors. We shall study the integral

$$F_\sigma(\alpha, \beta; P)=\int_\sigma \prod P_i(x_1, ..., x_k)^{\alpha_i} x_1^{\beta_1} \cdots x_k^{\beta_k}\, dx_1 \cdots dx_k,$$

where $P=(P_1, ..., P_m)\in\prod \mathbb{C}^{A_i}$, as a multivalued function in P. Since the integrand is also multivalued, we have to explain the meaning of this integral and the domain of integration.

2.2. Precise Definition of the Integral. Denote the region $(\mathbb{C}^*)^k - \bigcup\{P_i=0\}$ by $U(P)=U(P_1, ..., P_m)$. Consider the one-dimensional local system (i.e., locally constant sheaf of $\mathbb{C}$-vector spaces) $\mathscr{L}(\alpha, \beta, P)$ on $U(P)$ defined by monodromy exponents α_i around $\{P_i=0\}$ and β_j around $\{x_j=0\}$. We shall sometimes abbreviate $\mathscr{L}(\alpha, \beta, P)$ as $\mathscr{L}(P)$ or simply $\mathscr{L}$. A section of $\mathscr{L}$ over a simply connected region $U\subset U(P)$ can be viewed as a function $f\colon U\to\mathbb{C}$ such that f is a scalar multiple of some branch of $P^\alpha x^\beta=\prod P_i(x_1, ..., x_k)^{\alpha_i} x_1^{\beta_1}\cdots x_k^{\beta_k}$. A (singular) p-chain with coefficients in $\mathscr{L}$ is a finite formal sum $\sum(\delta, f_\delta)$, where each $\delta\colon \Delta^p\to U(P)$ is a singular p-simplex in $U(P)$, and f_δ is a section of $\delta^*(\mathscr{L})$ over Δ^p. For each k-chain σ we define the integral (1) as $\sum_\delta \int_{\Delta^k} f_\delta\, dx_1 \cdots dx_k$, where the x_j's are viewed as functions $x_j(\delta(t))$ on Δ^k.

Denote by $C_p(U(P), \mathscr{L})$ the space of p-chains defined above. The boundary operator $d\colon C_p(U(P), \mathscr{L})\to C_{p-1}(U(P), \mathscr{L})$ is defined in a standard way (see, e.g., [S]). Let $H_p(U(P), \mathscr{L})$ be the homology of this chain complex.

We shall consider the integrals $F_\sigma(\alpha, \beta; P)$ only when σ is a k-cycle, i.e., $d\sigma=0$. Then $F_\sigma(\alpha, \beta; P)$ depends only on the homology class of σ. We fix α, β and consider F_σ as a multivalued analytic function of the coefficients of all the P_i, i.e., on the space $\prod\mathbb{C}^{A_i}$. More precisely, choose some initial $P=(P_1, ..., P_m)$ and a k-cycle $\sigma=\sum(\delta, f_\delta)$ in $U(P)$. Then for P' sufficiently close to P all simplices δ occuring in σ will lie in $U(P')$. There is a unique k-cycle $\sigma'=\sum(\delta, f'_\delta)$ with coefficients in $\mathscr{L}(\alpha, \beta, P')$ such that the f'_δ are obtained from f_δ by analytic continuation.

We define the germ of our multivalued function by

$$F_\sigma(\alpha, \beta; P')=\int_{\sigma'} \prod P'_i(x_1, ..., x_k)^{\alpha_i} x_1^{\beta_1} \cdots x_k^{\beta_k}\, dx_1 \cdots dx_k.$$

2.3. *Remark.* The correspondence $\sigma\to\sigma'$ defines a mapping $G_{P,P'}\colon H_k(U(P), \mathscr{L}(P))\to H_k(U(P'), \mathscr{L}(P'))$ for P' sufficiently close to P.

Intuitively, P can be "more singular" than P'. Then $G_{P,P'}$ is not an isomorphism. When P and P' are generic, $G_{P,P'}$ is the isomorphism of parallel transport with respect to the Gauss–Manin connection.

2.4. LEMMA. *For any $\varphi_1, ..., \varphi_m \in \mathbb{Z}^k$ we have the equality*

$$F_\sigma(\alpha, \beta; x^{\varphi_1}P_1, ..., x^{\varphi_m}P_m) = F_\sigma\left(\alpha, \beta + \sum \alpha_i\varphi_i; P_1, ..., P_m\right).$$

This follows at once from the definitions.

According to this lemma we can and will assume that each A_i contains 0. We shall assume also that the union of A_i generates the Abelian group $\mathbb{Z}^k$ (otherwise the integral either can be reduced to this case or is equal to 0).

2.5. *From $(A_1, ..., A_m)$ to One Set of Monomials (The Cayley Trick).* We shall construct an A-hypergeometric system having F_σ as its solution. Consider the lattice $\mathbb{Z}^m$ with the basis $e_1, ..., e_m$ and let $A = \bigcup (\{e_i\} \times A_i) \subset \mathbb{Z}^m \times \mathbb{Z}^k = \mathbb{Z}^{m+k}$. Clearly this set lies in the hyperplane $h(u) = 1$, where h: $\mathbb{Z}^{m+k} \to \mathbb{Z}$ is the sum of the first m coordinates. On the level of Laurent polynomials this amounts to associating to a collection $(P_1, ..., P_m)$ of polynomials in $x = (x_1, ..., x_k)$ a new polynomial $P(y, x) = \sum y_i P_i(x)$, where $y = (y_1, ..., y_m)$. This substitution was used by Cayley in elimination theory.
The proof of the next lemma is straightforward.

2.6. LEMMA. *If the union of A_i generates $\mathbb{Z}^k$ as an Abelian group, and each A_i contains 0 then A generates $\mathbb{Z}^{m+k}$.*

We shall use the natural identification $\mathbb{C}^A = \prod \mathbb{C}^{A_i}$ and denote a sequence $(P_1, ..., P_m)$ and the corresponding polynomial $P(y, x)$ by the same letter P. Under this identification $\mathbb{C}^A_{\text{gen}}$ corresponds to the set of $(P_1, ..., P_m)$ such that all hypersurfaces $\{P_i = 0\} \subset (\mathbb{C}^*)^k$ are smooth, intersect each other transversely, and the same conditions hold "at infinity," i.e., on a suitable compactification.

2.7. THEOREM. *For any $P = (P_1, ..., P_m) \in \mathbb{C}^A$ and any $\sigma \in H_k(U(P), \mathscr{L}(\alpha, \beta, P))$ the function $P' \to F_\sigma(\alpha, \beta; P')$ in a neighborhood of P satisfies the A-hypergeometric system (2) with parameters γ, where $(\gamma_1, ..., \gamma_n) = (\alpha_1, ..., \alpha_m, -\beta_1 - 1, ..., -\beta_k - 1)$.*

Proof. First we verify the equations $\Box_a F_\sigma = 0$. For brevity denote the integrand in (1) by $P^\alpha x^\beta\, dx$. It suffices to show that $\Box_a(P^\alpha x^\beta) = 0$ for any $a \in L(A)$. Since $A = \bigcup (\{e_i\} \times A_i)$ we can regard each $a \in \mathbb{Z}^A$ as a collection

$(a(i))$, where $a(i) \in \mathbb{Z}^{A_i}$. It is clear that $a \in L(A)$ if and only if $a(i) \in L(A_i)$ and $\sum_{\varphi \in A_i} a(i)_\varphi = 0$ for all i.

Let $d(i) = \sum a(i)_\varphi$, $\eta(i) = \sum a(i)_\varphi \, \varphi$, both sums over $\{\varphi \in A_i : a(i)_\varphi > 0\}$. Obviously, summing up the same expressions over $\{\varphi \in A_i : a(i)_\varphi < 0\}$ we obtain $-d(i)$ and $-\eta(i)$, respectively. Now it is immediate that

$$\prod_{\omega:\, a_\omega > 0} \left(\frac{\partial}{\partial v_\omega}\right)^{a_\omega} (P^\alpha x^\beta)$$

$$= P^\alpha x^\beta \prod_i \alpha_i(\alpha_i - 1) \cdots (\alpha_i - d(i) + 1)\, P_i^{-d(i)} x^{\eta(i)}$$

$$= \prod_{\omega:\, a_\omega < 0} \left(\frac{\partial}{\partial v_\omega}\right)^{-a_\omega} (P^\alpha x^\beta),$$

$$\text{so} \qquad \Box_a(P^\alpha x^\beta) = 0, \quad \text{as required.}$$

It remains to verify that $Z_i F_\sigma = \alpha_i F_\sigma$, $i = 1, ..., m$, $Z_{m+j} F_\sigma = -(\beta_j + 1) F_\sigma$, $j = 1, ..., k$. All these equations are certain quasi-homogeneity conditions. In the integrated form they appear as

$$F_\sigma(\alpha, \beta; \lambda_1 P_1, ..., \lambda_m P_m) = \left(\prod \lambda_i^{\alpha_i}\right) F_\sigma(\alpha, \beta; P_1, ..., P_m),$$

$$F_\sigma(\alpha, \beta; P_1^{(\mu)}, ..., P_m^{(\mu)}) = \left(\prod \mu_j^{-\beta_j - 1}\right) F_\sigma(\alpha, \beta; P_1, ..., P_m),$$

where $\mu = (\mu_1, ..., \mu_k) \in (\mathbb{C}^*)^k$ and $P_i^{(\mu)}(x_1, ..., x_k) = P_i(\mu_1 x_1, ..., \mu_k x_k)$. Both these conditions for λ_i, μ_j sufficiently close to 1 follow immediately from definitions. Theorem is proven. ∎

2.8. *Remarks.* (a) In [GZK 1, GZK 2] we have constructed for any regular triangulation T of the polytope Q a certain domain $U(T)$ in $\mathbb{C}^A$ and a basis in the vector space of A-hypergeometric functions on $U(T)$ consisting of so-called Γ-series (essentially the hypergeometric series in the sense of Horn [BE]. By Theorem 2.6, F_σ can be expressed in $U(T)$ as a linear combination of these Γ-series with constant coefficients.

(b) The integrals we are considering are proper, i.e., taken over compact cycles. Therefore we do not have problems of convergence. It is more common in the theory of special functions to integrate over some non-compact regions naturally connected with the integrand. For example, if all P_i have real coefficients one can consider the integral of type (1) over some connected component of $\mathbb{R}^k \cap U(P)$. If such an integral has good properties of convergence Theorem 2.6 is also true for it since it is proved in a purely formal way.

2.9. *The Non-resonance Condition.* Let $A = \bigcup(\{e_i\} \times A_i) \subset \mathbb{Z}^m \times \mathbb{Z}^k = \mathbb{Z}^{m+k}$ be as above. It is clear that any set $A \subset \mathbb{Z}^n$ satisfying the conditions (a) and (b) from 1.1 can be transformed to such a form by an automorphism of $\mathbb{Z}^n$, at least for $m = 1$, $k = n - 1$. Denote by $K \subset \mathbb{R}^n$ the convex cone generated by $A \subset \mathbb{Z}^n$. Clearly, each face of K is a cone over some face of Q, the convex hull of A. For each face $\Gamma \subset K$ of codimension 1 let $\mathrm{Lin}(\Gamma) \subset \mathbb{C}^n$ be the $\mathbb{C}$-linear span of Γ.

We say that a vector of parameters $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{C}^n$ is non-resonant (for A) if for each face $\Gamma \subset K$ of codimension 1 we have $\gamma \notin \mathbb{Z}^n + \mathrm{Lin}(\Gamma)$.

Now we can formulate the converse statement to Theorem 2.7, namely that generalized Euler integrals form a complete set of solutions for any A-hypergeometric system. Denote by

$$E = E(\alpha, \beta, P) \colon H_k(U(P), \mathscr{L}(P)) \to \mathrm{Hyp}(\gamma)_P$$

the mapping $\sigma \to F_\sigma(\alpha, \beta; P)$.

2.10. THEOREM. *Suppose that α and β in Theorem 2.6 are such that γ is non-resonant for A. Then for each $P = (P_1, ..., P_m) \in \prod \mathbb{C}^{A_i}$ the mapping $E(\alpha, \beta, P)$ is an isomorphism.*

Note that the non-resonance of γ in Theorem 2.10 implies that all $\alpha_i \neq 0$. Our proof of Theorem 2.10 is based on the following

2.11. THEOREM. *If $P \in \mathbb{C}^A_{\mathrm{gen}}$ (see 1.3) and γ is non-resonant then the monodromy representation of $\pi_1(\mathbb{C}^A_{\mathrm{gen}}, P)$ on $\mathrm{Hyp}(\gamma)_P$ is irreducible.*

Theorems 2.10 and 2.11 will be proved in the next sections.

2.12. *Remark.* Let $z_1, ..., z_n$ be the coordinates on the open orbit S_0 of the torus in S (see 1.6). The non-resonance condition means that the multi-valued function z^γ on S_0 has non-trivial monodromy along each orbit $S(\Gamma)$ of codimension 1. It seems probable that a weaker condition suffices for Theorem 2.10 (but not for Theorem 2.11), viz., that z^γ has a singularity along each $S(\Gamma)$ for $\operatorname{codim} \Gamma = 1$. (Thus it can either ramify, or have a pole). This means that γ does not lie in $(\mathbb{Z}^n \cap K) + \mathrm{Lin}(\Gamma)$. Among these "semi-non-resonant" γ there are some integer points, namely $\gamma \in \mathbb{Z}^n$ such that $(-\gamma)$ lies strictly within K. The study of Euler integrals and hypergeometric functions for such γ is very important since they are connected with polylogarithms.

2.13. *The Sheaf $\mathscr{H}(\alpha, \beta)$.* Now we give sheaf-theoretic versions of Theorems 2.10 and 2.11. Let us introduce a constructible sheaf $\mathscr{H}(\alpha, \beta)$ on $\prod \mathbb{C}^{A_i}$ whose stalk at any point P is $H_k(U(P), \mathscr{L}(\alpha, \beta, P))$. Let $U = \{(P, x) \in (\prod \mathbb{C}^{A_i}) \times (\mathbb{C}^*)^k : x \in U(P)\}$ be the disjoint union of all $U(P)$.

There is the local system $\mathscr{L}(\alpha, \beta)$ on U whose restriction on each $U(P)$ is $\mathscr{L}(\alpha, \beta, P)$. Its sections are branches of functions of the form $(P, x) \to (\text{const.})\, P^{\alpha} x^{\beta}$. Let $\pi\colon U \to \prod \mathbb{C}^{A_i}$ be the projection. Define $\mathscr{H}(\alpha, \beta) = R^k \pi_! \mathscr{L}(\alpha, \beta)$. Here $R^k \pi_!$ is the kth direct image with proper supports; see [Bo]. By definition, the stalk $\mathscr{H}(\alpha, \beta)_P$ is $H^k_c(U(P), \mathscr{L}(\alpha, \beta, P))$, the *cohomology* with compact supports. By Poincaré duality, one has isomorphisms

$$
\begin{aligned}
H^k_c(U(P), \mathscr{L}(\alpha, \beta, P)) &= H^{2k-k}(U(P), \mathscr{L}(\alpha, \beta, P)^*)^* \\
&= H_k(U(P), \mathscr{L}(\alpha, \beta, P)).
\end{aligned}
$$

It is clear that the mapping $E(\alpha, \beta, P)$ from Theorem 2.10 is induced by a morphism of sheaves $E(\alpha, \beta) : \mathscr{H}(\alpha, \beta) \to \mathrm{Hyp}(\gamma)$ by taking stalks. Note that the mapping $G_{P,P'}$ (see remark 2.3) is a special case of the "transport" map defined for any constructible sheaf.

2.14. THEOREM. *In the conditions of Theorem* 2.10 *the morphism* $E(\alpha, \beta)$ *is an isomorphism of sheaves.*

Consider also the complex $R\pi_* \mathscr{L}(\alpha, \beta)$, the full direct image.

2.15. THEOREM. $R\pi_* \mathscr{L}(\alpha, \beta)$ *is an irreducible perverse sheaf and the canonical morphism* $R\pi_! \mathscr{L}(\alpha, \beta) \to R\pi_* \mathscr{L}(\alpha, \beta)$ *is an isomorphism.*

The definition and properties of perverse sheaves will be recalled in Section 3. Theorem 2.15 will be proved in Section 3, and Theorem 2.14 in Section 4.

2.16. *Remark.* We have a natural restriction morphism res: $R^k \pi_* \mathscr{L}(\alpha, \beta)_P \to H^k(U(P), \mathscr{L}(\alpha, \beta, P))$. In general, this morphism is not an isomorphism (in contrast with the case of $R^k \pi_!$ and H^k_c). Its image consists of cohomology classes of cocycles which can be extended to $H^k(U(P'), \mathscr{L}(\alpha, \beta, P'))$ for all P' close to P. By Poincaré duality, $H^k(U(P), \mathscr{L}(\alpha, \beta, P))$ is isomorphic to $H^{lf}_k(U(P), \mathscr{L}(\alpha, \beta, P))$, the homology defined by means of locally finite chains; see [VGZ]. For an "extendable" cycle $\sigma \in H^{lf}_k(U(P), \mathscr{L}(\alpha, \beta, P))$ we could define the Euler integral as in [VGZ]. But Theorem 2.15 means that the space of extendable cycles is just the image of the canonical map $H_k(U(P), \mathscr{L}(\alpha, \beta, P)) \to H^{lf}_k(U(P), \mathscr{L}(\alpha, \beta, P))$, and various "extensions" of a cycle σ correspond to various compact cycles homologous to σ.

2.17. EXAMPLE: HYPERGEOMETRIC FUNCTIONS ON THE GRASSMANNIAN $\mathrm{Gr}_{k+1}(\mathbb{C}^{m+k+1})$. These functions were originally defined in [G] in terms of generalized Euler integrals of type (1), where all P_i are (inhomogeneous) linear functions. In this case all A_i are equal to $\{0, e_1, ..., e_k\} \subset \mathbb{Z}^k$, where the e_j are standard basis vectors. The construction in 2.5 leads to the realization of $\mathbb{C}^A$ as the space of polynomials $P(y, x) = \sum_{i=1}^m v_{i0} y_i + \sum_{i=1}^m \sum_{j=1}^k v_{ij} y_i x_j$. The polytope Q is the product $\Delta^k \times \Delta^{m-1}$ of two simplices. It has $k+m+1$ faces of codimension 1. The corresponding non-resonance conditions have the form $\alpha_i \notin \mathbb{Z}$, $i = 1, ..., m$; $\beta_j \notin \mathbb{Z}$, $j = 1, ..., k$; $(\sum \alpha_i) + (\sum \beta_j) \notin \mathbb{Z}$, cf. [VGZ].

The set A can be transformed to a form $\bigcup(\{e_i'\} \times A_i')$, $i = 1, ..., m'$, $A_i' \subset \mathbb{Z}^{k'}$, in three different ways. The first was just described. The second has $m' = k+1$, $k' = m-1$, and the transformation is given by transposition of the matrix (v_{ij}). The third has $m' = 1$, $k' = m+k-1$. So we obtain three kinds of Euler integrals for the same hypergeometric system which are taken over cycles of different dimension. The relation between integrals of first two kinds is closely connected with the duality studied in [GG]. Both these integrals can be obtained from integrals of the third kind by an appropriate iterated integration.

3. Proof of Theorem 2.15

3.1. *Complexes of Sheaves Related to A.* Let $V = \mathbb{C}^A$, $S \subset V^*$ be as in Section 1. We denote the dual coordinate systems in V and V^* by (v_ω) and (ξ_ω), $\omega \in A$. Let $S_0 \subset S$ be the open orbit of the torus. We identify S_0 with $(\mathbb{C}^*)^n$ by means of coordinates $z = (z_1, ..., z_n)$ such that the corresponding point of S_0 is $(z^\omega)_{\omega \in A}$. For a complex vector $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{C}^n$ let $\mathscr{L}_{-\gamma}$ be the one-dimensional local system on S_0 which has monodromy exponents $(-\gamma_i)$ around $\{z_i = 0\}$. Sections of $\mathscr{L}_{-\gamma}$ are branches of functions of the form $\lambda z^{-\gamma}$, $\lambda \in \mathbb{C}^*$.

Let $j: S_0 \to V^*$ be the inclusion. On V^* we have the sheaf $j_! \mathscr{L}_{-\gamma}$ (extension by zero) and the complex of sheaves $Rj_* \mathscr{L}_{-\gamma}$ (direct image in the derived category). They are connected by the canonical morphism $c: j_! \mathscr{L}_{-\gamma} \to Rj_* \mathscr{L}_{-\gamma}$.

3.2. Proposition. *The vector γ is non-resonant for A if and only if $c: j_! \mathscr{L}_{-\gamma} \to Rj_* \mathscr{L}_{-\gamma}$ is an isomorphism (in the derived category).*

Proof. The fact that c is an isomorphism means that for any point $s \in S - S_0$ and a small neighborhood U of s we have $H^i(U \cap S_0, \mathscr{L}_{-\gamma}) = 0$ for all i.

Let s lie in the orbit $S_0(\Gamma) \subset S$, where Γ is some face of our polytope Q (see Proposition 1.6(a)). The structure of S near s was studied in [GZK 2, GZK 4]. Let us summarize necessary facts.

3.3. LEMMA. (a) *$U \cap S_0$ is homotopy equivalent to the disjoint union of several copies of the complex torus $H(\Gamma) = (\mathbb{C}^*)^{\dim \Gamma + 1}$.*

(b) *The restriction of $\mathscr{L}_{-\gamma}$ to each copy of $H(\Gamma)$ has the same vector of monodromy exponents $\gamma' \in (H(\Gamma)^\vee) \otimes \mathbb{C}$, where $H(\Gamma)^\vee$ is the character lattice of $H(\Gamma)$. There is a natural identification $H(\Gamma)^\vee = \mathbb{Z}^n/\mathrm{Lin}(\Gamma) \cap \mathbb{Z}^n$, such that γ' corresponds to the image of $-\gamma$ under the projection $\mathbb{C}^n \to \mathbb{C}^n/$ $\mathrm{Lin}(\Gamma)$.*

Proof. Part (a) follows from Proposition 2A.5 in [GZK 4]. (Note that the corresponding Lemma 1 in [GZK 2, Sect. 2.3] is stated incorrectly). Proof of (b) is straightforward. ▮

By (b), γ is non-resonant if and only if the restriction of $\mathscr{L}_{-\gamma}$ to each $H(\Gamma)$ is non-trivial. But it is well-known that a one-dimensional local system on a complex torus has no cohomology if and only if it is non-trivial. This completes the proof of Proposition 3.2. ▮

3.4. *Perversity.* Recall [BBD] that a perverse sheaf on V^* is a constructible complex $\mathscr{P}^\cdot$ of sheaves on V^* such that:

(Perv$^-$): The cohomology sheaf $\underline{H}^i(\mathscr{P}^\cdot)$ for $i \geqslant 0$ has support on a subvariety of codimension $\geqslant i$.

(Perv$^+$): If $X \subset V^*$ is a smooth locally closed subvariety of codimension d then the sheaves $\underline{H}^i_X(\mathscr{P}^\cdot)$ of hypercohomology with supports on X are zero for $i < d$.

Denote $N = \mathrm{Card}(A) = \dim V^*$. Then it follows from [BBD] (and is easy to see) that $j_! \mathscr{L}_{-\gamma}[n-N]$ and $Rj_* \mathscr{L}_{-\gamma}[n-N]$ are perverse sheaves. Here numbers in square brackets mean the shift of grading of complexes: the pth term of the complex $C^\cdot[i]$ is the $(p+i)$th term of $C^\cdot$.

3.5. THEOREM. *If γ is non-resonant then $j_! \mathscr{L}_{-\gamma}[n-N]$ is an irreducible perverse sheaf.*

Proof. Since $\mathscr{L}_{-\gamma}$ is 1-dimensional it is an irreducible local system on S_0. The intersection cohomology extension of $\mathscr{L}_{-\gamma}$ to V^*, denoted $j_{!*}(\mathscr{L}_{-\gamma})$ is, by definition, the image (in the Abelian category of perverse sheaves) of the canonical morphism $c\colon j_! \mathscr{L}_{-\gamma}[n-N] \to Rj_* \mathscr{L}_{-\gamma}[n-N]$; see [BBD]. By the general theory of [BBD], $j_{!*}(\mathscr{L}_{-\gamma})$ is an irreducible perverse sheaf on V^*. Since c is an isomorphism by Proposition 3.2, we obtain the desired statement. ▮

3.6. *The Geometric Fourier Transform* [Br, KS]. Denote by $G \subset V^* \times V$ the closed set $\{(\xi, v)\colon \mathrm{Re}(\xi, v) \geqslant 0\}$ and by p_1, p_2 the projections of $V^* \times V$ to V^* and V, respectively. Define a functor $\mathscr{F}$ from the derived category of sheaves on V^* to the similar category on V by $\mathscr{F}(\mathscr{P}) =$

$Rp_{2*}(\underline{R}\Gamma_G(p_1^*(\mathscr{P})))$. Here $\underline{R}\Gamma_G$ is the derived functor of the functor "subsheaf of sections with support in G." The functor $\mathscr{F}$ is called the *geometric Fourier transform.*

A sheaf $\mathscr{P}$ on V^* is called *conic* if it is locally constant on all orbits of $\mathbb{R}_+^*$. A complex $\mathscr{P}^\cdot$ of sheaves on V^* is called conic if all its cohomology sheaves $\underline{H}^i(\mathscr{P}^\cdot)$ are conic. Denote by $D^b_{\text{conic}}(V^*)$ the bounded derived category formed by such complexes. Let us summarize the main properties of $\mathscr{F}$.

3.7. THEOREM [B]. (a) *$\mathscr{F}$ takes $D^b_{\text{conic}}(V^*)$ to $D^b_{\text{conic}}(V)$ and is an equivalence of these categories.*

(b) *$\mathscr{F}$ takes constructible conic complexes on V^* to constructible conic complexes on V and defines an equivalence of corresponding derived categories.*

(c) *The functor $\mathscr{F}[N]$ takes conic perverse sheaves on V^* to conic perverse sheaves on V and defines an equivalence of corresponding categories.*

3.8. From now on we suppose that $A \subset \mathbb{Z}^n$ has the form $A = \bigcup (\{e_i\} \times A_i)$, where $n = m + k$, and $A_1, ..., A_m \subset \mathbb{Z}^k$ are finite subsets such that $\bigcap A_i$ contains 0 and $\bigcup A_i$ generates $\mathbb{Z}^k$ as an Abelian group (see 2.4). We shall use the notation from Sect. 2.

3.9. THEOREM. *Let $\alpha \in \mathbb{C}^m$, $\beta \in \mathbb{C}^k$ be such that $\gamma = (\alpha_1, ..., \alpha_m, -1-\beta_1, ..., -1-\beta_k)$ is non-resonant for A. Then there is an isomorphism $R\pi_*\mathscr{L}(\alpha, \beta)[k] \to \mathscr{F}(j_!\mathscr{L}_{-\gamma})[n]$ in the derived category of sheaves on V. In particular, $R\pi_*\mathscr{L}(\alpha, \beta)[k]$ is a perverse sheaf on V, and $R^k\pi_*\mathscr{L}(\alpha, \beta) = \underline{H}^n\mathscr{F}(j_!\mathscr{L}_{-\gamma})$.*

Proof. By definition, $\mathscr{F}(j_!\mathscr{L}_{-\gamma}) = Rp_{2*}(\underline{R}\Gamma_G(p_1^* j_!\mathscr{L}_{-\gamma})$. Since $j_!\mathscr{L}_{-\gamma} = j_*\mathscr{L}_{-\gamma}$, we have $\mathscr{F}(j_!\mathscr{L}_{-\gamma}) = Rq_{2*}(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^*\mathscr{L}_{-\gamma}))$, where q_1, q_2 are the projections of $S_0 \times V$ on the factors. We write a point of $V = \mathbb{C}^A$ in the form $P(y, x) = \sum y_i P_i(x)$, where $P_i \in \mathbb{C}^{A_i}$, $i = 1, ..., m$. The orbit S_0 is identified with $(\mathbb{C}^*)^n$ by means of coordinates $z = (z_1, ..., z_n) = (y_1, ..., y_m, x_1, ..., x_k)$, and $G \cap (S_0 \times V) = \{(y, x, P): \operatorname{Re} P(y, x) \geqslant 0\}$. Let s: $S_0 \times V \to (\mathbb{C}^*)^k \times V$ be the projection forgetting the y-coordinates. In 2.13 we introduced the set $U \subset (\mathbb{C}^*)^k \times V$ which is the union of all $U(P)$. Let ε: $U \to (\mathbb{C}^*)^k \times V$ be the embedding.

3.10. LEMMA. *We have an isomorphism $Rs_*(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^*\mathscr{L}_{-\gamma}))[n] = R\varepsilon_*\mathscr{L}(\alpha, \beta)[k]$ in the derived category of sheaves on $(\mathbb{C}^*)^k \times V$.*

Proof. Let $x \in (\mathbb{C}^*)^k$, $P = \sum y_i P_i(x) \in V$. Let us verify the required isomorphism at the level of stalks at (x, P). We distinguish two cases: $x \in U(P)$ and $x \notin U(P)$.

Case 1. $x \in U(P)$. The stalk at (x, P) of $R^i s_*(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^* \mathscr{L}_{-\gamma}))$ is just the ith cohomology of $s^{-1}(x, P)$ with coefficients in $(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^* \mathscr{L}_{-\gamma}))|_{s^{-1}(x, P)}$ which in our case coincides with $\underline{R}\Gamma_{G \cap (S_0 \times V) \cap s^{-1}(x, P)}(q_1^* \mathscr{L}_{-\gamma}|_{s^{-1}(x, P)})$.

Consider on $s^{-1}(x, P) = (\mathbb{C}^*)^m$ the local system $\mathscr{L}_\alpha$ and its direct image $R\tau_* \mathscr{L}_\alpha$, where $\tau: (\mathbb{C}^*)^m \to \mathbb{C}^m$ is the embedding. Denote by η_i the coordinates in $(\mathbb{C}^m)^*$ dual to y_i.

3.11. LEMMA. $\underline{R}\Gamma_{G \cap (S_0 \times V) \cap s^{-1}(x, P)}(q_1^* \mathscr{L}_{-\gamma}|_{s^{-1}(x, P)})$ *coincides with the stalk of the Fourier transform of* $R\tau_* \mathscr{L}_\alpha$ *at the point of* $(\mathbb{C}^m)^*$ *with coordinates* $\eta_i = P_i(x)$.

Proof. It suffices to note that $s^{-1}(x, P)$ is identified with $(\mathbb{C}^*)^m$ with coordinates $(y_1, ..., y_m)$, and $G \cap (S_0 \times V) \cap S^{-1}(x, P) = \{(y_1, ..., y_m)$: $\operatorname{Re} P(y, x) = \operatorname{Re} \sum y_i P_i(x) \geqslant 0\}$. ▮

Now Lemma 3.10 in case 1 follows from the next well-known fact

3.12. LEMMA [Br]. *Suppose that all* $\alpha_i \notin \mathbb{Z}$. *Then* $\mathscr{F}(R\tau_* \mathscr{L}_\alpha)[m] = R\tau'_* \mathscr{L}_{-1-\alpha}$, *where* τ' *is the embedding of* $(\mathbb{C}^*)^m$ *into* $(\mathbb{C}^m)^*$.

Case 2: $x \notin U(P)$. Denote the complex $Rs_*(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^* \mathscr{L}_{-\gamma}))$ by $\mathscr{P}$. We have to prove that $\mathscr{P} \cong R\varepsilon_* \varepsilon^* \mathscr{P}$.

3.13. LEMMA. *Let X be a topological space, $Y \subset X$ a closed subset, $U = X - Y$, ε: $U \to X$ the embedding, and $\mathscr{P}$ a complex of sheaves on X. Then the canonical morphism $\mathscr{P} \to R\varepsilon_* \varepsilon^* \mathscr{P}$ is an isomorphism if and only if for each $y \in Y$ we have $R\Gamma_{\{y\}}(\mathscr{P}) = 0$.*

Proof. Let i: $Y \to X$ be the embedding. The statement follows from the exact triangle $i_* i^! \mathscr{P} \to \mathscr{P} \to R\varepsilon_* \varepsilon^* \mathscr{P}$ (see [BBD]) and the fact that the stalk of $i^! \mathscr{P}$ at y is $R\Gamma_{\{y\}}(\mathscr{P}) = 0$. ▮

To prove that $R\Gamma_{\{(x, P)\}}(\mathscr{P})$ vanishes note that it coincides with the stalk at the point $(P_1(x), ..., P_m(x))$ of the Fourier transform of $\mathscr{L}_\alpha$. If all $\alpha_i \notin \mathbb{Z}$ and at least one $P_i(x) = 0$ then this stalk is zero by Lemma 3.12. Lemma 3.10 is proved. ▮

To finish the proof of Theorem 3.9 denote by $r: (\mathbb{C}^*)^k \times V \to V$ the projection. Then $q_2 = rs$, $\pi = r\varepsilon$, so $\mathscr{F}(j_! \mathscr{L}_{-\gamma})[n] = Rq_{2*}(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^* \mathscr{L}_{-\gamma}))[n] = Rr_* Rs_*(\underline{R}\Gamma_{G \cap (S_0 \times V)}(q_1^* \mathscr{L}_{-\gamma}))[n] = Rr_* R\varepsilon_* \mathscr{L}(\alpha, \beta)[k] = R\pi_* \mathscr{L}(\alpha, \beta)[k]$. ▮

Now we can complete the proof of Theorem 2.15. The first statement follows from Theorem 3.7c. To establish the second one note that

$R\pi_! \mathscr{L}(\alpha, \beta)$ is Verdier dual to $R\pi_*(\mathscr{L}(\alpha, \beta)^*)$ and hence also irreducible. Since perverse sheaves form an Abelian category and the morphism in question is non-zero, it must be an isomorphism. ∎

4. Study of the Hypergeometric $\mathscr{D}$-Module

4.1. *The Hypergeometric $\mathscr{D}$-Module.* Let $V = \mathbb{C}^A$, and denote $\mathscr{D}_V$ the sheaf of rings of linear differential operators with holomorphic coefficients on V. The operators Z_i and $\square_a$ from Section 1 are global sections of $\mathscr{D}_V$. We associate to the system (2) a sheaf of left $\mathscr{D}_V$-modules (or simply a $\mathscr{D}_V$-module) $\mathscr{M} = \mathscr{M}_\gamma = \mathscr{D}_V / (\sum \mathscr{D}_V(Z_i - \gamma_i) + \sum \mathscr{D}_V \square_a)$, see [Bj, K]. Consider the Fourier transform $\mathscr{N} = \mathscr{N}_\gamma$ of the $\mathscr{D}_V$-module $\mathscr{M}_\gamma$. It is the $\mathscr{D}_{V^*}$-module defined by

$$\mathscr{N}_\gamma = \mathscr{D}_{V^*} \Big/ \left(\sum \mathscr{D}_{V^*}(\hat{Z}_i - \gamma_i) + \sum \mathscr{D}_{V^*} \hat{\square}_a \right),$$

where $\hat{Z}_i = -\sum \omega_i(\xi_\omega(\partial/\partial \xi_\omega) + 1)$. The support of $\mathscr{N}$ is the subvariety $S \subset V^*$ defined in Section 2.

4.2. *Solution Complexes.* If $\mathscr{R}$ is any $\mathscr{D}$-module on a complex manifold Y then the *solution sheaf* of $\mathscr{R}$ is the sheaf $\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathscr{R}, \mathscr{O}_Y)$, see [K]. For the hypergeometric $\mathscr{D}$-module we have $\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathscr{M}_\gamma, \mathscr{O}_V) = \mathrm{Hyp}(\gamma)$, the sheaf of hypergeometric functions.

The *Solution complex* of $\mathscr{R}$ is defined as $\mathrm{Sol}(\mathscr{R}) = \underline{R\,\mathrm{Hom}}_{\mathscr{D}}(\mathscr{R}, \mathscr{O}_Y)$. If $\mathscr{R}$ is holonomic then $\mathrm{Sol}(\mathscr{R})$ is a perverse sheaf, see [Bo, BBD].

4.3. *$\mathscr{D}$-Module $\int \mathscr{R}_\gamma$.* We retain the notation of Section 3. Let $\mathscr{R}_\gamma$ be the $\mathscr{D}_{S_0}$-module defined by $\mathscr{R}_\gamma = \mathscr{D}_{S_0} / (\sum \mathscr{D}_{S_0}(\hat{Z}_i - \gamma_i))$. Then $\mathrm{Sol}(\mathscr{R}_\gamma)$ is isomorphic to the local system $\mathscr{L}_{-\gamma}$ defined in section 3. Let $\int_j \mathscr{R}_\gamma = \int \mathscr{R}_\gamma$ denote the $\mathscr{D}$-module direct image of $\mathscr{R}_\gamma$ under the inclusion $j: S_0 \to V^*$ [Bo]. By definition, $\int \mathscr{R}_\gamma$ is obtained from the sheaf-theoretic meromorphic direct image $j_*^{\mathrm{mer}} \mathscr{R}_\gamma$ (defined for quasi-coherent sheaves of $\mathscr{O}$-modules) by adding formal derivatives of its sections in directions transversal to S. Therefore, $j_*^{\mathrm{mer}} \mathscr{R}_\gamma$ is a subsheaf of $\int \mathscr{R}_\gamma$. For the solution complex we have $\mathrm{Sol}(\int \mathscr{R}_\gamma) = j_! \mathscr{L}_{-\gamma}$.

4.4. Proposition. *If γ is non-resonant then the $\mathscr{D}$-module $\int \mathscr{R}_\gamma$ is irreducible.*

Proof. By Riemann–Hilbert correspondence [Bo] it suffices to verify that $\mathrm{Sol}(\int \mathscr{R}_\gamma) = j_! \mathscr{L}_{-\gamma}$ is an irreducible perverse sheaf on V^*. But this is Theorem 3.5. ∎

4.5. *Morphism* $\mathscr{E}: \mathscr{N}_\gamma \to \int \mathscr{R}_\gamma$. Let $u \in H^0(S_0, \mathscr{R}_\gamma)$ be the canonical generator of $\mathscr{R}_\gamma$ (i.e., the image of $1 \in \mathscr{D}_{S_0}$). It defines a section u' of $j_*^{\mathrm{mer}} \mathscr{R}_\gamma \subset \int \mathscr{R}_\gamma$, which satisfies the system (2′). Let u'' be the canonical generator of $\mathscr{N}_\gamma$. We define $\mathscr{E}$ by the formula $\mathscr{E}(u'') = u'$.

4.6. THEOREM. *If* γ *is non-resonant for* A *then* $\mathscr{E}: \mathscr{N}_\gamma \to \int \mathscr{R}_\gamma$ *is an isomorphism.*

Proof. Since $\int \mathscr{R}_\gamma$ is irreducible for non-resonant γ, it follows that $\mathscr{E}$ is surjective. Let $\mathscr{K} = \operatorname{Ker} \mathscr{E}$. It is also a holonomic regular $\mathscr{D}$-module.

For any holonomic $\mathscr{D}$-module $\mathscr{R}$ on V^* let us denote by $SS(\mathscr{R})$ its characteristic variety [K]. It is a Lagrangian subvariety in T^*V^*. Denote also by $\underline{SS}(\mathscr{R})$ the characteristic cycle of $\mathscr{R}$ (see [K]). It is a formal sum of irreducible components of $SS(\mathscr{R})$ with certain nonnegative multiplicities. We have $\underline{SS}(\mathscr{N}_\gamma) = \underline{SS}(\mathscr{K}) + \underline{SS}(\int \mathscr{R}_\gamma)$. Therefore to show that $\mathscr{K} = 0$ it suffices to prove that $\underline{SS}(\mathscr{N}_\gamma) = \underline{SS}(\int \mathscr{R}_\gamma)$.

In [GZK 2] it was proved that any irreducible component of $SS(\mathscr{N}_\gamma)$ is the closure of conormal bundle to an orbit $S_0(\Gamma)$ for some face $\Gamma \subset Q$. We shall denote this variety $\operatorname{Con}(\Gamma)$. The multiplicity of $\operatorname{Con}(\Gamma)$ in $\underline{SS}(\mathscr{N}_\gamma)$ was also calculated in [GZK2, Theorem 5].

On the other hand, the calculation of $\underline{SS}(\int \mathscr{R}_\gamma)$ can be deduced from the results of Kouchnirenko [Kou]. First we note that $\underline{SS}(\mathscr{N}_\gamma) \geqslant \underline{SS}(\int \mathscr{R}_\gamma)$, and so each irreducible component of $SS(\int \mathscr{R}_\gamma)$ is some $\operatorname{Con}(\Gamma)$. Let $c(\Gamma)$ be the multiplicity of $\operatorname{Con}(\Gamma)$ in $\underline{SS}(\int \mathscr{R}_\gamma)$. Following [Gi], it can be calculated by means of the vanishing cycles functor Φ. Namely, let (ξ, w) be a generic point of $\operatorname{Con}(\Gamma)$, where $\xi \in S_0(\Gamma) \subset V^*$, $w \in T_\xi^* V^*$. Let g be a function on V^* vanishing along $S_0(\Gamma)$ and satisfying $d_\xi g = w$. Let Φ_g be the vanishing cycles functor acting from constructible complexes of sheaves on V^* to constructible complexes on the hypersurface $g = 0$. Then we have $c(\Gamma) = \chi(\Phi_g(j_! \mathscr{L}_\gamma)_\xi)$, the Euler characteristic of the stalk at ξ of the complex Φ_g.

Take for g a linear function $g(\eta) = \sum_{\omega \in A} v_\omega \eta_\omega$. Clearly, g vanishes along $S_0(\Gamma)$ if and only if $v_\omega = 0$ for $\omega \in \Gamma$. In coordinates $x_1, ..., x_n$ on S_0 the linear function g becomes a Laurent polynomial $f(x) = \sum v_\omega x^\omega$. The number $\chi(\Phi_g(j_! \mathscr{L}_\gamma)_\xi)$ is then identified with the Euler characteristic of the Milnor fiber of f with coefficients in $\mathscr{L}_\gamma$. This number has been calculated in [Kou] in terms of volumes of certain Newton polytopes. It turns out that the answer is the same as that of [GZK 2] for $\mathscr{N}_\gamma$. This proves that $\mathscr{K} = \operatorname{Ker} \mathscr{E} = 0$ and hence $\mathscr{E}$ is an isomorphism. ∎

4.7. *Proof of Theorem* 2.11. By Theorem 4.6 and Proposition 4.4, $\mathscr{N}_\gamma$ is an irreducible $\mathscr{D}_{V^*}$-module. Therefore, $\mathscr{M}_\gamma$, being the Fourier transform of $\mathscr{N}_\gamma$, is itself irreducible. So it must be the intersection cohomology extension of a $\mathscr{D}$-module corresponding to some irreducible local system on the open part $U \subset V$ consisting of smooth points for $\mathcal{M}_\gamma$; see [Bo]. This open part is precisely the generic stratum, whence the theorem. ▮

4.8. *Proof of Theorems* 2.10 *and* 2.14. The sheaves $\mathcal{H}(\alpha, \beta)$ and $\mathrm{Hyp}(\gamma)$ are 0th cohomology sheaves of perverse sheaves $R\pi_! \mathcal{L}(\alpha\, \beta)[k]$ and $\mathrm{Sol}(\mathcal{M}_\gamma)$ respectively: see 3.7. By Theorems 2.15 and 3.9, $R\pi_! \mathcal{L}(\alpha, \beta)[k-n]$ is the geometric Fourier transform $\mathcal{F}(j_! \mathcal{L}_{-\gamma})$. Since $j_! \mathcal{L}_{-\gamma}[n-N]$ is isomorphic to the solution complex of $\mathcal{N}_\gamma$, we can apply a theorem of Brylinsky [Br] to obtain that $\mathrm{Sol}(\mathcal{M}_\gamma)$ is isomorphic to $\mathcal{F}(j_! \mathcal{L}_{-\gamma})[n]$. Hence $\mathrm{Sol}(\mathcal{M}_\gamma)$ and $R\pi_! \mathcal{L}(\alpha, \beta)[k]$ are irreducible perverse sheaves isomorphic to each other. Therefore, they are the intersection cohomology extensions $\tau_{!*}$ (where $\tau\colon \mathbb{C}^A_{\mathrm{gen}} \to \mathbb{C}^A$ is the embedding) of their restrictions to $\mathbb{C}^A_{\mathrm{gen}}$. Our map $E(\alpha, \beta)\colon \mathcal{H}(\alpha, \beta) \to \mathrm{Hyp}(\gamma)$ considered over $\mathbb{C}^A_{\mathrm{gen}}$ yields, by functoriality of $\tau_{!*}$, a morphism $\tau_{!*}(E(\alpha, \beta) \mid \mathbb{C}^A_{\mathrm{gen}})$ of irreducible perverse sheaves. This must be an isomorphism since it is non-zero. But one has the isomorphism $\underline{H}^0(\tau_{!*}\mathcal{P}) = R^0\tau_* \mathcal{P}$ for any local system $\mathcal{P}$ on $\mathbb{C}^A_{\mathrm{gen}}$. It follows that $\mathcal{H}(\alpha, \beta)$ and $\mathrm{Hyp}(\gamma)$ coincide with the 0th direct images of their restrictions to $\mathbb{C}^A_{\mathrm{gen}}$, i.e., $\mathcal{H}(\alpha, \beta) = R^0\tau_* \tau^* \mathcal{H}(\alpha, \beta)$, $\mathrm{Hyp}(\gamma) = R^0\tau_* \tau^* \mathrm{Hyp}(\gamma)$. Also we have the equality $E(\alpha, \beta) = R^0\tau_* \tau^* E(\alpha, \beta)$. Hence $E(\alpha, \beta)$ must be an isomorphism on the whole $\mathbb{C}^A$. Theorem 2.14 and hence Theorem 2.10 are proven. ▮

4.9. *Remark.* Our proof of Theorem 2.14 is rather roundabout. This is due to the fact that we do not have an explicit description of the isomorphism $\phi\colon \mathcal{F}(\mathrm{Sol}(\mathcal{R}))[N] \to \mathrm{Sol}(\hat{\mathcal{R}})$ which is the translation of Brylinsky's isomorphism [Br] given originally for de Rham complexes to the language of solution complexes. It would be natural to expect that ϕ can be given by explicit integrals of type $\int \Xi(\xi) \exp(i\langle \xi, v\rangle)\, d\xi$ applied to suitable generalized solutions Ξ of the $\mathcal{D}$-module $\mathcal{R}$.

REFERENCES

[A] K. Aomoto, On the structure of integrals of power products of linear functions, *Sci. Papers College Gen. Ed. Univ. Tokyo* **27** (1977), 49–61.

[BE] H. Bateman and A. Erdelyi, "Higher transcendental functions," Vol. 1, McGraw–Hill, New York, 1953.

[Bj] J. E. Bjork, "Rings of Differential Operators," Mathematical Library, Vol. 21, Elsevier, Amsterdam/New York, 1979.

[Bo] A. Borel (Ed.), "Seminar on Intersection Homology," Birkhäuser, Boston, 1984.

[Br] J.-L. Brylinsky, Transformations canoniques et transformation de Fourier, *Astérisque* (1987), no. 140.

[G] I. M. Gelfand, A general theory of hypergeometric functions, *Doklady Akad. Nauk SSSR* **288** (1986), 14–18. [In Russian]

[GG] I. M. Gelfand and M. I. Graev, A duality theorem for general hypergeometric functions, *Doklady Akad. Nauk SSSR* **289** (1986), 19–23. [In Russian]

[GGZ] I. M. Gelfand, M. I. Graev, and A. V. Zelevinsky, Holonomic systems of equations and series of hypergeometric type, *Doklady Akad. Nauk SSSR* **295** (1987), 14–19. [In Russian]

[GZK 1] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, Equations of hypergeometric type and Newton polytopes, *Doklady Akad. Nauk SSSR* **300** (1988), 529–534. [In Russian]

[GZK 2] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, Hypergeometric functions and toric varieties, *Funktsional. Anal. i Prilozhen.* **23** (1989), 12–26. [In Russian]

[GZK 3] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, A-discriminants and Cayley–Koszul complexes, *Doklady Akad. Nauk SSSR* **307** (1989), 1307–1310. [In Russian]

[GZK 4] I. M. Gelfand, A. V. Zelevinsky and M. M. Kapranov, Discriminants of polynomials in several variables and triangulations of Newton polytopes, *Algebra i anal.* **2** (1990), 1–62. [In Russian]

[Gi] V. A. Ginsburg, Characteristic varieties and vanishing cycles, *Invent. Math.* **84** (1986), 327–402.

[K] M. Kashiwara, "Systems of Micro-Differential Equations," Birkhäuser, Boston, 1983.

[KS] M. Kashiwara and P. Shapira, Microlocal study of sheaves. *Astérisque* no. 128.

[Kou] A. G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, *Invent. Math.* **32** (1976), 1–31.

[MV] R. MacPherson and K. Vilonen, Elementary constructions of perverse sheaves, *Invent. Math.* **84** (1986), 403–435.

[S] N. Steenrod, Homology with local coefficients, *Ann. of Math.* **44** (1945), 610–627.

[VGZ] V. A. Vasiliev, I. M. Gelfand, and A. V. Zelevinsky, General hypergeometric functions on complex Grassmannians, *Funktsional. Anal. i Prilozhen.* **21** (1987), 23–38. [In Russian]