ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 226 (2011) 2233–2304

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Integral Menger curvature for surfaces

Paweł Strzelecki [a,*], Heiko von der Mosel [b]

[a] *Instytut Matematyki, Uniwersytet Warszawski, ul. Banacha 2, PL-02-097 Warsaw, Poland*
[b] *Institut für Mathematik, RWTH Aachen, Templergraben 55, D-52062 Aachen, Germany*

Received 6 November 2009; accepted 16 September 2010

Available online 13 October 2010

Communicated by Tristan Rivière

Abstract

We develop the concept of integral Menger curvature for a large class of nonsmooth surfaces. We prove uniform Ahlfors regularity and a $C^{1,\lambda}$-a priori bound for surfaces for which this functional is finite. In fact, it turns out that there is an explicit length scale $R > 0$ which depends only on an upper bound E for the integral Menger curvature $\mathscr{M}_p(\Sigma)$ and the integrability exponent p, and *not* on the surface Σ itself; below that scale, each surface with energy smaller than E looks like a nearly flat disc with the amount of bending controlled by the (local) $\mathscr{M}_p$-energy. Moreover, integral Menger curvature can be defined a priori for surfaces with self-intersections or branch points; we prove that a posteriori all such singularities are excluded for surfaces with finite integral Menger curvature. By means of slicing and iterative arguments we bootstrap the Hölder exponent λ up to the optimal one, $\lambda = 1 - (8/p)$, thus establishing a new geometric 'Morrey–Sobolev' imbedding theorem.

As two of the various possible variational applications we prove the existence of surfaces in given isotopy classes minimizing integral Menger curvature with a uniform bound on area, and of area minimizing surfaces subjected to a uniform bound on integral Menger curvature.

MSC: 28A75; 49Q10; 49Q20; 53A05; 53C21; 46E35

Keywords: Menger curvature; Surface theory; Geometric Morrey–Sobolev embedding; Uniform Ahlfors regularity; Self-avoidance energy

---

* Corresponding author.
E-mail addresses: pawelst@mimuw.edu.pl (P. Strzelecki), heiko@instmath.rwth-aachen.de (H. von der Mosel).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.09.016

Contents

1. Introduction

For three different non-collinear points $x, y, z \in \mathbb{R}^n$ the expression

$$R(x, y, z) := \frac{|x-y||x-z||y-z|}{4A(x, y, z)}, \tag{1.1}$$

where $A(x, y, z)$ is the area of the triangle with vertices at x, y and z, provides the radius of the uniquely defined circumcircle through x, y, and z. This gives rise to a family of *integral Menger curvatures*,[1] that is, geometric curvature energies of the form

$$\mathscr{M}_p(E) := \int\limits_E \int\limits_E \int\limits_E \frac{1}{R^p(x, y, z)}\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y)\, d\mathscr{H}^1(z), \quad p \geqslant 1, \tag{1.2}$$

defined on one-dimensional Borel sets $E \subset \mathbb{R}^n$. According to a remarkable result of J.C. Léger [12] such sets E with Hausdorff measure $\mathscr{H}^1(E) \in (0, \infty)$ and with finite integral Menger curvature $\mathscr{M}_p(E)$ for some $p \geqslant 2$, are 1-rectifiable in the sense of geometric measure theory. To be precise, $\mathscr{H}^1$-almost all of E is contained in a countable union of Lipschitz graphs. Ahlfors-regular[2] one-dimensional Borel sets $E \subset \mathbb{R}^2$ satisfying the local condition

$$\mathscr{M}_2\big(E \cap B(\xi, r)\big) \leqslant Cr \quad \text{for all } \xi \in \mathbb{R}^2,\ r \in (0, r_0] \tag{1.3}$$

turn out to be *uniformly rectifiable*, i.e., they are contained in the graph of *one* bi-Lipschitz map $f : \mathbb{R} \to \mathbb{R}^2$; see [23, Theorem 39] referring to work of P. Jones. M. Melnikov and J. Verdera [19,20] realized that $\mathscr{M}_2$ is a crucial quantity in harmonic analysis to characterize removable sets for bounded analytic functions; see e.g. the surveys [17,18,36].

[1] Coined after K. Menger who generalized expression (1.1) to metric spaces as a foundation of a metric coordinate free geometry; see [21,4].

[2] A set E of Hausdorff dimension 1 is said to be *Ahlfors-regular* if and only if there is a constant $C_E \geqslant 1$ such that $C_E^{-1} R \leqslant \mathscr{H}^1(E \cap B(x, R)) \leqslant C_E R$ for every $x \in E$ and $R > 0$, where $B(x, R)$ denotes a closed ball of radius R.

If one considers the $\mathscr{M}_p$-energy on rectifiable closed curves $E = \gamma(\mathbb{S}^1) \subset \mathbb{R}^3$ the following *geometric Morrey-Sobolev imbedding theorem* was proven in [31, Theorem 1.2], and this result may be viewed as a counterpart to J.C. Léger's regularity result on a higher regularity level:

If $\mathscr{M}_p(\gamma)$ is finite for some $p \in (3, \infty]$ and if the arclength parametrization Γ of the curve γ is a local homeomorphism then $\gamma(\mathbb{S}^1)$ is diffeomorphic to the unit circle $\mathbb{S}^1$, and Γ is a finite covering of $\gamma(\mathbb{S}^1)$ of class $C^{1,1-(3/p)}$.

In fact, even the stronger local version holds true [31, Theorem 1.3], which may be viewed as a *geometric Morrey-space imbedding* and whose superlinear growth assumption (1.4) is the counterpart of (1.3):

If the arclength parametrization Γ is a local homeomorphism, and if

$$\int_{B(\tau_1,r)} \int_{B(\tau_2,r)} \int_{B(\tau_3,r)} \frac{ds\,dt\,d\sigma}{R^2(\Gamma(s),\Gamma(t),\Gamma(\sigma))} \leqslant Cr^{1+2\beta} \tag{1.4}$$

holds true for all $r \in (0, r_0]$ and all arclength parameters τ_1, τ_2, τ_3, then Γ is a $C^{1,\beta}$-covering of the image $\gamma(\mathbb{S}^1)$ which itself is diffeomorphic to the unit circle.

From the results on one-dimensional sets and in particular on curves it becomes apparent that integral Menger curvature $\mathscr{M}_p$ exhibits regularizing and self-avoidance effects (as already suggested in [10] and [2]). These effects become stronger with increasing p, in fact, one has

$$\lim_{p\to\infty} \big(\mathscr{M}_p(\gamma)\big)^{1/p} = \frac{1}{\inf_{\sigma\neq s\neq t\neq\sigma} R(\Gamma(s),\Gamma(t),\Gamma(\sigma))} =: \frac{1}{\triangle[\gamma]},$$

where $\triangle[\gamma]$ is the notion of *thickness* of γ introduced by O. Gonzalez and J.H. Maddocks [10] who were motivated by analytical and computational issues arising in the natural sciences such as the modelling of knotted DNA molecules. In fact, it was shown in [11] and [27] that closed curves with finite energy $1/\triangle[\gamma]$, i.e. with positive thickness, are *exactly* those embeddings with a $C^{1,1}$-arclength parametrization, which lead to variational applications for nonlinearly elastic curves and rods with positive thickness; see also [28,29]. We generalized this concept of thickness in [32] and [33] to a fairly general class of nonsmooth surfaces $\Sigma \subset \mathbb{R}^n$ with the central result *that surfaces with positive thickness $\triangle[\Sigma]$ are in fact $C^{1,1}$-manifolds with a uniform control on the size of the local $C^{1,1}$-graph patches depending only on the value of $\triangle[\Sigma]$.* Uniform estimates on sequences then allow for the treatment of various energy minimization problems in the context of thick (and therefore embedded) surfaces of prescribed genus or isotopy class; see [33, Theorem 7.1].

In the present situation we ask the question:

Is it possible to extend the definition of integral Menger curvature $\mathscr{M}_p$ for $p < \infty$ to surfaces with similar regularizing and self-avoidance effects as in the curve case?

The most natural generalization of $\mathscr{M}_p$ to two-dimensional closed surfaces $\Sigma \subset \mathbb{R}^3$ would be to replace the circumcircle radius $R(x, y, z)$ of three points x, y, z in (1.2) by the circumsphere radius $R(\xi, x, y, z)$ of the tetrahedron $T := (\xi, x, y, z)$ spanned by the four non-coplanar points ξ, x, y, z. This radius is given by

$$\frac{1}{2R(T)} = \frac{|\langle z_3, z_1 \times z_2\rangle|}{||z_1|^2 z_2 \times z_3 + |z_2|^2 z_3 \times z_1 + |z_3|^2 z_1 \times z_2|}, \tag{1.5}$$

where $z_1 = \xi - z$, $z_2 = x - z$, $z_3 = y - z$. This would lead to the geometric curvature energy

$$\int_\Sigma \int_\Sigma \int_\Sigma \int_\Sigma \frac{d\mathscr{H}^2(\xi)\, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z)}{R^p(\xi, x, y, z)}, \tag{1.6}$$

which in principle would serve our purpose: all our results stated below extend to this energy. But – although the integrand is trivially constant if Σ happens to be a round sphere – there are smooth surfaces with straight nodal lines (such as the graph of the function $f(x, y) := xy$) where the integrand is not pointwise bounded; see Appendix B. This is a problem since we want to consider arbitrarily large p, and we envision a whole family of integral Menger curvatures that are finite on *any* closed smooth surface for *any* value of p.

Rewriting (1.1) as

$$R(x, y, z) = \frac{|x - z||y - z|}{2\,\mathrm{dist}(z, L_{xy})},$$

where L_{xy} denotes the straight line through x and y, one is naively tempted to consider 4-*point-integrands* of the form

$$\left(\frac{\mathrm{dist}(\xi, \langle x, y, z\rangle)}{M(|\xi - x|, |\xi - y|, |\xi - z|)^\alpha}\right)^p, \tag{1.7}$$

where $\langle x, y, z\rangle$ denotes the affine 2-plane through generic non-collinear points $x, y, z \in \mathbb{R}^3$. Here, $\alpha \geqslant 1$ is a power and the function $M : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *mean*, i.e. M is monotonically increasing with respect to each variable and satisfies the inequality

$$\min\{a, b, c\} \leqslant M(a, b, c) \leqslant \max\{a, b, c\}.$$

Again, all our results that will be stated below would hold if we worked with integrands as in (1.7) for $\alpha = 2$. This is very similar to a suggestion of J.C. Léger [12, p. 833] who proposes a general integrand for d-dimensional sets; for $d = 2$ his choice boils down to (1.7) with M being the geometric mean and $\alpha = 3$. However, the situation for such integrands, due to the lack of symmetry w.r.t. permutations of the 4 points, is even worse than for inverse powers of the circumsphere radius: for any choice of $\alpha > 1$ there are sufficiently large $p = p(\alpha)$ such that even a round sphere has infinite energy. This singular behaviour is caused by small tetrahedra for which the plane through (x, y, z) is almost perpendicular to the surface. See Appendix B for more details.

Roughly speaking, the trouble with (1.5) or (1.7) for surfaces comes from the fact that various 'obviously equivalent' formulae for $1/R$ for triangles (relying on the sine theorem) are no longer equivalent for tetrahedra. To obtain a whole scale of surface integrands which penalize wrinkling, folding, appearance of narrow tentacles, self-intersections etc. but stay bounded on smooth surfaces, one should make a choice here. In their pioneering work [13,14] dealing with d-rectifiability and least square approximation of d-regular measures, G. Lerman and J.T. Whitehouse suggest a whole series of high-dimensional counterparts of the one-dimensional Menger curvature. Their ingenious discrete curvatures are based, roughly speaking, on the so-called polar sine function scaled by some power of the diameter of the simplex, and can be used to obtain powerful and very general characterizations of rectifiability of measures. (In [14, Secs. 1.5 and 6] they also note that the integrand suggested by Léger does not fit into their setting.)

Motivated by this and by the explicit formula for the circumsphere radius, we are led to consider another 4-point integrand with symmetry and with fewer cancellations in the denominator. For a tetrahedron T consider the function

$$\mathcal{K}(T) := \begin{cases} \frac{V(T)}{A(T)(\operatorname{diam} T)^2} & \text{if the vertices of } T \text{ are not co-planar,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

where $V(T)$ denotes the volume of T and $A(T)$ stands for the total area, i.e., the sum of the areas of all four triangular faces of T. Thus, up to a constant factor $\mathcal{K}$ is the ratio of the minimal height of T to the square of its diameter, which is similar but not identical to the numerous curvatures considered by Lerman and Whitehouse in [14]. The difference is that our $\mathcal{K}$ scales like the inverse of length whereas their d-dimensional curvatures, cf. e.g. the definition of c_{MT} in [14, p. 327], for $d = 2$ scale like the inverse of the *cube* of length. Such scaling enforces too much singularity for our purposes; we explain that in Remark 5.2 in Section 5.

This leads us to the *integral Menger curvature for two-dimensional surfaces* $\Sigma \subset \mathbb{R}^3$,

$$\mathscr{M}_p(\Sigma) := \int_\Sigma \int_\Sigma \int_\Sigma \int_\Sigma \mathcal{K}^p(T)\, d\mathscr{H}^2 \otimes d\mathscr{H}^2 \otimes d\mathscr{H}^2 \otimes d\mathscr{H}^2(T), \tag{1.9}$$

which is finite for any C^2-surface for any finite p, since $\mathcal{K}(T)$ is bounded on the set of all nondegenerate tetrahedra with vertices on such a surface; see Appendix A.

To keep a clear-cut situation in the introduction we state our results here for closed Lipschitz surfaces only and refer the reader to Definition 2.4 in Section 2.2 for the considerably more general class $\mathscr{A}$ of admissible surfaces, and to Sections 3, 5, and 6 for the corresponding theorems in full generality. Let us just remark, however, that our admissibility class $\mathscr{A}$ contains surfaces that are not even topological submanifolds of $\mathbb{R}^3$: e.g. a sphere with the north and south pole glued together. The finiteness of $\mathscr{M}_p(\Sigma)$ has therefore topological, measure-theoretic and analytical consequences.

Theorem 1.1 *(Uniform Ahlfors regularity and a diameter bound). There exists an absolute constant $\alpha > 0$ such that for any $p > 8$, every $E > 0$, and for every closed compact and connected Lipschitz surface $\Sigma \subset \mathbb{R}^3$ with $\mathscr{M}_p(\Sigma) \leqslant E$ the following estimates hold*:

$$\operatorname{diam} \Sigma \geqslant \left(\frac{\alpha^{5p}}{E}\right)^{\frac{1}{p-8}},$$

$$\mathscr{H}^2\big(\Sigma \cap B(x, R)\big) \geqslant \frac{\pi}{2} R^2 \quad \textit{for all } x \in \Sigma \textit{ and } R \in \big(0, \big(\alpha^{5p}/E\big)^{1/(p-8)}\big]. \tag{1.10}$$

General Lipschitz surfaces may have conical singularities with a very small opening angle, but finite $\mathscr{M}_p$-energy controls uniformly the lower density quotient. These quantitative lower estimates for diameter and density quotient resemble L. Simon's results [30, Lemma 1.1 and Corollary 1.3] for smooth embedded two-dimensional surfaces of finite Willmore energy, derived by means of the first variation formulas. Here, in contrast, we set up an intricate algorithm (see Theorem 3.3 and its proof in Section 4), starting with a growing double cone and continuing with an increasingly complicated growing set centrally symmetric to a surface point, to scan the possibly highly complex exterior and interior domain bounded by Σ in search for three more complementing surface points to produce a "nice" tetrahedron whose size is controlled in terms of the energy. Along the way, the construction allows for large projections onto affine 2-planes which leads to the uniform estimate (1.10).

The case $p = 8$ yields a result which may be interpreted as a two-dimensional variant of Fenchel's theorem on the total curvature of closed curves [8]:

Theorem 1.2 *(Fenchel for surfaces). There is an absolute constant $\gamma_0 > 0$ such that $\mathscr{M}_8(\Sigma) \geqslant \gamma_0$ for any closed compact connected Lipschitz surface $\Sigma \subset \mathbb{R}^3$.*

The exponent $p = 8$ is a limiting one here: $\mathscr{M}_8$ is scale invariant. Invoking scaling arguments, it is easy to see that any cone over a smooth curve must have infinite $\mathscr{M}_p$-energy for every $p \geqslant 8$.

Uniform control over the lower Ahlfors regularity constant as in Theorem 1.1 permits us to prove the existence of a field of tangent planes for finite energy surfaces Σ (coinciding with the classical tangent planes at points of differentiability of Σ), and quantitatively control its oscillation:

Theorem 1.3 *(Oscillation of the tangent planes). For any closed compact and connected Lipschitz surface $\Sigma \subset \mathbb{R}^3$ with $\mathscr{M}_p(\Sigma) \leqslant E$ for some $p > 8$ the tangent plane $T_x\Sigma$ is defined everywhere and depends continuously on x: there are constants $\delta_1 = \delta_1(p) > 0$ and $A = A(p) \geqslant 0$ such that*

$$\sphericalangle(T_x\Sigma, T_y\Sigma) \leqslant A E^{\frac{1}{p+16}} |x - y|^{\frac{p-8}{p+16}} \tag{1.11}$$

whenever $|x - y| \leqslant \delta_1(p) E^{-1/(p-8)}$.

One might compare this theorem with Allard's famous regularity theorem [1, Theorem 8.19] for varifolds: Supercritical integrability assumptions (with exponent $p >$ dimension) on the generalized mean curvature are replaced here by integrability assumptions on our four-point Menger curvature integrand $\mathcal{K}$ for $p > 8 = 4 \cdot$ dimension – with possible extension to metric spaces, since our integrand may be expressed in terms of distances only. To prove Theorem 1.3 (see Section 5 for all details), we start with a technical lemma, ascertaining that the so-called P. Jones' β-numbers of Σ, measuring the distance from Σ to the best 'approximating plane' and defined by

$$\beta_\Sigma(x, r) := \inf\left\{ \sup_{y \in \Sigma \cap B(x,r)} \frac{\operatorname{dist}(y, F)}{r} : F \text{ is an affine 2-plane through } x \right\}, \tag{1.12}$$

can be estimated by const $\cdot\, r^{(p-8)/(p+16)}$ at small scales. This estimate is uniform, i.e. depends only on p and on the energy bounds, due to Theorem 1.1. For a wide class of *Reifenberg flat* *sets with vanishing constant*, see G. David, C. Kenig and T. Toro [5] or D. Preiss, X. Tolsa and Toro [24], this would be enough to guarantee the desired result. However, at this stage we cannot ensure that the surface we consider is Reifenberg flat with vanishing constant; it might be just a Lipschitz surface with some folds or conical singularities which are not explicitly excluded in Theorem 1.1. Reifenberg flatness, introduced by E.R. Reifenberg [26] in his famous paper on the Plateau problem in high dimensions, requires not only some control of β's, but also a stronger fact: one needs to know that the Hausdorff distance between the approximating planes and Σ is small at small scales. To get such control, we use some elements of the proof of Theorem 1.1 to guarantee the existence of large projections of Σ onto planes, and, proceeding iteratively, combine this with the decay of β's to reach the desired conclusion. The proof is presented in Section 5; it is self-contained and independent of [5] and [24].

Once Theorem 1.3 is established, we know that in a small scale, depending solely on p and on the energy bound, the surface is a graph of a $C^{1,\kappa}$ function. Slicing arguments similar to, but technically more intricate than those in the proof of optimal Hölder regularity for curves in [31, Theorem 1.3], are employed in Section 6 to bootstrap the Hölder exponent from $\kappa = (p-8)/(p+16)$ to $(p-8)/p$ and to prove the following result.

Theorem 1.4 *(Optimal Hölder exponent). Any closed, compact and connected Lipschitz surface Σ in $\mathbb{R}^3$ with $\mathscr{M}_p(\Sigma) \leqslant E < \infty$ for some $p > 8$ is an orientable $C^{1,1-(8/p)}$-manifold with local graph representations whose domain size is controlled solely in terms of E and p.*

We expect that $1 - 8/p$ is the optimal exponent, like the corresponding optimal exponent $1 - 3/p$ in the curve case in [31, Theorem 1.3]; see the example for curves in [34, Section 4.2].

The last section deals with sequences of surfaces with a uniform bound on their $\mathscr{M}_p$-energy. Using a combination of Blaschke's selection theorem and Vitali covering arguments with balls on the scale of uniformly controlled local graph representations we can establish the following compactness result.

Theorem 1.5 *(Compactness for surfaces with equibounded $\mathscr{M}_p$-energy). Let $\{\Sigma_j\}$ be a sequence of closed, compact and connected Lipschitz surfaces containing $0 \in \mathbb{R}^3$ with*

$$\mathscr{M}_p(\Sigma_j) \leqslant E \quad \textit{for all } j \in \mathbb{N} \quad \textit{and} \quad \sup_{j\in\mathbb{N}} \mathscr{H}^2(\Sigma_j) \leqslant A,$$

for some $p > 8$. Then there is a compact $C^{1,1-8/p}$-manifold Σ without boundary embedded in $\mathbb{R}^3$, and a subsequence j', such that $\Sigma_{j'}$ converges to Σ in C^1, and such that

$$\mathscr{M}_p(\Sigma) \leqslant \liminf_{j\to\infty} \mathscr{M}_p(\Sigma_j).$$

Instead of the uniform area bound one could also assume a uniform diameter bound.

Using this compactness result and the self-avoidance effects of integral Menger curvature we will prove that one can minimize area in the class of closed, compact and connected Lipschitz surfaces of fixed genus under the constraint of equibounded energy. For given $g \in \mathbb{N}$ let M_g be a closed, compact and connected reference surface of genus g that is smoothly embedded in $\mathbb{R}^3$, and consider the class $\mathscr{C}_E(M_g)$ of closed, compact and connected Lipschitz surfaces $\Sigma \subset \mathbb{R}^3$ ambiently isotopic to M_g with $\mathscr{M}_p(\Sigma) \leqslant E$ for all $\Sigma \in \mathscr{C}_E(M_g)$.

Theorem 1.6 *(Area minimizers in a given isotopy class). For each $g \in \mathbb{N}$, $E > 0$ and each fixed reference surface M_g the class $\mathscr{C}_E(M_g)$ contains a surface of least area.*

We can also minimize the integral Menger curvature $\mathscr{M}_p$ itself in a given isotopy class with a uniform area bound, i.e. in the class $\mathscr{C}_A(M_g)$ of closed, compact and connected Lipschitz surfaces $\Sigma \subset \mathbb{R}^3$ ambiently isotopic to M_g with $\mathscr{H}^2(\Sigma) \leqslant A < \infty$.

Theorem 1.7 *($\mathscr{M}_p$-minimizers in a given isotopy class). For each $g \in \mathbb{N}$, $A > 0$, there exists a surface $\Sigma \in \mathscr{C}_A(M_g)$ with*

$$\mathscr{M}_p(\Sigma) = \inf_{\mathscr{C}_A(M_g)} \mathscr{M}_p.$$

The proofs of Theorems 1.5–1.7 are given in Section 7.

Remark 1.8. It can be checked that our Theorems 1.1 and 1.2 can be proven for a large class of integrands including the two-dimensional c_{MT} and other curvatures of Lerman and Whitehouse, and even the one suggested by Léger. (One just has to check what is the critical scaling-invariant exponent, and work above this exponent.) However, Theorems 1.3 and 1.4, and consequently also Theorems 1.5, 1.6, and 1.7 seem to fail for any choice of integrand $\mathcal{K}_s(T)$ which scales like the inverse of length to some power $1+s$, $s > 0$. Such a choice enforces too much singularity for large p, and the methods we employ to prove Hölder regularity of the unit normal show that the only surface with $\int \mathcal{K}_s^p \, d\mu$ finite for all p would be (a piece of) the flat plane. See Remark 5.2 in Section 5.

Remark 1.9. Our work is related to the theory of uniformly rectifiable sets of G. David and S. Semmes, see their monograph [6]. Numerous equivalent definitions of these sets involve subtle conditions stating how well, in an average sense, the set can be approximated by planes. One of the deep ideas behind this is to try and search for the analogies between classes of sets and function spaces. It turns out then that various approximation or imbedding theorems for function spaces have geometric counterparts for sets, see e.g. the introductory chapter of [6]. Speaking naively and vaguely, David and Semmes work in the realm which corresponds to the subcritical case of the Sobolev imbedding theorem: there is no smoothness but subtle tools are available to give nontrivial control of the rate of approximation of a function by linear functions (or rather: a set by planes). Here, we are in the supercritical realm. For exponents larger than the critical $p = 8$ related to scale invariance, excluding conical singularities, finiteness of our curvature integrands gives continuity of tangent planes, with precise local control of the oscillation. Note that the exponent $1 - 8/p$ in Theorem 1.6 is computed according to Sobolev's recipe: the domain of integration has dimension 8 and we are dealing with the p'th power of 'curvature'.

2. Notation. The class of admissible surfaces

2.1. Basic notation

Balls, planes and slabs. $B(a, r)$ denotes always the *closed* ball of radius r, centered at a. When $a = 0 \in \mathbb{R}^3$, we often write just B_r instead of $B(0, r)$.

For non-collinear points $x, y, z \in \mathbb{R}^3$ we denote by $\langle x, y, z\rangle$ the affine 2-plane through x, y, and z. If H is a 2-plane in $\mathbb{R}^3$, then π_H denotes the orthogonal projection onto H. For an affine plane $F \subset \mathbb{R}^3$ such that $0 \notin F$, we write σ_F to denote the central projection from 0 onto F.

If F is an affine plane in $\mathbb{R}^3$ and $d > 0$, then we denote the infinite slab about F by

$$U_d(F) := \{y \in \mathbb{R}^3\colon \operatorname{dist}(y, F) \leqslant d\}.$$

Cones. Let $\varphi \in (0, \frac{\pi}{2})$ and $w \in \mathbb{S}^2$. We set

$$C(\varphi, w) := \{y \in \mathbb{R}^3\colon |y \cdot w| \geqslant |y| \cos\varphi\}$$

describing the infinite double-sided cone of opening angle 2φ whose axis is determined by w, and we define $C_r(\varphi, w) := B(0, r) \cap C(\varphi, w)$. We also distinguish between the two conical halves

$$C^+(\varphi, w) := \{y \in \mathbb{R}^3\colon y \cdot w \geqslant |y| \cos\varphi\}, \qquad C^-(\varphi, w) := \{y \in \mathbb{R}^3\colon -y \cdot w \geqslant |y| \cos\varphi\},$$

and set $C_r^\pm(\varphi, w) := B(0, r) \cap C^\pm(\varphi, w)$.

Rotations in $\mathbb{R}^3$. Throughout, we fix an orientation of $\mathbb{R}^3$. Assume that $u, v \in \mathbb{S}^2$ are orthogonal and $u \times v = w \in \mathbb{S}^2$. We write $R(\varphi, w)$ to denote the rotation which, in the orthonormal basis (u, v, w), is represented by the matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this formula gives in fact a linear map which does not depend on the choice of orthonormal vectors u, v with $u \times v = w$.

Segments. Whenever $z \in \mathbb{R}^3$, $s > 0$ and $w \in \mathbb{S}^2$, we set

$$I_{s,w}(z) := \{z + tw\colon |t| \leqslant s\}$$

(this is the segment of length $2s$, centered at z and parallel to w).

Tetrahedra. Since we deal with an integrand defined on quadruples of points in $\mathbb{R}^3$, and in various places we need to estimate that integrand on specific quadruples satisfying some additional conditions, we introduce some notation now to shorten the statements of several results in Sections 3–6.

By a *tetrahedron* T we mean a quadruple of points, $T = (x_0, x_1, x_2, x_3)$ with $x_i \in \mathbb{R}^3$ for $i = 0, 1, 2, 3$. By a triangle Δ we mean a triple of points, $\Delta = (x_0, x_1, x_2)$, $x_i \in \mathbb{R}^3$. We say that $\Delta = \Delta(T)$ is the base of T iff $\Delta = (x, y, z)$ and $T = (x, y, z, w)$ for some $x, y, z, w \in \mathbb{R}^3$.

For $T = (x_0, x_1, x_2, x_3)$ and $T = (x_0', x_1', x_2', x_3')$ we set

$$\|T - T'\| := \min_{\sigma \in S_4}\Big[\max_{0 \leqslant i \leqslant 3}|x_{\sigma(i)} - x_i'|\Big],$$

where $|x_{\sigma(i)} - x_i'|$ denotes the Euclidean norm and S_4 is the symmetric group of all permutations of sets with four elements. We write $\mathscr{B}_r(T) := \{T'\colon \|T - T'\| \leqslant r\}$.

To investigate the local and global behaviour of a surface, we often estimate its $\mathscr{M}_p$-energy on $\mathscr{B}_\varepsilon(T) \cap \Sigma$ where either T resembles, roughly speaking, a regular tetrahedron or at least its base $\Delta(T)$ resembles, again roughly, a regular triangle. Here are the appropriate definitions.

Definition 2.1. Let $\theta \in (0,1)$ and $d > 0$. We say that $T = (x_0, x_1, x_2, x_3)$ is (θ, d)-voluminous, and write $T \in \mathscr{V}(\theta, d)$, if and only if

(i) $x_i \in B(x_0, 2d)$ for all $i = 1, 2, 3$;
(ii) $\theta d \leqslant |x_i - x_j|$ for all $i \neq j$, where $i, j = 0, 1, 2, 3$;
(iii) $\sphericalangle(x_1 - x_0, x_2 - x_0) \in [\theta, \pi - \theta]$;
(iv) $\operatorname{dist}(x_3, \langle x_0, x_1, x_2\rangle) \geqslant \theta d$.

Definition 2.2. Let $\theta \in (0,1)$ and $d > 0$. We say that $\Delta = (x_0, x_1, x_2)$ is (θ, d)-wide, and write $\Delta \in \mathscr{S}(\theta, d)$, if and only if

(i) $x_i \in B(x_0, 2d)$ for $i = 1, 2$;
(ii) $\theta d \leqslant |x_i - x_j|$ for $i \neq j$, where $i, j = 0, 1, 2$;
(iii) $\sphericalangle(x_1 - x_0, x_2 - x_0) \in [\theta, \pi - \theta]$.

Remark 2.3. Similar classes of simplices have been used by Lerman and Whitehouse, see [14, Sec. 3]. The class of T with $\Delta(T) \in \mathscr{S}(\theta, d)$ differs from their class of 2-separated tetrahedra as the minimal face area of T with $\Delta(T) \in \mathscr{S}(\theta, d)$ does not have to be comparable to the square of $\operatorname{diam} T$. This plays a role in Section 5 and Section 6.

2.2. The class of admissible surfaces

Throughout the paper we consider only compact and closed surfaces.

Definition 2.4. We say that a compact connected subset $\Sigma \subset \mathbb{R}^3$ such that $\Sigma = \partial U$ for some bounded domain $U \subset \mathbb{R}^3$ is *an admissible surface*, and write $\Sigma \in \mathscr{A}$, if the following two conditions are satisfied:

(i) There exists a constant $K = K(\Sigma)$ such that

$$\infty > \mathscr{H}^2\big(\Sigma \cap B(x, r)\big) \geqslant K^{-1} r^2 \quad \text{for all } x \in \Sigma \text{ and all } 0 < r \leqslant \operatorname{diam} \Sigma;$$

(ii) There exists a dense subset $\Sigma^* \subset \Sigma$ with the following property: for each $x \in \Sigma^*$ there exists a vector $v = v(x) \in \mathbb{S}^2$ and a radius $\delta_0 = \delta_0(x) > 0$ such that

$$B(x, \delta_0) \cap \big(x + C^+(\pi/4, v)\big) \subset U \cup \{x\},$$
$$B(x, \delta_0) \cap \big(x + C^-(\pi/4, v)\big) \subset \big(\mathbb{R}^3 \setminus \overline{U}\big) \cup \{x\}.$$

Condition (ii) seems to be rather rigid because of the symmetry requirement. We could have used some smaller angle φ_0 instead of $\varphi_0 = \pi/4$ with the only effect that the absolute constants in Theorems 3.1–3.3, 5.4, and 6.1 would change, but we stick to $\varphi_0 = \pi/4$ for the sake of simplicity.

Condition (i) excludes sharp cusps around an isolated point of Σ but allows for isolated conical singularities and various cuspidal folds along arcs.

Note that this is a large class of surfaces, and if $\Sigma \in \mathscr{A}$, then Σ does not have to be an embedded topological manifold. Consider for example a sphere on which two distinct points have been identified, or, more generally, a sphere with $2N$ distinct smooth arcs and identify N pairs of these arcs.

Here are further examples.

Example 2.5 *(C^1 surfaces).* If Σ is a C^1 manifold which bounds a domain U, then $\Sigma \in \mathscr{A}$. One can take $\Sigma^* \equiv \Sigma$; by definition of differentiability, for each point $x \in \Sigma$ condition (ii) is satisfied for $v(x) =$ the inner normal to Σ at x, and one can choose a uniform lower bound for the numbers $\delta_0(x)$, i.e. we can always pick a $\delta_0(x) \geqslant \delta_0 = \delta_0(\Sigma) > 0$.

Example 2.6 *(Lipschitz surfaces).* If $\Sigma = \partial U$ is a Lipschitz manifold, then $\Sigma \in \mathscr{A}$. We can take $\Sigma^* =$ the set of all points where Σ has a classically defined tangent plane. By Rademacher's theorem, Σ^* is a set of full surface measure, hence it is dense. Obviously, $\delta_0(x)$ does depend on $x \in \Sigma^*$ now. It is an easy exercise to check (with a covering argument using compactness of Σ) that condition (i) is also satisfied.

Example 2.7 *($W^{2,2}$ surfaces).* If $\Sigma = \partial U$ is locally a graph of a $W^{2,2}$ function and condition (i) is satisfied, then $\Sigma \in \mathscr{A}$. This follows from Toro's [35] theorem on the existence of bi-Lipschitz parametrizations for such surfaces.

Example 2.8. If a compact, connected surface $\Sigma = \partial U$ is locally a graph of an AC^2-function (see J. Malý's paper [16] for a definition of absolutely continuous functions of several variables) and if (i) is satisfied – which is a necessary assumption as graphs of AC^2 functions may have cusps – then Σ is admissible. (AC^2 functions are differentiable a.e. and this implies condition (ii) of Definition 2.4.)

2.3. The energy and two simple estimates of the integrand

As mentioned in the introduction, we consider the energy

$$\mathscr{M}_p(\Sigma) := \int_{\Sigma^4} \mathcal{K}^p(T)\, d\mu(T), \qquad \Sigma \in \mathscr{A}, \tag{2.1}$$

where

$$\mathcal{K}(T) := \begin{cases} \frac{V(T)}{A(T)(\operatorname{diam} T)^2} & \text{if the vertices of } T \text{ on } \Sigma \text{ are not co-planar,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $V(T)$ denotes the volume of T and $A(T)$ the total area, i.e. the sum of the areas of all four triangular faces of T. For the sake of brevity we write

$$d\mu(\xi, x, y, z) := d\mathscr{H}^2(\xi)\, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z). \tag{2.2}$$

If $T = (x_0, x_1, x_2, x_3)$ and one sets $z_i = x_i - x_0$ for $i = 1, 2, 3$, then we have

$$\mathcal{K}(T) = \frac{1}{3} \cdot \frac{|z_3 \cdot (z_1 \times z_2)|}{[|z_1 \times z_2| + |z_2 \times z_3| + |z_1 \times z_3| + |(z_2 - z_1) \times (z_3 - z_2)|](\operatorname{diam} T)^2}. \tag{2.3}$$

We will mostly not work with (2.3) directly. In almost all proofs in Sections 3–6, we use iteratively two simple estimates of $\mathcal{K}$ on appropriate classes of tetrahedra.

Lemma 2.9. *If $T \in \mathscr{V}(\theta, d)$, then*

$$\mathcal{K}(T) > \frac{1}{50^2}\theta^4 d^{-1}.$$

Lemma 2.10. *If $T = (x_0, x_1, x_2, x_3)$ is such that $\Delta(T) = (x_0, x_1, x_2) \in \mathscr{S}(\theta, d)$, $x_3 \in B(x_0, 2d)$ and* $\operatorname{dist}(x_3, \langle x_0, x_1, x_2\rangle) \geqslant \kappa d$, *then*

$$\mathcal{K}(T) > \frac{1}{50^2}\theta^3 \kappa d^{-1}.$$

Proof of Lemma 2.9. Let $T = (x_0, x_1, x_2, x_3)$, $z_i := x_i - x_0$ for $i = 1, 2, 3$. Using conditions (ii) and (iii) of Definition 2.1, we obtain $|z_1 \times z_2| \geqslant \theta^2 d^2 \sin\theta \geqslant \frac{2}{\pi}\theta^3 d^2$ and by (iv)

$$\left|\frac{z_3 \cdot (z_1 \times z_2)}{|z_1 \times z_2|}\right| = \operatorname{dist}\bigl(x_3, \langle x_0, x_1, x_2\rangle\bigr) \overset{\text{Def. 2.1(iv)}}{\geqslant} \theta d. \tag{2.4}$$

Therefore we can estimate

$$\begin{aligned}
\mathcal{K}(T) &\overset{(2.4)}{\geqslant} \frac{1}{3(\operatorname{diam} T)^2} \cdot \frac{\theta d}{1 + \frac{|z_2 \times z_3|}{|z_1 \times z_2|} + \frac{|z_1 \times z_3|}{|z_1 \times z_2|} + \frac{|(z_2 - z_1) \times (z_3 - z_2)|}{|z_1 \times z_2|}} \\
&\geqslant \frac{1}{3(4d)^2} \cdot \frac{\theta d}{1 + 2 \cdot \frac{(2d)^2}{(2/\pi)\theta^3 d^2} + \frac{(4d)^2}{(2/\pi)\theta^3 d^2}} \\
&= \frac{\theta^4}{48d[\theta^3 + 12\pi]} > \frac{\theta^4}{50^2 d}. \qquad \square
\end{aligned}$$

The proof of Lemma 2.10 is identical. One just replaces (2.4) by $\operatorname{dist}(x_3, \langle x_0, x_1, x_2\rangle) \geqslant \kappa d$.

3. From energy bounds to uniform Ahlfors regularity

The main result of this section is the following.

Theorem 3.1 *(Energy bounds imply uniform Ahlfors regularity). There exists an absolute constant $\alpha > 0$ such that for every $p > 8$, every $E > 0$ and every $\Sigma \in \mathscr{A}$ with $\mathscr{M}_p(\Sigma) \leqslant E$ the following holds*:

Whenever $x \in \Sigma$, then

$$\mathscr{H}^2\bigl(B(x, R) \cap \Sigma\bigr) \geqslant \pi R^2/2$$

for all radii

$$R \leqslant R_0(E, p) := \left(\frac{\alpha^{5p}}{E}\right)^{\frac{1}{p-8}}. \tag{3.1}$$

Note that the value of $R_0(E, p)$ depends on E and p, but *not on* Σ itself, which is by no means obvious. Even infinitely smooth surfaces can have long 'fingers' which contribute a lot to the diameter but very little to the area. The point is that fixing an energy bound E we can be sure that 'fingers' cannot appear on Σ at a scale smaller than $R_0(E, p)$. Moreover, a general sequence Σ_i of C^∞-surfaces could in principle gradually form a tip approaching a cusp singularity as $i \to \infty$ (in fact, it is not difficult to produce examples of sequences of smooth surfaces with uniformly bounded area and infinitely many cusp or hair-like singularities in the limit), whereas this cannot happen according to Theorem 3.1 for a sequence of smooth admissible surfaces with equibounded $\mathscr{M}_p$-energy.

This fact plays a crucial role later on, in the derivation of uniform estimates for the oscillation of the tangent in Section 5. These estimates in turn allow us to prove in Section 7 compactness for sequences of surfaces having equibounded energy.

The scale-invariant limiting case $p = 8$ leads to the following result which can be viewed as a naive counterpart of the Gauß–Bonnet theorem for closed surfaces, or the Fenchel theorem for closed curves: one needs a fixed amount of energy to 'close' the surface. Our estimate of this necessary energy quantum is by no means sharp; it would be interesting to know the optimal value of that constant.

Theorem 3.2. *There exists an absolute constant $\gamma_0 > 0$ such that $\mathscr{M}_8(\Sigma) > \gamma_0$ for every surface $\Sigma \in \mathscr{A}$.*

The proof of both theorems relies on a preparatory technical result which might be of interest on its own, since it allows us to find for any given admissible surface (no matter how "crooked" its shape might look) a good tetrahedron with vertices on the surface, i.e. a voluminous tetrahedron in the sense of Definition 2.1. This result is completely independent of Menger curvature, but in our context it will allow us to prove $\mathscr{M}_p$-energy estimates from below.

Theorem 3.3 *(Good tetrahedra with vertices on Σ). There exist two absolute constants $\alpha, \eta \in (0, 1)$ such that*

$$1 > 2\eta > 40\alpha > 0 \tag{3.2}$$

with the following property: For every surface $\Sigma \in \mathscr{A}$ and every $x_0 \in \Sigma^*$ one can find a positive stopping distance $d_s(x_0) \in (\delta_0(x_0), \operatorname{diam} \Sigma]$ *and a triple of points $(x_1, x_2, x_3) \in \Sigma \times \Sigma \times \Sigma$ such that*

(i) $T = (x_0, x_1, x_2, x_3) \in \mathscr{V}(\eta, d_s(x_0))$,
(ii) *whenever $\|T' - T\| \leqslant \alpha d_s(x_0)$, we have $T' \in \mathscr{V}(\frac{\eta}{2}, \frac{3}{2} d_s(x_0))$.*

Moreover, for each $r \in (0, d_s(x_0)]$ there is an affine plane $H = H(r)$ passing through x_0 such that

$$\pi_H\big(\Sigma \cap B(x_0, r)\big) \supset H \cap B(x_0, r/\sqrt{2}) \tag{3.3}$$

and therefore we have

$$\mathscr{H}^2\big(\Sigma \cap B(x_0, r)\big) \geqslant \frac{\pi}{2} r^2 \quad \textit{for all } r \in \big(0, d_s(x_0)\big]. \tag{3.4}$$

The proof of this result is elementary but tedious. We give it in the next section. We also state one direct corollary of that proof for sake of further reference.

Proposition 3.4 *(Large projections and forbidden conical sectors). Let $p > 8$, $E > 0$, and $\partial U = \Sigma \in \mathscr{A}$ with $\mathscr{M}_p(\Sigma) \leqslant E$. Assume that $R_0 = R_0(E, p)$ is given by* (3.1). *For each $x \in \Sigma$ and $r < R_0$ there exists a plane H passing through x and a unit vector $v \in \mathbb{S}^2$, $v \perp H$, such that*

$$D := H \cap B(x, r/\sqrt{2}) \subset \pi_H\big(\Sigma \cap B(x, r)\big), \tag{3.5}$$

$$\operatorname{int} C_r^+(\varphi_0, v) \setminus B(x, r/2) \subset U, \tag{3.6}$$

$$\operatorname{int} C_r^-(\varphi_0, v) \setminus B(x, r/2) \subset \mathbb{R}^3 \setminus \overline{U}, \tag{3.7}$$

where $\varphi_0 = \pi/4$.

In the remaining part of this section we show how to derive Theorems 3.1 and 3.2 from Theorem 3.3. We begin with an auxiliary result which gives an estimate for the infimum of stopping distances considered in Theorem 3.3. Note that for Σ of class C^1, compact and closed, property (i) below is obvious: we have $d_s(x_0) > \delta_0(x_0)$, and, as mentioned in Example 2.5, in this case one can in fact choose a positive δ_0 independent of x_0.

Proposition 3.5. *Assume that $p > 8$, $\Sigma \in \mathscr{A}$ and $\mathscr{M}_p(\Sigma) < \infty$. Then*

(i) *The stopping distances $d_s(x_0)$ given by Theorem* 3.3 *have a positive greatest lower bound,*

$$d(\Sigma) := \inf_{x_0 \in \Sigma^*} d_s(x_0) > 0.$$

(ii) *We have*

$$\mathscr{M}_p(\Sigma) \geqslant \alpha^{5p} d(\Sigma)^{8-p}. \tag{3.8}$$

Proof. To prove (i), we argue by contradiction. Assume that $d(\Sigma) = 0$ and set

$$\varepsilon := \frac{1}{2}\left(\frac{\alpha^{5p}}{K(\Sigma)^4 \mathscr{M}_p(\Sigma)}\right)^{1/(p-8)}, \tag{3.9}$$

where $K(\Sigma)$ is the constant from Definition 2.4(i). Select $x_0 \in \Sigma^*$ with $d_s(x_0) =: d_0 < \varepsilon$. Pick x_1, x_2, x_3 whose existence is guaranteed by Theorem 3.3. Perturbing these points slightly, by at most $\alpha d_0/2$, we may assume that

$$x_i \in \Sigma^*, \quad i = 0, 1, 2, 3, \tag{3.10}$$

$$T = (x_0, x_1, x_2, x_3) \in \mathscr{V}\left(\eta/2, \frac{3}{2} d_0\right), \tag{3.11}$$

$$\|T' - T\| < \alpha d_0/2 \quad \Rightarrow \quad T' \in \mathscr{V}\left(\eta/2, \frac{3}{2} d_0\right). \tag{3.12}$$

Integrating over all T' close to T, we now estimate the energy as follows:

$$\begin{aligned}\mathscr{M}_p(\Sigma) &\geqslant \int\limits_{\Sigma^4\cap\mathscr{B}_{\alpha d_0/2}(T)} \mathcal{K}^p(T')\,d\mu(T') \\ &> \frac{1}{K(\Sigma)^4}\left(\frac{\alpha d_0}{2}\right)^8\left[\frac{1}{50^2}\left(\frac{\eta}{2}\right)^4(3d_0/2)^{-1}\right]^p \\ &> \frac{1}{K(\Sigma)^4}d_0^{8-p}\left(\frac{\alpha\eta^4}{50^2\cdot 2^6}\right)^p \\ &\geqslant \frac{\alpha^{5p}}{K(\Sigma)^4}d_0^{8-p} \quad \text{as } \eta/20\geqslant\alpha. \end{aligned} \tag{3.13}$$

(We have used Definition 2.4(i) and Lemma 2.9 in the second inequality.)

This gives a contradiction with (3.9) and the choice of d_0, as (3.13) implies $d_0 > 2\varepsilon$.

(ii) Now we shall show that $d(\Sigma)$ is not only strictly positive, but has a lower bound depending only on the energy. Fix $\varepsilon > 0$ small and pick $x_0 \in \Sigma^*$ with $d_0 := d_s(x_0) < (1+\varepsilon)d(\Sigma)$. As in the first part of the proof, take x_1, x_2, x_3 given by Theorem 3.3. Perturbing these points slightly, we may assume that (3.10)–(3.12) are satisfied. Moreover, by (3.2)

$$\frac{\alpha d_0}{2} < \frac{d_0}{80} < d(\Sigma) \leqslant d_s(x_i) \quad \text{for } i = 1, 2, 3,$$

so that by (3.4)

$$\mathscr{H}^2\big(\Sigma\cap B(x_i,\alpha d_0/2)\big) \geqslant \frac{\pi}{2}\left(\frac{\alpha d_0}{2}\right)^2, \quad i = 0, 1, 2, 3.$$

Using this information, we again estimate the energy as in (3.13), replacing now the constant $1/K(\Sigma)$ by an absolute one, $\frac{\pi}{2}$. This yields

$$\begin{aligned}\mathscr{M}_p(\Sigma) &> \left(\frac{\pi}{2}\right)^4\alpha^{5p}d_0^{8-p} \\ &\geqslant \alpha^{5p}(1+\varepsilon)^{8-p}d(\Sigma)^{8-p}.\end{aligned}$$

Upon letting $\varepsilon \to 0$, we conclude the whole proof. □

Proof of Theorem 3.1. Inequality (3.8) implies that

$$d(\Sigma) \geqslant \left(\frac{\alpha^{5p}}{\mathscr{M}_p(\Sigma)}\right)^{\frac{1}{p-8}}.$$

Combining this estimate with (3.4), we see that

$$\mathscr{H}^2\big(\Sigma\cap B(x,r)\big) \geqslant \frac{\pi}{2}r^2, \quad r\in\big(0,d(\Sigma)\big] \tag{3.14}$$

holds for all $x \in \Sigma^*$. Since Σ^* is dense in Σ, (3.14) must in fact hold for *all* $x \in \Sigma$.

Proof of Theorem 3.2. We shall construct inductively a (possibly finite) sequence of tetrahedra with vertices in Σ^*.

Initially, we pick an arbitrary point $x_0 = x_0^1 \in \Sigma^*$. Let $d_1 := d_s(x_0^{(1)}) > 0$. Use Theorem 3.3 and density of Σ^* to find a tetrahedron

$$T_1 = \left(x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) \in \mathscr{V}(\eta/2, 3d_1/2) \cap \left(\Sigma^*\right)^4 \tag{3.15}$$

such that

$$\left\|T' - T_1\right\| \leqslant \frac{\alpha d_1}{2} \quad \Rightarrow \quad T' \in \mathscr{V}(\eta/2, 3d_1/2). \tag{3.16}$$

Assume that $T_1, T_2, \dots, T_k$ have been already defined, $T_j = (x_0^{(j)}, x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$ for $j = 1, \dots, k$, so that the following properties are satisfied:

$$d_j := d_s\left(x_0^{(j)}\right) < \frac{\alpha d_{j-1}}{4}, \quad j = 2, \dots, k, \tag{3.17}$$

$$T_j \in \mathscr{V}(\eta/2, 3d_j/2) \cap \left(\Sigma^*\right)^4, \tag{3.18}$$

$$\|T - T_j\| \leqslant \frac{\alpha d_j}{2} \quad \Rightarrow \quad T \in \mathscr{V}(\eta/2, 3d_j/2), \tag{3.19}$$

$$x_0^{(j)} = x_{i(j)}^{(j-1)} \quad \text{for some } i(j) \in \{1, 2, 3\}. \tag{3.20}$$

(The last property simply means that T_j and T_{j-1} have one vertex in common.) Now for $y \in \Sigma$, let

$$R_*(y) = \sup\left\{r > 0\colon \mathscr{H}^2\left(\Sigma \cap B(y, \varrho)\right) \geqslant \pi \varrho^2/2 \text{ for all } \varrho \in (0, r]\right\}.$$

We consider the following stopping condition:

$$R_*\left(x_i^{(k)}\right) \geqslant \frac{\alpha d_k}{4} =: r_k \quad \text{for all } i \in \{1, 2, 3\}. \tag{3.21}$$

For a fixed value of k, there are two cases possible.

Case 1. Condition (3.21) does hold. We then estimate the energy, integrating over small balls centered at vertices of T_k. This yields

$$\begin{aligned} \mathscr{M}_8(\Sigma) &\geqslant \int\limits_{\Sigma^4 \cap \mathscr{B}_{r_k}(T_k)} \mathcal{K}^8(T)\, d\mu(T) \\ &> \left(\frac{\pi}{2}\right)^4 r_k^8 \left[\frac{1}{50^2}\left(\frac{\eta}{2}\right)^4 (3d_k/2)^{-1}\right]^8 \quad \text{by Lemma 2.9} \\ &=: \gamma_0 > 0, \end{aligned}$$

where the constant γ_0 depends *only* on the choice of α and η (note that the ratio $r_k/d_k = \alpha/4$ is constant). This is the desired estimate of $\mathscr{M}_8(\Sigma)$.

Case 2. Condition (3.21) fails. Then we choose $i(k) \in \{1, 2, 3\}$ such that by (3.2)

$$R_*\big(x_{i(k)}^{(k)}\big) < r_k = \frac{\alpha d_k}{4} < \frac{1}{160} d_k.$$

We set $x_0^{(k+1)} := x_{i(k)}^{(k)}$ and $d_{k+1} := d_s(x_0^{(k+1)})$. The choice of $i(k)$ gives

$$d_{k+1} < \alpha d_k/4 < d_k/160. \tag{3.22}$$

Again, we use Theorem 3.3 and density of Σ^* to find the next tetrahedron

$$T_{k+1} = \big(x_0^{(k+1)}, x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}\big) \in \mathscr{V}(\eta/2, 3d_{k+1}/2) \cap \big(\Sigma^*\big)^4$$

such that (3.19) is satisfied for $j = k + 1$. Thus, we have increased the length of sequence of tetrahedra satisfying (3.17)–(3.20).

Note that if the stopping condition (3.21) is satisfied for some $k = 1, 2, \ldots$, then we are done. The only possibility left to consider is that (3.21) fails for each k. We then have an infinite sequence of tetrahedra satisfying (3.17)–(3.20). To prove that this also gives the desired result, we shall show later that

$$\text{the sets } \Sigma^4 \cap \mathscr{B}_{r_k}(T_k), \quad k = 1, 2, \ldots, \text{ are pairwise disjoint.} \tag{3.23}$$

Assuming (3.23) for the moment, we have by Definition 2.4(i) and Lemma 2.9

$$\begin{aligned}
\mathscr{M}_8(\Sigma) &\geqslant \sum_{k=1}^{\infty} \int_{\Sigma^4 \cap \mathscr{B}_{r_k}(T_k)} \mathcal{K}^8(T)\, d\mu(T) \\
&> \sum_{k=1}^{\infty} \frac{1}{K(\Sigma)^4} r_k^8 \left[\frac{1}{50^2} \left(\frac{\eta}{2}\right)^4 (3d_k/2)^{-1} \right]^8 \\
&= \frac{\gamma_1}{K(\Sigma)^4} \sum_{k=1}^{\infty} 1 \\
&= +\infty,
\end{aligned}$$

where γ_1 denotes some constant depending *only* on the choice of α and η (again, note that $r_k/d_k = \alpha/4$ for each k).

It remains to prove (3.23). Since $T_k \in \mathscr{V}(\eta/2, 3d_k/2)$ for each k, we have by virtue of Part (i) of Definition 2.1

$$\big|x_0^{(k+1)} - x_0^{(k)}\big| = \big|x_{i(k)}^{(k)} - x_0^{(k)}\big| \leqslant 3d_k,$$

so that (3.22) implies for each $m > k$

$$\begin{aligned}\left|x_0^{(m)} - x_0^{(k)}\right| &\leqslant 3d_k + 3d_{k+1} + \cdots \\ &< 3d_k\bigl(1 + 160^{-1} + 160^{-2} + \cdots\bigr) \\ &< 4d_k. \end{aligned} \tag{3.24}$$

For $m = k$ (3.24) holds trivially. Also by definition of $\mathscr{V}(\eta/2, 3d_{k-1}/2)$ we have

$$\begin{aligned}\left|x_0^{(k)} - x_0^{(k-1)}\right| &= \left|x_{i(k-1)}^{(k-1)} - x_0^{(k-1)}\right| \\ &\geqslant \frac{\eta}{2}\frac{3d_{k-1}}{2} \\ &> 15\alpha d_{k-1} \quad \text{as } \eta > 20\alpha. \end{aligned} \tag{3.25}$$

Using (3.24), (3.25), and the condition $4d_k < \alpha d_{k-1}$, we obtain

$$\begin{aligned}\left|x_0^{(m)} - x_0^{(k-1)}\right| &\geqslant \left|x_0^{(k)} - x_0^{(k-1)}\right| - \left|x_0^{(m)} - x_0^{(k)}\right| \\ &> 15\alpha d_{k-1} - 4d_k \\ &> 14\alpha d_{k-1} \end{aligned}$$

for each $m \geqslant k$. The last inequality readily implies that $B_{r_m}(x_0^{(m)})$ and $B_{r_{k-1}}(x_0^{(k-1)})$ are disjoint for all $m \geqslant k$, as

$$r_m + r_{k-1} = \frac{\alpha}{4}(d_m + d_{k-1}) < \frac{\alpha d_{k-1}}{2}.$$

Thus, the sets $\mathscr{B}_{r_m}(T_m)$ and $\mathscr{B}_{r_{k-1}}(T_{k-1})$ are disjoint in $(\mathbb{R}^3)^4$, which proves (3.23).

The whole proof of Theorem 3.2 is complete now. □

4. Good tetrahedra: proof of Theorem 3.3

The proof of Theorem 3.3 is lengthy but elementary. It is of algorithmic nature and, at each of finitely many steps, requires a case inspection which from a geometric point of view is not so complicated but nevertheless includes three different cases (and one of them has to be divided into three further subcases). The crucial task is to find a triple (x_1, x_2, x_3) such that the x_i's $(i = 0, 1, 2, 3)$ satisfy conditions (i) and (3.3) of the theorem. Condition (ii) follows then from simple estimates based on elementary linear algebra; for sake of completeness, we present the details of that part in Section 4.3.

Here are a few informal words about the main idea of the proof.

Assume for a while that $\Sigma = \partial U$ is of class C^1. To find a candidate for x_1, we look at the surface $M_\rho = \partial B_\rho \cap C$, where $\rho > 0$, B_ρ is centered at x_0, and C is a double cone with vertex x_0, fixed opening angle, and axis given by $n(x_0)$, the normal to Σ at x_0. For small $\rho > 0$, x_0 is the only point of Σ in $C_\rho := B_\rho \cap C$. (If $\Sigma \in \mathscr{A}$ is not C^1, then the existence of an appropriate cone follows from Part (ii) of Definition 2.4.)

It is clear that as ρ increases, the growing cone C_ρ must hit Σ for some (possibly large) $\rho = \rho_1 > 0$, at some $x_1 \in \Sigma \setminus \{x_0\}$, $x_1 \in M_{\rho_1}$. If the point of the first hit, x_1, is close to the center of one of the two "lids" $\mathscr{M}_{\rho_1}$ of the cone C_{ρ_1}, then we can use the fact that the two components

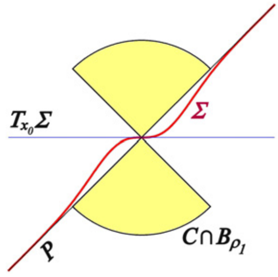

Fig. 1. A little kink determines $T_{x_0}\Sigma$.

U^+, U^- of $\operatorname{int} C_{\rho_1}$ are on two different sides of Σ to select a voluminous tetrahedron with two of its vertices at x_0 and x_1, and all edges $\approx \rho_1$. To convince yourself that this is indeed plausible, note that there are many segments perpendicular to $T_{x_0}\Sigma$, with one endpoint in U^+ and the other in U^-; each such segment intersects Σ and therefore contains a candidate for one of the remaining vertices. And, as we shall check later, many of those candidates are good enough for our purposes.

However, it might happen that for this particular intermediate value of $\rho_1 > 0$—somewhere between $\operatorname{diam}\Sigma$ and the infinitesimal scale where a smooth Σ is very close to the tangent plane—most points of $\Sigma \cap B_{\rho_1}$ are very close to a fixed plane P which might be completely different from $T_{x_0}\Sigma$, due to a little kink of Σ near to x_0. In fact, such a plane might be tangent to ∂C, and $\Sigma \cap B_{\rho_1}$ would look pretty flat at all length scales $\approx \rho_1$.

If this were the case, then x_1 would be located close to the rim of $C \cap B_{\rho_1}$, and one could not expect to find a good tetrahedron with vertices $x_i \in \Sigma \cap B_{\rho_1}$ and edges $\approx \rho_1$. But then, one might rotate C around an axis contained in $T_{x_0}\Sigma$, away from such a plane P, to a new position C' chosen so that two connected components of $C' \cap (B_{\rho_1} \setminus B_{\rho_1/2})$ are still on two different sides of Σ. One could look for possible vertices of a good tetrahedron in $C' \cap B_\rho$ for $\rho > \rho_1$, enlarging the radius ρ until $C' \cap (B_\rho \setminus B_{\rho_1})$ hits the surface again. This would happen for some radius $\rho_2 > \rho_1$.

It might turn out again that at scales comparable to ρ_2 large portions of Σ are almost flat, close to a single fixed plane P' which is tangent to $\partial C'$ so that it is not at all evident how to indicate a voluminous tetrahedron with vertices $x_i \in \Sigma \cap B_{\rho_2}$ and edges $\approx \rho_2$. One could try then to iterate the reasoning, rotating portions of the cones if necessary.

Several steps like that might be needed if, for example, x_0 were at the end of a long tip that spirals many times—in such cases the points of Σ that we hit, enlarging the consecutive cones, might not convey enough information about the shape of the surface. We make all this precise (including a stopping mechanism, a procedure which allows one to select appropriate rotations at each step of the iteration, and a bound on the number of steps) in Section 4.2, using Definition 2.4(ii) to construct the desired cones for small radii. Before, in Section 4.1, we state two elementary geometric lemmata which are then used to obtain (i) and (3.3) for various quadruples (x_0, x_1, x_2, x_3).

Without loss of generality we suppose throughout Section 4 that $x_0 = (0, 0, 0) \in \mathbb{R}^3$.

4.1. Slanted planes and good vertical segments

Suppose that we have a fixed a cone $C = C(\varphi_0, v)$ in $\mathbb{R}^3$, where $v \in \mathbb{S}^2$ and $\varphi_0 \in (0, \frac{\pi}{4}]$. We also fix an auxiliary angle $\varphi_1 \in (0, \frac{\pi}{2}]$.

Throughout this subsection, we say that a segment I is *vertical* (with respect to the cone C) if I is parallel to v, i.e., $I = I_{s,v}(z)$ for some $s > 0$ and $z \in \mathbb{R}^3$. Any plane $P = \langle 0, y_1, y_2 \rangle$ whose unit normal n satisfies $0 < |n \cdot v| < 1$ is called *slanted*. We say that I is *good* (*for* P) iff $\operatorname{dist}(I, P) \approx \operatorname{diam} I$, up to constants depending *only* on the angles φ_i.

We state and prove two elementary lemmata which give quantitative estimates of the distance between good vertical segments I and slanted planes spanned by 0 and two other points y_1, y_2. In the first lemma both y_i have to be in $C \cap F$, on the same affine plane F whose normal equals the cone axis of C, i.e. with unit normal $n_F = v$. In the second lemma we keep one of the y_i's in C and allow the other one to belong to a portion of C', where C' is a cone congruent to C but rotated by an angle $\gamma \in (0, \varphi_0/2]$.

To fix the whole setting, pick a radius $\rho > 0$. Set $h = \rho \cos\varphi_0$ and $r = \rho \sin\varphi_0$. Moreover, set $H := v^\perp \subset \mathbb{R}^3$, and let $\pi_H\colon \mathbb{R}^3 \to H$ be the orthonormal projection onto H. Let σ_F denote the central projection from 0 to the affine plane $F := H + hv$.

Lemma 4.1 *(Slanted planes and good vertical segments, I). Suppose that $P = \langle 0, y_1, y_2 \rangle \subset \mathbb{R}^3$ is spanned by* 0 *and two other points $y_1 \neq y_2 \in F \cap C_\rho(\varphi_0, v)$ such that there is an angle $\varphi_1 \in (0, \pi)$ such that*

$$\pi > \measuredangle\big(\pi_H(y_1), \pi_H(y_2)\big) \geqslant \varphi_1 \quad \textit{and} \quad \pi_H(y_i) \neq 0 \quad \textit{for } i = 1, 2.$$

Then, there exists a point $z \in H \cap \partial B_r$ such that

$$\operatorname{dist}\big(I_{h,v}(z), P\big) \geqslant c_0 \rho, \tag{4.1}$$

where the constant

$$c_0 := c_0(\varphi_0, \varphi_1) = \frac{1}{2}\left(1 - \cos\frac{\varphi_1}{2}\right) \sin 2\varphi_0 > 0. \tag{4.2}$$

Proof. Let $z_i := \pi_H(y_i)$ for $i = 1, 2$. Consider the 2-dimensional disk (see Fig. 2)

$$D := H \cap B_r \ni z_1, z_2.$$

Let $\gamma := H \cap \partial B_r$ be the boundary of D in H. We select $z \in \gamma$ such that $z \perp z_2 - z_1$ and the segment $[0, z]$ has a common point with the straight line l which passes through z_1 and z_2. By elementary planar geometry, we have

$$\begin{aligned} d &:= \operatorname{dist}(z, l) \\ &\geqslant r\left(1 - \cos\frac{\varphi_1}{2}\right) \\ &= \rho \sin\varphi_0 \left(1 - \cos\frac{\varphi_1}{2}\right). \end{aligned} \tag{4.3}$$

Now, let ψ denote the angle between v and P. It is easy to see that we have $0 < \psi < \varphi_0$ since $\varphi_1 \in (0, \pi/2]$ and $y_1 \neq y_2 \in F \cap C_\rho(\varphi_0, v)$. Thus,

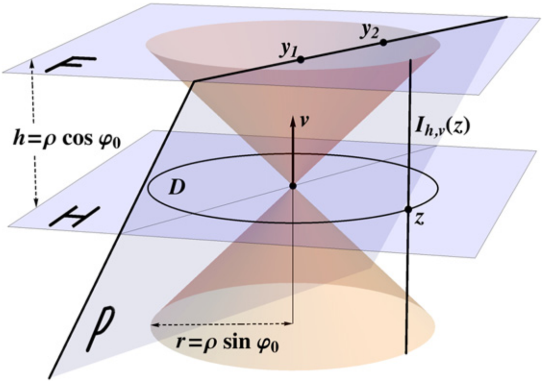

Fig. 2. The setting in Lemma 4.1: a double cone and three planes H, F, P.

$$\begin{aligned}\operatorname{dist}\bigl(I_{h,v}(z), P\bigr) &= d\cos\psi \geqslant d\cos\varphi_0\\ &\overset{(4.3)}{\geqslant} \rho\cos\varphi_0\,\sin\varphi_0\left(1-\cos\frac{\varphi_1}{2}\right)\\ &= c_0\rho,\end{aligned}$$

where the constant c_0 is given by (4.2). □

Lemma 4.2 *(Slanted planes and good vertical segments, II). Let $y_1 \in F \cap B_\rho$, assume $\pi_H(y_1) \neq 0$ and set $u = \pi_H(y_1)/|\pi_H(y_1)|$. Let $w := u \times v$ and consider the family of rotations $R_s := R(s\varphi_0, w)$, where $s \in [0, \frac{1}{2}]$. Then, for any point*

$$y_2 \in \bigcup_{s\in[0,1/2]} R_s\bigl(C_\rho(\varphi_0, v) \setminus \operatorname{int} B_{\rho/2}\bigr) \quad \text{such that } y_2 \cdot u < 0 < y_2 \cdot v \tag{4.4}$$

there exists a point $z \in H \cap \partial B_r$ such that $\operatorname{dist}(I_{h,v}(z), \langle 0, y_1, y_2\rangle) \geqslant c_1\rho$. One can take

$$c_1 \equiv c_1(\varphi_0) = \frac{1}{16}\sin 2\varphi_0 > 0.$$

Proof. Consider the two-dimensional disk $D := F \cap B_\rho$ and its boundary circle $\gamma = F \cap \partial B_\rho$. Note that the radius of D equals $r = \rho\sin\varphi_0$. The key point is to observe what the union of all the central projections $\sigma_F(R_s(D))$, $s \in [0, 1/2]$, looks like. The rest will follow from the previous lemma.

Without loss of generality we assume that $v = (0, 0, 1) \in \mathbb{S}^2$ and $y_1 = (a, 0, h) \in \mathbb{R}^3$ for some $a \in (0, r]$. Then $u = \pi_H(y_1)/|\pi_H(y_1)| = (1, 0, 0)$ and $w = u \times v = (0, -1, 0)$. In the standard basis of $\mathbb{R}^3$ – which is $(u, -w, v)$ – the rotations $R_s = R(s\varphi_0, w)$ are given by

$$R_s = \begin{pmatrix} \cos s\varphi_0 & 0 & -\sin s\varphi_0\\ 0 & 1 & 0\\ \sin s\varphi_0 & 0 & \cos s\varphi_0 \end{pmatrix}.$$

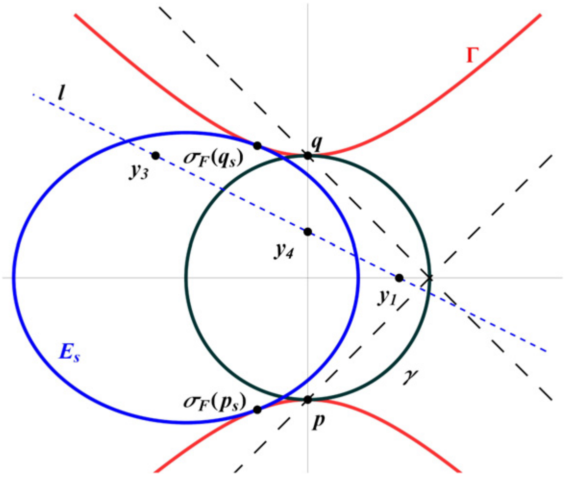

Fig. 3. The situation in F.

Now, consider the points $p = (0, -r, h)$ and $q = (0, r, h)$ in $\gamma \subset F$. Let $p_s = R_s(p)$ and $q_s = R_s(q)$, $s \in [0, \frac{1}{2}]$. Since the axis of rotation contains w, the angles $\measuredangle(p_s, w)$ and $\measuredangle(q_s, -w)$ are constant for all s and equal $\frac{\pi}{2} - \varphi_0$. Thus, as s goes from 0 to $\frac{1}{2}$, the points p_s, q_s move along arcs of vertical circles on $\partial C(\frac{\pi}{2} - \varphi_0, w)$. Hence, the central projections $\sigma_F(p_s)$ and $\sigma_F(q_s)$ trace arcs of two branches of the hyperbola

$$\Gamma := F \cap \partial C\left(\frac{\pi}{2} - \varphi_0, w\right).$$

(In fact, as s goes from 0 to $\frac{1}{2}$, the point $\sigma_F(R_s(x))$ moves along a hyperbola in F for each $x \in D$, except the x's that lie on the diameter of D parallel to u.)

Note also that, for each $s \in [0, \frac{1}{2}]$, the central projection $\sigma_F(R_s(D))$ is equal to an ellipse E_s which is tangent to both arms of Γ at $\sigma_F(p_s)$ and $\sigma_F(q_s)$.

Suppose now that y_2 satisfies (4.4). Since

$$\sigma_F\bigl(R_s\bigl(C_\rho(\varphi_0, v) \setminus \operatorname{int} B_{\rho/2}\bigr)\bigr) = \sigma_F\bigl(R_s(D)\bigr),$$

and the plane $P = \langle 0, y_1, y_2\rangle$ contains the line through 0 and y_2, we have $y_3 := \sigma_F(y_2) \in P$. Therefore $P = \langle 0, y_1, y_3\rangle$.

As $y_2 \cdot u < 0 < y_2 \cdot v$, the first coordinate of $y_3 = \sigma_F(y_2)$ is negative. Hence, the line l which passes through y_3 and y_1 in F, and satisfies $l = P \cap F$, contains a point $y_4 \in P \cap F$ on the diameter of D whose endpoints are p and q. Thus, $\langle 0, y_1, y_2\rangle = P = \langle 0, y_1, y_4\rangle$. If y_4 is not in the center of D (as on Fig. 3 above), then the desired claim follows from the previous lemma, applied for $P = \langle 0, y_1, y_4\rangle$ and $\varphi_1 = \pi/2$. If $y_4 =$ the center of D, then the plane P is vertical and one can take e.g. $z = \pi_H(p)$ to conclude the proof. In that case one has

$$\operatorname{dist}\bigl(I_{h,v}(z), P\bigr) = r = \rho \sin\varphi_0 > \frac{\rho}{16}\sin 2\varphi_0 = \rho c_1. \qquad \square$$

4.2. Looking for good vertices x_1, x_2, x_3: the iteration

Throughout this subsection we assume that $0 = x_0 \in \Sigma^* \subset \Sigma = \partial U$, where $U \Subset \mathbb{R}^3$ is bounded; Σ belongs to the class $\mathscr{A}$ of all admissible surfaces as defined in Definition 2.4. Fix $\varphi_0 = \frac{\pi}{4}$. Proceeding iteratively, we shall construct four finite sequences:

- of compact, connected, centrally symmetric sets $S_0 \subset T_1 \subset S_1 \subset T_2 \subset S_2 \subset \cdots \subset S_{N-1} \subset T_N \subset S_N \subset \mathbb{R}^3$,
- of unit vectors $v_0, \dots, v_N, v_0^*, \dots, v_{N-1}^* \in \mathbb{S}^2$ such that $\measuredangle(v_i, v_i^*) = \varphi_0/2 = \pi/8$ for each $i = 0, \dots, N-1$,
- of two-dimensional subspaces $H_i = (v_i)^\perp \subset \mathbb{R}^3$, $i = 0, \dots, N$,
- and of radii $\rho_0 < \rho_1 < \cdots < \rho_N$, where $\rho_N =: d_s(x_0)$, so ρ_N will provide the desired stopping distance for x_0 as claimed in Theorem 3.3.

These sequences will be shown to satisfy the following properties:

(A) **(Diameter of S_i grows geometrically).** We have $S_i \subset B_{\rho_i} \equiv B(0, \rho_i)$ and $\operatorname{diam} S_i = 2\rho_i$ for $i = 0, \dots, N$. Moreover

$$\rho_i > 2\rho_{i-1} \quad \text{for } i = 1, \dots, N. \tag{4.5}$$

(B) **(Large 'conical caps' in S_i and T_i).**

$$S_i \setminus B_{\rho_{i-1}} = C_{\rho_i}(\varphi_0, v_i) \setminus B_{\rho_{i-1}} \quad \text{for } i = 1, \dots, N, \tag{4.6}$$

and

$$T_{i+1} \subset B_{\rho_i} \quad \text{and} \quad S_i \subset T_{i+1} \quad \text{for } i = 0, \dots, N-1. \tag{4.7}$$

(C) **(Relation between S_i and T_{i+1}).** For each $i = 0, \dots, N-1$ there is a unit vector $w_i \perp v_i$ and a continuous one-parameter family of rotations R_s^i with axis parallel to w_i and rotation angle $s\varphi_0$, $s \in [0, 1/2]$, such that

$$T_{i+1} = S_i \cup \bigcup_{s \in [0,1/2]} R_s^i\big(C_{\rho_i}(\varphi_0, v_i) \setminus \operatorname{int} B_{\rho_i/2}\big). \tag{4.8}$$

(D) **(Σ does not enter the interior of S_i or T_{i+1}).**

$$\Sigma \cap \operatorname{int} S_i = \emptyset \quad \text{for } i = 0, \dots, N, \tag{4.9}$$

$$\Sigma \cap \operatorname{int} T_{i+1} = \emptyset \quad \text{for } i = 0, \dots, N-1. \tag{4.10}$$

Moreover, we have

$$\Sigma \cap \partial B_r \cap C\big(\varphi_0, v_i^*\big) = \emptyset \quad \text{for } \rho_i \leqslant r \leqslant 2\rho_i,\ i = 0, \dots, N-1, \tag{4.11}$$

and

$$\partial B_t \cap C^+(\varphi_0, v_i) \subset U \quad \text{and} \quad \partial B_t \cap C^-(\varphi_0, v_i) \subset \mathbb{R}^3 \setminus \overline{U}, \tag{4.12}$$

for all $t \in (\rho_{i-1}, \rho_i)$ and $i = 1, \dots, N$.

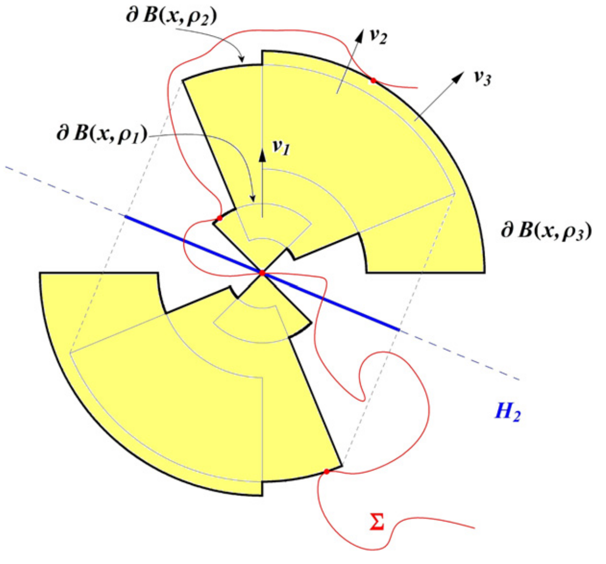

Fig. 4. A possible outcome of the iterative construction. Here, $x = x_0$ is at the center of the picture and we have $N = 3$. The position of the disk $B_{r_2} \cap H_2$, containing the annulus A_2 mentioned in condition (4.14) of (F), is marked with a thick line.

(E) **(Points of $\Sigma \setminus \{x_0\}$ on ∂S_i).** The intersection $\Sigma \cap \partial B_{\rho_i} \cap \partial S_i$ is nonempty for each $i = 1, \ldots, N$.

(F) **(Big projections of $B_{\rho_i} \cap \Sigma$ onto H_i).** For $t \in [\rho_{i-1}, \rho_i]$, $i = 1, \ldots, N$ we have

$$\pi_{H_i}(\Sigma \cap B_t) \supset H_i \cap B_{t \sin\varphi_0}. \tag{4.13}$$

Moreover, for $r_i = \rho_i \sin\varphi_0$, $i = 1, \ldots, N$,

$$I_{|z|,v_i}(z), \quad z \in A_i := H_i \cap (B_{r_i} \setminus \operatorname{int} B_{r_i/2}) \text{ contains at least one point of } \Sigma. \tag{4.14}$$

Once this is achieved (see Fig. 4 for a possible outcome of the construction), Condition (E) implies that

$$\mathscr{H}^2\big(\Sigma \cap B(x_0, r)\big) \geqslant \mathscr{H}^2\big(D^2(0, r\sin\varphi_0)\big) = \pi r^2/2 \quad \text{for } 0 < r \leqslant \rho_N =: d_s(x_0),$$

where $D^2(p, s)$ denotes a planar disk with center p and radius s. We shall also show that it is possible to select $x_j \in B_{2\rho_N}$ ($j = 1, 2, 3$) with the desired properties listed in Theorem 3.3.

Start of the iteration. We set $S_0 := \emptyset$ and $T_1 := \emptyset$, $\rho_0 := 0$ and $v_0^* := v_1 := v(x_0)$, where $v(x_0) \in \mathbb{S}^2$ is given by Definition 2.4(ii). For v_0 we take any unit vector with the angle condition $\sphericalangle(v_0, v_0^*) = \sphericalangle(v_0, v_1) = \varphi_0/2 = \pi/8$. Then we have $H_0 := (v_0)^\perp$ and $H_1 := (v_1)^\perp$. Moreover, we use the convention that our closed balls are defined as

$$B_r = B(0, r) := \overline{\{y \in \mathbb{R}^3 \colon |y| < r\}}$$

so that the closed ball B_0 of radius zero is the empty set. Notice that for a complete iteration start we need to define ρ_1 and S_1 in order to check conditions (4.5) in (A), (4.6) in (B), (4.9) and (4.12) for $i = 1$, Condition (E), and (4.13) and (4.14) constituting Condition (F). All the other conditions within the list (A)–(D) are immediate for $i = 0$. We set

$$K_t^1 := C_t(\varphi_0, v_1). \tag{4.15}$$

With growing radii t the sets K_t^1 describe larger and larger double cones with constant opening angle $2\varphi_0 = \pi/2$ and fixed axis v_1. Now we define

$$\rho_1 := \inf\{t > \rho_0 = 0\colon\ \Sigma \cap K_t^1 \cap \partial B_t \neq \emptyset\}, \tag{4.16}$$

and notice that by definition of the set $\mathscr{A}$ of admissible surfaces (see Definition 2.4(ii)) one has $\rho_1 > \delta_0(x_0) > 0 = 2\rho_0$, which takes care of (4.5) in Condition (A) for $i = 1$. Set $S_1 := K^1_{\rho_1}$, then we see that $S_1 = C_{\rho_1}(\varphi_0, v_1) \subset B_{\rho_1}$ with $\operatorname{diam} S_1 = 2\rho_1$, so all properties of (A) hold for $i = 1$. Moreover,

$$S_1 = C_{\rho_1}(\varphi_0, v_1) = C_{\rho_1}(\varphi_0, v_1) \setminus B_{\rho_0},$$

since $B_{\rho_0} = B_0 = \emptyset$, thus (4.6) in (B) holds for $i = 1$. The definition of $\rho_1 > 0$ (see (4.16)) implies (4.9) for $i = 1$, notice that $\operatorname{int} S_1$ is the union of two disjoint open cones centrally symmetric to but not containing $x_0 = 0 \in \Sigma$. For the proof of (4.12) for $i = 1$ we observe that for each $t \in (0, \rho_1)$ we have by definition of ρ_1 that

$$B_t \cap C^+(\varphi_0, v_1) \subset U \cup \{0\} \quad \text{and} \quad B_t \cap C^-(\varphi_0, v_1) \subset \big(\mathbb{R}^3 \setminus \overline{U}\big) \cup \{0\}, \tag{4.17}$$

which is even stronger than (4.12). Condition (E) holds for $i = 1$, too, by definition of ρ_1 and the fact that Σ is a closed set. For $i = 1$ we will prove (4.14) even for *all* $z \in D_1 := H_1 \cap B_{r_1}$, which would immediately imply (4.13) of Condition (F).[3] From (4.17) we also infer that every segment $I_{|z|, v_1}(z)$, for $z \in H_1 \cap (B_{r_1} \setminus \{0\})$ with $|z| < r_1$, has one endpoint in U, and the other in $\mathbb{R}^3 \setminus \overline{U}$, which implies that $I_{|z|, v_1}(z)$ intersects the closed surface Σ in at least one point for these z. For $z = 0 = x_0 \in \Sigma$ this is trivially also true, and for $z \in D_1$ with $|z| = r_1$ we approximate $z_k \to z$ as $k \to \infty$ with points $z_k \in D_1$ and $|z_k| < r_1$ to find a sequence $\xi_k \in \Sigma \cap I_{|z_k|, v_1}(z_k)$ which converges to some surface point $\xi \in \Sigma \cap I_{|z|, v_1}(z)$. This completes the proof of (4.14) even for all $z \in H_1 \cap B_{r_1}$ and hence of (F) for $i = 1$.

To summarize this first step, we have defined the sets $S_0 \subset T_1 \subset S_1 \subset \mathbb{R}^3$, the unit vectors $v_0, v_1, v_0^* \in \mathbb{S}^2$ with $\measuredangle(v_0, v_0^*) = \varphi_0/2$, and the corresponding subspaces $H_0 = (v_0)^\perp$, and $H_1 = (v_1)^\perp$, and finally radii $\rho_0 = 0 < \rho_1$ without having made the decision if $N = 1$ or $N > 1$. In addition we have now proved the first two items in Condition (A) for $i = 0, 1$, and (4.5) for $i = 1$. Moreover, we have verified (4.9) for $i = 0, 1$, and all other statements in the list of properties (B)–(F) are established for the respective smallest index i. Note, however, that we have not defined v_1^* yet.

[3] Alternatively, one could look for $t \in (0, \rho_1)$ at the (longer) vertical segments $I_{\psi(t), v_1}(z)$, $\psi(t) := \sqrt{t^2 - |z|^2}$, whose endpoints are contained in $\partial B_t \cap C^+(\varphi_0, v_1)$, and in $\partial B_t \cap C^-(\varphi_0, v_1)$, respectively, use (4.12) for $i = 1$ as proved just before, to conclude that $I_{\psi(t), v_1}$ intersects Σ for each $t \in (0, \rho_1)$. This proves (4.13) for $t \in (0, \rho_1)$, the statement for $t = 0 = \rho_0$ is trivial, and for $t = \rho_1$ use continuity, and the fact that Σ is a closed set. This is actually the argument we repeat in the induction step $j \mapsto j + 1$ later on, since there we have less explicit information about S_j.

Stopping criteria and the iteration step. For the decision whether to stop the iteration or to continue it with step number $j+1$ for $j \geqslant 1$, we may now assume that the sets

$$S_0 \subset T_1 \subset S_1 \subset T_2 \subset S_2 \subset \cdots \subset T_j \subset S_j \subset \mathbb{R}^3,$$

and unit vectors $v_0, \dots, v_j, v_0^*, \dots, v_{j-1}^*$ with $\sphericalangle(v_i, v_i^*) = \varphi_0/2$ for $i = 0, \dots, j-1$, are defined. We also have at this point a sequence of radii $\rho_0 = 0 < \rho_1 < \cdots < \rho_j$ satisfying (4.5) for $i = 1, \dots, j$. The first two conditions in (A) may be assumed to hold for $i = 0, \dots, j$. In (B) we may suppose (4.6) for $i = 1, \dots, j$, in contrast to (4.7) which holds for $i = 0, \dots, j-1$. Similarly, we may now work with (4.8) in (C), (4.10) and (4.11) in (D) for all $i = 0, \dots, j-1$, whereas we may use (4.9) in (D) for $i = 0, \dots, j$, (4.12), Condition (E), and (4.13) and (4.14) in (F) now for $i = 1, \dots, j$.

Now we are going to study the various geometric situations that allow us to stop the iteration here, in which case we set $N := j$, $d_s(x_0) := \rho_j = \rho_N$, so that (3.3) and (3.4) stated in Theorem 3.3 can be extracted for $H := H_j$ directly from Condition (F). Indeed, (4.13) for $t := \rho_j = \rho_N$ yields (3.3) since $\varphi_0 = \pi/4$. How to find the remaining vertices x_1, x_2, x_3 such that Statement (i) of Theorem 3.3 holds for the tetrahedron $T = (x_0, x_1, x_2, x_3)$ will be explained later in detail for each case in which we stop the iteration. Moreover, we will convince ourselves that the only case in which the iteration cannot be stopped, can happen only finitely many times. But each time this happens we have to define unit vectors $v_j^*, v_{j+1} \in \mathbb{S}^2$, with $\sphericalangle(v_j, v_j^*) = \varphi_0/2$, and $H_{j+1} := (v_{j+1})^\perp$, a new radius ρ_{j+1}, new sets $T_{j+1} \subset S_{j+1}$ containing S_j, and then check all the properties listed in (A)–(F).

The different geometric situations depend on how the surface hits the "roof" of the current centrally symmetric set S_j, that is, where the points of the nonempty intersection in Condition (E) lie:

Case 1. **(Central hit.)** By this we mean that $\Sigma \cap \partial B_{\rho_j} \cap C(\frac{3}{4}\varphi_0, v_j)$ is nonempty.

Case 2. **(No central hit but nice distribution of intersection points.)** By this we mean that Case 1 does not hold but there exist two different points $x_1, x_2 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ such that

$$\sphericalangle\big(\pi_{H_j}\big(\sigma(x_1)\big), \pi_{H_j}\big(\sigma(x_2)\big)\big) \geqslant \frac{\pi}{3}, \tag{4.18}$$

where π_{H_j} denotes the orthogonal projection onto the current plane $H_j = (v_j)^\perp$.

In Cases 1 and 2, we can find triples of points (x_1, x_2, x_3) with all the desired properties and stop the iteration right away. Below, in Sections 4.2.1 and 4.2.2, we indicate how to select the x_i's in each of these cases, and present the necessary estimates.

If neither Case 1 nor Case 2 occurs, then we have to deal with

Case 3. **(Antipodal position.)** $\Sigma \cap \partial B_{\rho_j} \cap C(\frac{3}{4}\varphi_0, v_j)$ is empty and for *any two different points* $x_1, x_2 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ we have

$$\sphericalangle\big(\pi_{H_j}\big(\sigma(x_1)\big), \pi_{H_j}\big(\sigma(x_2)\big)\big) < \frac{\pi}{3}. \tag{4.19}$$

(Intuitively, Case 3 corresponds to the situation alluded to in the introduction to Section 4: at this stage we have to take into account the possibility that most points of $\Sigma \cap B_{\rho_j}$ are close to some fixed 2-plane containing the segment with endpoints x_0, x_1.) Now this third case is more complicated, we will distinguish three further subcases, of which two will allow us to stop the iteration here. Only the third subcase will force us to continue the iteration.

To make this precise, let us fix some point $x_1 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$. Such a point does exist according to Condition (E). Set $u_j := \pi_{H_j}(x_1)/|\pi_{H_j}(x_1)|$ and $w_j := u_j \times v_j$, and consider the family of rotations

$$R_s^j := R(s\varphi_0, w_j), \quad s \in [0, 4]. \tag{4.20}$$

Consider the union of rotated conical caps

$$G_t^j := \bigcup_{0 \leqslant s \leqslant t} R_s\big(C_{\rho_j}(\varphi_0, v_j) \setminus \operatorname{int} B_{\rho_j/2}\big), \quad t \in \left[0, \frac{1}{2}\right]. \tag{4.21}$$

Let

$$t_0 := \sup\left\{t \in \left[0, \frac{1}{2}\right]: G_t \cap (\Sigma \setminus S_j) = \emptyset\right\}. \tag{4.22}$$

(Intuitively: we rotate the conical cap "away from the intersection $\Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$" and look for new points of Σ in the rotated set.) There are now three subcases possible. To describe them, let $v_j^* := R_{1/2}(v_j)$ (this will be the new v_{j+1} in the third subcase).

Subcase 3(a). $G_{t_0}^j \cap (\Sigma \setminus S_j) \neq \emptyset$. Then $j = N$; we stop the iteration and select x_2 and x_3, the remaining vertices of a good tetrahedron, using Lemma 4.2 to obtain the desired estimates; see Section 4.2.3 for the computations.

Intuitively, Subcase 3(a) corresponds to the situation where we initially suspect that the surface might be similar to the one with a little kink (see Fig. 1 at the beginning of Section 4). Condition (4.19) alone does not exclude this – but here, rotating a portion of the cone slightly, we find new points of Σ and detect that Σ is not flat at scale ρ_j.

Subcase 3(b). We have $t_0 = 1/2$ and $G_{1/2}^j \cap (\Sigma \setminus S_j) = \emptyset$. However,

$$\Sigma \cap \big(C_{2\rho_j}\big(\varphi_0, v_j^*\big) \setminus C_{\rho_j}\big(\varphi_0, v_j^*\big)\big) \neq \emptyset. \tag{4.23}$$

Again, $j = N$; we stop the iteration and select x_2 and x_3. For details, see Section 4.2.3.

Informally: here we rotate a portion of the cone slightly and do not find new points of Σ. However, there are other points of the surface at comparable distances, again allowing us to exclude the possibility that Σ is close to being flat at scale ρ_j.

Subcase 3(c). We have $t_0 = 1/2$ and

$$G_{1/2}^j \cap (\Sigma \setminus S_j) = \emptyset. \tag{4.24}$$

Moreover, (4.23) is violated, i.e.,

$$\Sigma \cap \big(C_{2\rho_j}\big(\varphi_0, v_j^*\big) \setminus C_{\rho_j}\big(\varphi_0, v_j^*\big)\big) = \emptyset. \tag{4.25}$$

If this is the case, then we are unable to exclude the possibility that (most of) Σ is nearly flat at the given scale, and the iteration goes on. We set $T_{j+1} := S_j \cup G^j_{1/2}$, $v_{j+1} := v_j^* = R_{1/2}(v_j)$, $H_{j+1} := (v_{j+1})^{\perp}$, and

$$K_t^{j+1} := C_t(\varphi_0, v_{j+1}), \tag{4.26}$$

and define

$$\rho_{j+1} := \inf\big\{t > \rho_j\colon\ \Sigma \cap K_t^{j+1} \cap \partial B_t \neq \emptyset\big\}. \tag{4.27}$$

Notice that condition (4.25) in the context of this subcase guarantees that $\rho_{j+1} > 2\rho_j$ which verifies (4.5) in Condition (A) for $i = j+1$. Now we define

$$S_{j+1} := T_{j+1} \cup \big(K^{j+1}_{\rho_{j+1}} \setminus \operatorname{int} B_{\rho_j}\big), \tag{4.28}$$

and check that Conditions (A)–(F) are satisfied. Indeed, $S_{j+1} \subset S_j \cup K^{j+1}_{\rho_{j+1}} \subset B_{\rho_j} \cup B_{\rho_{j+1}}$ by Condition (A) for $i = j$, which implies that (A) holds for $i = j+1$ as well. Next,

$$S_{j+1} \setminus B_{\rho_j} = K^{j+1}_{\rho_{j+1}} \setminus B_{\rho_j} = C_{\rho_{j+1}}(\varphi_0, v_{j+1}) \setminus B_{\rho_j},$$

since $S_j \subset B_{\rho_j}$ by Condition (A) for $i = j$. Hence (4.6) holds for $i = j+1$. As $G_t^j \subset B_{\rho_j}$ for all $t \in [0, 1/2]$ we have $T_{j+1} \subset S_j \cup B_{\rho_j} \subset B_{\rho_j}$ because of Condition (A) for $i = j$. The second item in (4.7) is a direct consequence of the definition of T_{j+1}, whence (4.7) holds for $i = j$. Condition (C) holds also for $i = j$ by definition of T_{j+1}. Using (4.9) for $i = j$, (4.24), and the definition of $\rho_{j+1} > 2\rho_j$ in (4.27) we infer that (4.9) holds for $i = j+1$, and (4.10) for $i = j$. Relation (4.11) for each $r \in (\rho_j, \rho_{j+1}]$ is an immediate consequence of (4.25). For $r = \rho_j$, however, we have to use (4.24) in combination with the fact that all surface points in $\Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ are in antipodal position described by (4.19), so that $\Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j^*) = \emptyset$.

Now we turn to the proof of (4.12) for $i = j+1$. The definition (4.27) of ρ_{j+1} implies that

$$\partial B_t \cap C^+(\varphi_0, v_{i+1}) \subset U, \quad \text{or} \quad \partial B_t \cap C^+(\varphi_0, v_{i+1}) \subset \mathbb{R}^3 \setminus \overline{U} \tag{4.29}$$

for all $t \in (\rho_j, \rho_{j+1})$. Now (4.24) together with (4.12) implies that

$$\partial B_{\rho_j} \cap C^+(\varphi_0, v_{i+1}) \subset U,$$

which excludes the second alternative in (4.29). Condition (E) holds for $i = j+1$ by the definition of ρ_{j+1} and the fact that Σ is a closed set. For the proof of (4.13) for $i = j+1$ we look for $t \in (\rho_j, \rho_{j+1})$ at the vertical segments $I_{\psi(t), v_{j+1}}(z)$, $\psi(t) := \sqrt{t^2 - |z|^2}$, $z \in B_{t \sin\varphi_0} \cap H_{j+1}$. The endpoints of these segments lie in $\partial B_t \cap C^+(\varphi_0, v_{j+1})$, and in $\partial B_t \cap C^-(\varphi_0, v_{j+1})$, respectively. Now we use (4.12) for $i = j+1$ to conclude that $I_{\psi(t), v_{j+1}}(z)$ intersects Σ for each $t \in (\rho_j, \rho_{j+1})$. This proves (4.13) for $t \in (\rho_j, \rho_{j+1})$. For $t = \rho_j$ and $t = \rho_{j+1}$ use continuity, and the fact that Σ is a closed set. Finally, to prove (4.14) for $i = j+1$ note that (4.5) together with (4.12) for $i = j+1$ imply that the two endpoints of the vertical segments $I_{|z|,v_{j+1}}$ for $z \in A_{j+1}$ lie in the different open connected components U and $\mathbb{R}^3 \setminus \overline{U}$. This suffices to conclude that these segments intersect Σ, which finishes the proof of all conditions in the list (A)–(F) in the iteration step.

Since we have established Condition (E) in the iteration step and (4.5) holds, too, we can deduce that Subcase 3(c) can happen only finitely many times, depending on the position x_0 on Σ and on the shape and size of Σ:

$$\operatorname{diam} \Sigma \geqslant \rho_i > 2\rho_{i-1} > \cdots > 2^{i-1}\rho_1 > 2^{i-1}\delta_0(x_0),$$

whence the maximal number of iteration steps is bounded by

$$1 + \log\bigl(\operatorname{diam} \Sigma / \delta_0(x_0)\bigr) / \log 2.$$

This concludes the Subcase 3(c). Now we have to analyze the geometric situation in the remaining Cases 1, 2, and 3(a) and 3(b), to extract surface points x_1, x_2, x_3, so that the selected tetrahedron $T = (x_0, x_1, x_2, x_3)$ (with $x_0 = 0$) satisfies Part (i) of Theorem 3.3. Part (ii) then follows from an easy perturbation argument; see Corollary 4.4.

4.2.1. Case 1 (Central hit): the details

We fix a point $x_1 \in \Sigma \cap \partial B_{\rho_j}$ such that

$$x_1 \cdot v_j = \pm\rho_j \cos\gamma_1, \quad 0 \leqslant \gamma_1 \leqslant \frac{3}{4}\varphi_0,$$

and we are going to select suitable points $x_2, x_3 \in \Sigma \cap B_{\rho_j}$ so that condition (i) of Theorem 3.3 is satisfied. This will justify our decision to stop the iteration by having set $N := j$ and $d_s(x_0) := \rho_j = \rho_N$. Without loss of generality, rotating the coordinate system if necessary, let us suppose that $v_j = (0, 0, 1) \in \mathbb{R}^3$ and $\pi_{H_j}(x_1) \in H_j$ is equidistant from $z_1 := (0, r_j, 0)$ and $z_2 := (0, -r_j, 0)$, where we recall from Condition (F) for $i = j$ that $r_j = \rho_j \sin\varphi_0$. (In other words, we assume w.l.o.g. that the second coordinate of x_1 is zero.)

Condition (4.14) in (F) for $i = j$ guarantees the existence of a point $x_2 \in \Sigma \cap I_{h_j,v_j}(z_2)$, where $h_j = \cos\varphi_0 = r_j$. Now let $P := \langle 0, x_1, x_2\rangle$. Then $\pi_{H_j}(x_2) \perp x_1$ and we have

$$\rho_j |x_2| \bigl|\cos\sphericalangle(x_1, x_2)\bigr| = |x_1 \cdot x_2| = \bigl|x_1 \cdot \bigl(x_2 - \pi_{H_j}(x_2)\bigr)\bigr|,$$

which yields

$$\bigl|\cos\sphericalangle(x_1, x_2)\bigr| \leqslant \frac{|x_2 - \pi_{H_j}(x_2)|}{|x_2|} \leqslant \sin\biggl(\frac{\pi}{2} - \varphi_0\biggr) = 1/\sqrt{2}.$$

Thus, Definition 2.1(iii) is satisfied for $x_0 = 0, x_1, x_2$, for every $\theta \leqslant \pi/4$. To select x_3, we consider two subcases.

Subcase 1(a). If the points z_1, z_2 and $\pi_{H_j}(x_1)$ are collinear, then we simply have $P = \langle 0, x_1, z_2\rangle$. We then use (F) for $i = j$ to select $x_3 \in \Sigma \cap I_{h_j,v_j}(z_3)$, where $z_3 := (r_j, 0, 0)$ belongs to the two-dimensional disk $D_j := D^2(0, r_j)$ in H_j. Thus,

$$\rho_j \sin\varphi_0 \leqslant |x_k - x_i| \leqslant 2\rho_j \quad \text{for } k \neq i,\ k, i = 0, 1, 2, 3,$$

which establishes conditions (i) and (ii) of Definition 2.1 for $d := d_s(x_0) = \rho_j$ and any $\theta \leqslant \sin\varphi_0 = 1/\sqrt{2}$ Finally, $\operatorname{dist}(x_3, P) = r_j = \rho_j \sin\varphi_0$, and this takes care of Part (iv) of Definition 2.1 so that $T = (x_0, x_1, x_2, x_3) \in \mathscr{V}(\eta, d_s(x_0)$ for any $\eta < 1/\sqrt{2}$, i.e. in this subcase Part (i) of Theorem 3.3 is satisfied for any $\eta < 1/2$.

Subcase 1(b). If the points z_1, z_2 and $\pi_{H_j}(x_1)$ are non-collinear, then we consider the line segment $J := F_j \cap B_{\rho_j} \cap P$ contained in the affine plane $F_j := H_j + h_j v_j$. Since $x_1 \in C(\frac{3}{4}\varphi_0, v_j)$ and $y_1 := \sigma_{F_j}(x_1) \in J$, it is easy to check that, no matter where x_2 has been chosen, J (and P) contains points $y_2 \in F_j$ such that

$$\measuredangle\big(\pi_{H_j}(y_2), \pi_{H_j}(y_1)\big) \geqslant \arccos\left(\cot\varphi_0 \tan\frac{3}{4}\varphi_0\right) > \frac{\pi}{5}.$$

Therefore, we may apply Lemma 4.1 with $\varphi_0 = \frac{\pi}{4}$ and $\varphi_1 := \pi/5$ to select a point $x_3 \in \Sigma$ on one of the vertical segments $I_{h_j,v_j}(z)$, $z \in \gamma_j :=$ the boundary of D_j in H_j, so that

$$\eta\rho_j < \operatorname{dist}(x_3, P) \quad \text{and} \quad \eta\rho_j < |x_k - x_i| \leqslant 2\rho_j \quad \text{for } k \neq i, k, i = 0, 1, 2, 3.$$

where $\eta := 1/100 < \pi/200 \leqslant \frac{1}{2}(1 - \cos\frac{\pi}{10}) = c_0(\pi/4, \pi/5)$ (and we used $1 - \cos x \geqslant x^2/\pi$, $x \in [0, \frac{\pi}{2}]$, for the first inequality). This verifies conditions (i), (ii), and (iv) of Definition 2.1 for each $\theta \leqslant \eta = 1/100$, and we have seen before that Part (iii) of that Definition holds for all $\theta \leqslant \pi/4$. Hence Part (i) of Theorem 3.3 is also satisfied for $\eta := 1/100$ in this subcase, which completes our considerations for Case 1.

4.2.2. Case 2 (No central hit but nice distribution of intersection points): the details

The setting. As in Case 1, we have stopped the iteration, set $N := j$, $d_s(x_0) := \rho_j = \rho_N$. Let $H_j = (v_j)^\perp$ and $F_j = H_j + h_j v_j$, and let $\sigma \equiv \sigma_{F_j}$ denote the central projection from 0 to F_j.

Recall that we now have

$$\Sigma \cap \partial B_{\rho_j} \cap C\left(\frac{3}{4}\varphi_0, v_j\right) = \emptyset \tag{4.30}$$

but we assume that there are *two different* points $x_1, x_2 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ such that

$$\measuredangle\Big(\pi_{H_j}\big(\sigma(x_1)\big), \pi_{H_j}\big(\sigma(x_2)\big)\Big) \geqslant \frac{\pi}{3}. \tag{4.31}$$

Let $y_k = \sigma(x_k)$, $k = 1, 2$. Since the plane $P = \langle 0, x_1, x_2\rangle = \langle 0, y_1, y_2\rangle$, we can apply Lemma 4.1 with $\varphi_0 = \pi/4$, $\varphi_1 = \frac{\pi}{3}$ to select a third point $x_3 \in \Sigma$ on a vertical segment $I_{h_j,v_j}(z_3)$ (using (4.14) for $i = j$), where $z_3 \in \gamma_j := \partial B_{r_j} \cap H_j$, the outer boundary of A_j in H_j. This gives

$$\eta_1\rho_j \leqslant \operatorname{dist}(x_3, P) \quad \text{and} \quad \eta_1\rho_j \leqslant |x_k - x_i| \leqslant 2\rho_j \quad \text{for } k \neq i, k, i = 0, 1, 2, 3$$

where now we have $\eta_1 = c_0(\pi/4, \pi/3) = \frac{1}{2}(1 - \cos\frac{\pi}{6}) = \frac{1}{4}$.

It remains to verify that the angle $\sphericalangle(x_1 - x_0, x_2 - x_0) = \sphericalangle(x_1, x_2)$ is in $[\eta_2, \pi - \eta_2]$ for some absolute constant $\eta_2 > 0$ (possibly smaller than η_1), to verify condition (iii) in Definition 2.1. This is intuitively obvious but we give the details (without aiming at the best possible bounds).

Let us suppose first that the two scalar products $x_k \cdot v_j$ $(k = 1, 2)$ have the same sign. Write

$$x_k = u_k + w_k, \qquad u_k := \pi_{H_j}(x_k) \quad \text{for } k = 1, 2,$$

and let $a_k := |w_k|/\rho_j \equiv |w_k|/|x_k|$ for $k = 1, 2$. Note that since $|x_1| = |x_2| = \rho_j$ and (4.30) is satisfied, we have in fact

$$a_k \leqslant \sin\left(\frac{\pi}{2} - \frac{3}{4}\varphi_0\right) = \frac{5\pi}{16}, \quad k = 1, 2. \tag{4.32}$$

Moreover, we have

$$\sphericalangle(u_1, u_2) = \sphericalangle\big(\pi_{H_j}\big(\sigma(x_1)\big), \pi_{H_j}\big(\sigma(x_2)\big)\big) \geqslant \frac{\pi}{3}, \tag{4.33}$$

(the first equality in (4.33) holds since the scalar products of x_k, $k = 1, 2$, with v_j are of the same sign). Set $\psi := \sphericalangle(x_1, x_2)$. Then, since the scalar products $x_k \cdot v_j$ $(k = 1, 2)$ have the same sign, we have $w_1 \cdot w_2 = |w_1| \cdot |w_2| > 0$, and therefore

$$\begin{aligned} 0 \leqslant \cos\psi = \frac{x_1 \cdot x_2}{|x_1| \cdot |x_2|} &= \frac{(u_1 \cdot u_2) + (w_1 \cdot w_2)}{\rho_j^2} = \frac{|u_1| \cdot |u_2| \cos\sphericalangle(u_1, u_2)}{\rho_j^2} + a_1 a_2 \\ &\leqslant \frac{1}{2}\big(1 - a_1^2\big)^{1/2}\big(1 - a_2^2\big)^{1/2} + a_1 a_2 \quad \text{by (4.33)} \\ &\overset{(*)}{\leqslant} (1 - \lambda)\big(\big(1 - a_1^2\big)^{1/2}\big(1 - a_2^2\big)^{1/2} + a_1 a_2\big) \\ &\leqslant 1 - \lambda \quad \text{by Young's inequality,} \end{aligned}$$

provided that we can choose $\lambda \in (0, \frac{1}{2})$ so that $(*)$ holds, i.e., equivalently,

$$\lambda a_1 a_2 \leqslant \left(\frac{1}{2} - \lambda\right)\big(1 - a_1^2\big)^{1/2}\big(1 - a_2^2\big)^{1/2}. \tag{4.34}$$

Now, (4.32) implies that the left-hand of (4.34) does not exceed $\lambda \sin^2 \frac{5\pi}{16}$ whereas the right-hand side is certainly greater than $(\frac{1}{2} - \lambda)\cos^2 \frac{5\pi}{16}$. Thus, (4.34) holds for every $\lambda \leqslant \frac{1}{2}\cos^2 \frac{5\pi}{16}$, e.g. for $\lambda = \frac{1}{2}\cos^2 \frac{\pi}{3} = \frac{1}{8}$ and then with strict inequality. This gives $\cos\psi \in [0, \frac{7}{8})$, i.e.,

$$\eta_2 \leqslant \psi = \sphericalangle(x_1, x_2) \leqslant \frac{\pi}{2},$$

for $\eta_2 := \arccos \frac{7}{8} \simeq 0.505 > 1/4$.

If the two scalar products $x_k \cdot v_j$ $(k = 1, 2)$ have different signs, we consider $\tilde{x}_2 = -x_2$. Since the central projections $\sigma(x_2)$ and $\sigma(\tilde{x}_2)$ coincide, we can apply the previous reasoning to x_1 and $\tilde{x}_2$, to obtain $\sphericalangle(x_1, \tilde{x}_2) \in [\eta_2, \frac{\pi}{2}]$, i.e. $\sphericalangle(x_1, x_2) \in [\frac{\pi}{2}, \pi - \eta_2]$.

With the choice $\eta := \min\{\eta_1, \eta_2\} = \eta_1 = 1/4$ we have verified that the tetrahedron $T = (x_0, x_1, x_2, x_3)$ satisfies all conditions of Definition 2.1, hence is of class $\mathscr{V}(\eta, d_s(x_0))$ for $\eta = 1/4$, which proves Part (i) of Theorem 3.3 also in Case 2. This concludes the proof in Case 2.

4.2.3. Case 3 (Antipodal position): the details

We deal with Subcases 3(a) and 3(b), where we have stopped the iteration, have set $N := j$, with stopping distance $d_s(x_0) := \rho_j = \rho_N$. Recall that $H_j = (v_j)^\perp$, $F_j = H_j + h_j v_j$, and $\sigma \equiv \sigma_{F_j}$ is the central projection from 0 to F_j.

As in Case 2, $\Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ is nonempty but we have

$$\Sigma \cap \partial B_{\rho_j} \cap C\left(\frac{3}{4}\varphi_0, v_j\right) = \emptyset.$$

However, in this Case condition (4.18) is violated, i.e. *for every two* points $x_1, x_2 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ we have

$$\sphericalangle\big(\pi_{H_j}\big(\sigma(x_1)\big), \pi_{H_j}\big(\sigma(x_2)\big)\big) < \frac{\pi}{3}. \tag{4.35}$$

We have already fixed $x_1 \in \Sigma \cap \partial B_{\rho_j} \cap C(\varphi_0, v_j)$ and assume now without loss of generality that $v_j = (0, 0, 1)$, $x_1 \cdot v_j > 0$, and $u := u_j = \pi_{H_h}(x_1)/|\pi_{H_j}(x_1)| = (1, 0, 0)$. Hence the unit vector $w := w_j = (0, -1, 0)$ determines the axis of the rotations R_s^j defined in (4.20) which in turn were used to rotate conical caps to obtain the sets G_t^j and the stopping rotational angle t_0 (see (4.21) and (4.22)). On this basis the three subcases in Case 3 were distinguished. Let us describe in some detail how we choose x_2 and x_3 in Subcase 3 (a) and (b).

Stopping the iteration in Subcase 3(a)

Let us first note that $t_0 > 0$. To see this, set

$$X^j := \{y \in \mathbb{R}^3\colon (y \cdot v_j)(y \cdot u) \leqslant 0\}, \qquad Y^j := X^j \cap \big(C_{\rho_j}(\varphi_0, v_j) \setminus \operatorname{int} B_{\rho_j/2}\big),$$

and note that if $R_s(C_{\rho_j}(\varphi_0, v_j) \setminus \operatorname{int} B_{\rho_j/2})$ contains a new point y of Σ, i.e. a point $y \in \Sigma \setminus S_j$, then we have in fact $y \in R_s(Y^j)$. However, this cannot happen for s arbitrarily close to 0, as in Case 3 we have

$$\operatorname{dist}\big(Y^j, \Sigma \cap X^j\big) > 0$$

due to (4.35), (4.9) and (4.12) for $i = j$ in (D), and (4.5) for $i = j$.

We choose $x_2 \in G_{t_0} \cap (\Sigma \setminus K^j_{\rho_j})$. It is easy to see that if $x_2 \cdot v_j$ and $x_1 \cdot v_j$ have the same sign, then

$$\begin{aligned}\frac{3}{16}\pi = \frac{3}{4}\varphi_0 \leqslant \sphericalangle(x_1, x_2) &\leqslant \sphericalangle(x_1, v_j) + \sphericalangle\big(v_j, R_{t_0}(v_j)\big) + \sphericalangle\big(R_{t_0}(v_j), x_2\big)\\ &\leqslant \varphi_0 + t_0\varphi_0 + \varphi_0 \leqslant \frac{5}{2}\varphi_0 = \frac{5}{8}\pi.\end{aligned} \tag{4.36}$$

If the scalar products $x_2 \cdot v_j$ and $x_1 \cdot v_j$ have different signs, then (4.36) holds with $\tilde{x}_2 = (-x_2)$ instead of x_2, so that in either case we have

$$\frac{3}{16}\pi \leqslant \sphericalangle(x_1, x_2) \leqslant \pi - \frac{3}{16}\pi = \frac{13}{16}\pi, \tag{4.37}$$

and condition (iii) of Definition 2.1 holds with $\theta := 3\pi/16$.

Now, take $P = \langle 0, x_1, x_2\rangle = \langle 0, \sigma_{F_j}(x_1), \sigma_{F_j}(x_2)\rangle$ and apply Lemma 4.2 in connection with (4.14) for $i = j$ in (F) to find the last good vertex x_3 on one of the segments $I_{h_j,v_j}(z)$, where z runs along the circle γ_j bounding the disk $H_j \cap B_{r_j}$, $r_j = \rho_j \sin\varphi_0$. Then $\operatorname{dist}(x_3, P) \geqslant c_1(\varphi_0)\rho_j$ where $c_1(\varphi_0) = \frac{1}{16}\sin 2\varphi_0 = \frac{1}{16}$, which verifies condition (iv) of Definition 2.1 with $\theta := 1/16$. Conditions (i) and (ii) of that definition are easily checked, so that $T = (x_0, x_1, x_2, x_3) \in \mathscr{V}(\eta, d_s(x_0))$ (and therefore Part (i) of Theorem 3.3 is shown) for $\eta = 1/16$ in Subcase 3(a).

4.2.4. *Stopping the iteration in Subcase 3(b)*

Use (4.23) to select a point $x_2 \in \Sigma \cap (C_{2\rho_j}(\varphi_0, v_j^*) \setminus C_{\rho_j}(\varphi_0, v_j^*))$.

Assume first that $x_2 \cdot v_j^* > 0$. Since, by the definition of R_s and $v_j^* = R_{1/2}(v_j)$, we have

$$\sphericalangle\big(x_1, v_j^*\big) = \sphericalangle(x_1, v_j) + \sphericalangle\big(v_j, v_j^*\big) \in \left[\frac{5}{4}\varphi_0, \frac{3}{2}\varphi_0\right],$$

and $\sphericalangle(x_2, v_j^*) \leqslant \varphi_0$, two applications of the triangle inequality for the spherical metric give

$$\sphericalangle(x_1, x_2) \in \left[\frac{1}{4}\varphi_0, \frac{5}{2}\varphi_0\right] = [\pi/16, 5\pi/8]$$

in that case. If $x_2 \cdot v_j^* < 0$, then we estimate the angle $\sphericalangle(x_1, -x_2)$ in the same way. This yields

$$\sphericalangle(x_1, x_2) \in [\pi/16, 15\pi/16],$$

no matter what is the sign of $x_2 \cdot v_j^*$, which yields condition (iii) of Definition 2.1 for $\theta = \pi/16$. Note that this estimate for the angle implies an estimate for the distance, $\rho_j \sin(\pi/16) \leqslant |x_2 - x_1|$ being part of condition (ii) in Definition 2.1 for $\theta = \sin \pi/16$.

To select x_3, we argue similarly to the proof of Lemma 4.2.

Consider the affine plane $F \equiv F_j = H_j + h_j v_j$, $h_j = \rho_j \cos\varphi_0$. Let $\sigma \equiv \sigma_F$ be the central projection from 0 to F. Set

$$E := \sigma\big(C_{2\rho_j}\big(\varphi_0, v_j^*\big)\big) \subset F;$$

this is a filled ellipse in F. We have $y_2 = \sigma(x_2) \in E$. Consider now the point $y_1 = \sigma(x_1) \in F$. The plane $P = \langle 0, x_1, x_2\rangle$ is equal to $\langle 0, y_1, y_2\rangle$. The straight line $l = P \cap F$ passes through y_1, y_2, and has to intersect ∂E and l_2, where the straight line

$$l_2 := P_2 \cap F \quad \text{for } P_2 := \big(R(7\pi/8, w)(v_j)\big)^\perp = \big(R_{7/2}(v_j)\big)^\perp,$$

is tangent to ∂E in F, and the direction of l_2 is perpendicular to v_j and to $u = (1, 0, 0)$. Let y_3 be that point in $\partial E \cap l$—which in general contains two points—which is closer to y_1, and let

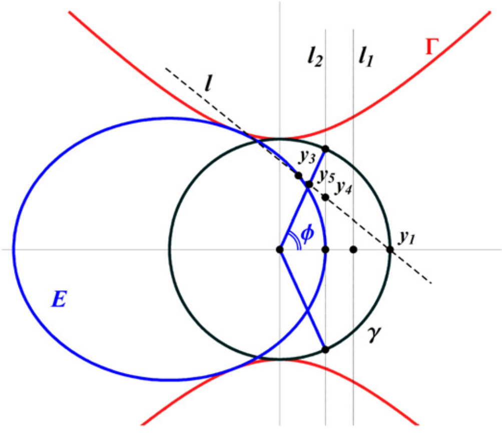

Fig. 5. The configuration in F discussed above. The (slanted, dashed) line l passes through $y_1 = \sigma(x_1)$ and some other point (not shown) belonging to the ellipse E. The four points depicted on l are, from right to left, y_1, y_4, y_5 and y_3. Note that y_4 is always situated between y_3 (which is on the boundary of the ellipse) and y_1. The position of y_5, which is chosen on l so that the angle $\sphericalangle(\pi_{H_j}(y_1), \pi_{H_j}(y_5)) = \phi$, may change, depending on the slope of l and position of $y_1 = \sigma(x_1)$ (a special case $\sigma(x_1) = x_1 \in F$ is shown here). For some positions of x_2 considered in Subcase 3(b), when l is not so close to a tangent to E, we might obtain the order: y_1, then $y_4 \in l_2$, then $y_3 \in \partial E$, and finally y_5 satisfying (4.38).

$\{y_4\} := l_2 \cap l$. Then it is easy to see that y_4 lies on l between y_3 and y_1. Therefore, l contains a point y_5 such that (see Fig. 5)

$$\sphericalangle\big(\pi_{H_j}(y_5), \pi_{H_j}(y_1)\big) = \phi := \arccos\left[\cot\varphi_0\left(\tan\frac{\pi}{8}\right)\right]$$
$$= \arccos\left(\tan\frac{\pi}{8}\right) = 1.1437\ldots, \tag{4.38}$$

and we have $P = \langle 0, x_1, x_2\rangle = \langle 0, y_1, y_5\rangle$.

Applying Lemma 4.1 with $\varphi_1 := \phi$, we find a point $z_3 \in H_j \cap \partial B_{r_j}$, $r_j = \rho_j \sin\varphi_0$, and because of (4.14) for $i = j$ in (F) the last vertex $x_3 \in I_{h_j, v_j}(z_3) \cap \Sigma \subset B_{\rho_j}$ of a good tetrahedron. The estimate from Lemma 4.1 gives now

$$\operatorname{dist}(x_3, P) \geqslant c_0(\varphi_0, \phi)\rho_j = 0.0795\ldots \cdot \rho_j.$$

Since $c_0(\varphi_0, \phi) < \cos(\pi/16)$, it is easy to see that all the distances $d_{ik} := |x_i - x_k|$, $i \neq k$, satisfy

$$0.0795\ldots \cdot \rho_j \leqslant d_{ik} \leqslant 3\rho_j.$$

All the conditions of Definition 2.1 are verified now, and we conclude that $T = (x_0, x_1, x_2, x_3) \in \mathscr{V}(\eta, d_s(x_0))$ for $\eta := c_0(\varphi_0, \phi) = 0.0795$ and $d_s(x_0) = \rho_j = \rho_N$, which implies the validity of Part (i) of Theorem 3.3 for this last Case where the iteration was stopped. Part (ii) follows from Corollary 4.4 below.

4.3. Estimates for perturbed tetrahedra

Lemma 4.3. *Assume that* $x_0 = 0, x_1, x_2, x_3 \in \mathbb{R}^3$ *satisfy*

(i) $\eta d \leqslant |x_i - x_j| \leqslant \eta^{-1} d$ *for all* $i \neq j$, $i, j = 0, 1, 2, 3$,
(ii) $\operatorname{dist}(x_3, \langle x_0, x_1, x_2\rangle) \geqslant \eta d$,
(iii) $\eta \leqslant \measuredangle(x_1 - x_0, x_2 - x_0) \leqslant \pi - \eta$,

where $\eta \in (0, \frac{1}{2})$ *and* $d > 0$. *Then, there exists a number* $\varepsilon = \varepsilon(\eta) \in (0, 1/4)$ *such that*

$$\operatorname{dist}\big(y_3, \langle y_0, y_1, y_2\rangle\big) \geqslant \frac{1}{2}\eta d \tag{4.39}$$

whenever $y_i \in B_{\varepsilon d}(x_i)$ *for* $i = 0, 1, 2, 3$.

Proof. W.l.o.g. we may assume that $x_0 = 0$. Let $y_i = x_i + v_i$ with $|v_i| \leqslant \varepsilon d$ for $i = 0, 1, 2, 3$; we shall fix $\varepsilon \in (0, 1/4)$ later on. Since the left-hand side of (4.39) is invariant under translations, it is enough to prove (4.39) for the quadruple (y_0, y_1, y_2, y_3) shifted by $-v_0$. Thus, from now on we suppose that

$$y_0 = x_0 = 0, \quad y_j = x_j + w_j, \quad \text{where } |w_j| \leqslant 2\varepsilon d \quad \text{for } j = 1, 2, 3.$$

By (iii), (i), and the fact that $\eta < 1/2$, we have

$$d^2\eta^4 \leqslant d^2\eta^2 \sin\eta \leqslant |x_1 \times x_2| \leqslant |x_1||x_2| \leqslant d^2\eta^{-2}.$$

Moreover, $y_1 \times y_2 = (x_1 \times x_2) + v$, where the remainder vector v satisfies by Assumption (i)

$$\begin{aligned}|v| = |w_1 \times x_2 + x_1 \times w_2 + w_1 \times w_2| &\overset{\text{(i)}}{\leqslant} 2 \cdot 2\varepsilon d \cdot d\eta^{-1} + (2\varepsilon d)^2 \\ &\leqslant d^2\eta^{-1}\big(4\varepsilon + 4\varepsilon^2\big) < 5\varepsilon d^2\eta^{-1}\end{aligned}$$

(the last inequality is satisfied for all $\varepsilon \in (0, 1/4)$ and $0 < \eta \leqslant 1$). Thus,

$$|y_1 \times y_2| \leqslant \frac{3}{2}|x_1 \times x_2|$$

if $|v| \leqslant \frac{1}{2}d^2\eta^4 \leqslant \frac{1}{2}|x_1 \times x_2|$, and the last condition is satisfied whenever

$$10\varepsilon \leqslant \eta^5. \tag{4.40}$$

Since $y_0 = 0 = x_0$, for all such choices of ε we have according to Assumption (ii)

$$\begin{aligned}\operatorname{dist}\big(y_3, \langle y_0, y_1, y_2\rangle\big) &= \frac{|\langle y_3, y_1 \times y_2\rangle|}{|y_1 \times y_2|} \\ &\geqslant \frac{2|\langle y_3, y_1 \times y_2\rangle|}{3|x_1 \times x_2|} \geqslant \frac{2|\langle x_3, x_1 \times x_2\rangle|}{3|x_1 \times x_2|} - R \\ &\overset{\text{(ii)}}{\geqslant} \frac{2}{3}d\eta - R,\end{aligned}$$

where, by the triangle inequality,

$$\begin{aligned}0 \leqslant R &\leqslant \frac{2}{3|x_1 \times x_2|}\left(|w_3||x_1||x_2| + |w_3||v| + |x_3||v|\right)\\ &\leqslant \frac{2}{3}\left(d^2\eta^4\right)^{-1}\left(2\varepsilon d \cdot d^2\eta^{-2} + \varepsilon d \cdot d^2\eta^4 + d\eta^{-1} \cdot 5\varepsilon d^2\eta^{-1}\right)\\ &< 6\varepsilon d\eta^{-6}\end{aligned}$$

as $0 < \eta < 1$ hence $\eta^4 < \eta^{-2}$ for the last inequality. Choosing $\varepsilon = \varepsilon(\eta) \in (0, 1/4)$ so small that $R \leqslant 6\varepsilon d\eta^{-6} \leqslant \frac{1}{6}d\eta$ in addition to the requirement in (4.40), we conclude the proof. □

Corollary 4.4. *Given $d > 0$ one finds for any $\eta \in (0, 1/2)$ a constant $\alpha = \alpha(\eta) \in (0, \eta/20)$ such that for all tetrahedra $T \in \mathscr{V}(\eta, d)$ one has*

$$T' \in \mathscr{V}\left(\frac{\eta}{2}, \frac{3}{2}d\right) \quad \textit{for all } \left\|T - T'\right\| \leqslant \alpha d.$$

We omit the proof since it relies on simple distance estimates using the triangle inequality and on Lemma 4.3.

4.4. Large projections and forbidden conical sectors

It is clear that Conditions (A)–(F) stated at the beginning of Section 4.2 combined with the lower bound for stopping distances obtained in Proposition 3.5 imply the statement of Proposition 3.4 for all points $x \in \Sigma^*$.

Using density of Σ^* and closedness of Σ it is easy to see that Proposition 3.4 does hold also for all $x \in \Sigma \setminus \Sigma^*$.

Indeed, fix $x \in \Sigma$ and $r < R_0 = R_0(E, p)$. Choose a sequence of $x_i \to x$, $x_i \in \Sigma^*$. For each x_i, let H_i and v_i be the plane and unit vector whose existence is given by Proposition 3.4 for points of Σ^*. Set $D_i := H_i \cap B(x_i, r/\sqrt{2})$.

Passing to subsequences if necessary, we can assume that H_i and v_i converge as $i \to \infty$ to a plane H and a unit vector v. We shall show that H and v satisfy the requirements of Proposition 3.4 for x and r.

For each $w \in D := H \cap B(x, r/\sqrt{2})$ we select $w_i \in D_i$ with $|w_i - x_i| = |w - x|$ such that $w_i \to w$ as $i \to \infty$. By (3.5) applied for x_i, Σ contains points $y_i = w_i + t_i v_i$ where the coefficients t_i satisfy

$$|t_i|^2 \leqslant r^2 - |w_i - x_i|^2 = r^2 - |w - x|^2.$$

Again, without loss of generality we can assume that $t_i \to t$ as $i \to \infty$, so that

$$y_i = w_i + t_i v_i \to y = w + tv, \quad |t|^2 \leqslant r^2 - |w - x|^2.$$

It is clear that $y \in \Sigma \cap B(x, r)$ and $\pi_H(y) = w$ so that (3.5) holds at x.

Finally, if one of (3.6)–(3.7) were violated with our choice of H and v, then the respective condition would be violated for x_i, r, H_i and v_i for all i sufficiently large, a contradiction.

5. Uniform flatness and oscillation of the tangent planes

Throughout this section we assume that $\Sigma = \partial U$ is a closed, compact admissible surface in $\mathbb{R}^3$, with

$$\mathscr{M}_p(\Sigma) < E < \infty$$

for some $p > 8$. As was shown before in Theorem 3.1, all such Σ are Ahlfors regular with bounds depending only on the energy, i.e. there exists an $R_0 = R_0(E, p) > 0$ whose precise value was given in (3.1) such that

$$\mathscr{H}^2\big(\Sigma \cap B(x, R)\big) \geqslant \frac{\pi}{2} R^2 \quad \text{for all } x \in \Sigma \text{ and } R \in (0, R_0]. \tag{5.1}$$

We shall show that each such Σ is in fact a manifold of class C^1. To this end, we shall show that the tangent plane to Σ exists and satisfies an a priori Hölder estimate. This a priori estimate allows to cover Σ by a finite number of balls, with radii depending only on p and the bound for energy, such that in each of these balls Σ is a graph of a C^1 function with Hölder continuous derivatives, see Corollary 5.7. This fact will be used also later in Section 7 when dealing with sequences of admissible surfaces with equibounded energy.

Our aim in this section will be to estimate the so-called *beta numbers*; see e.g. the introductory chapter of [6],

$$\beta_\Sigma(x, r) := \inf\left\{ \sup_{y \in \Sigma \cap B(x,r)} \frac{\operatorname{dist}(y, F)}{r} \colon F \text{ is an affine plane through } x \right\} \tag{5.2}$$

for small radii r and points $x \in \Sigma$, and to show that

$$\beta_\Sigma(x, r) \leqslant C(E, p) r^\kappa \tag{5.3}$$

where $\kappa = \kappa(p) = (p - 8)/(p + 16) > 0$. One of the issues is that we want to have such estimates for all $r < R_1(E, p)$ where $R_1(E, p)$ is a constant that does not depend on Σ.

It is known that for the class of Reifenberg flat sets with vanishing constant uniform estimates like (5.3) imply $C^{1,\kappa}$ regularity, cf. for example David, Kenig and Toro [5, Section 9], or Preiss, Tolsa and Toro [24, Def. 1.2 and Prop. 2.4]. In our case, we a priori know that $\Sigma \in \mathscr{A}$ and this information by itself does not imply Reifenberg flatness. However, we establish (5.3) inductively; while doing that, we can simultaneously ensure that Σ is Reifenberg flat with a vanishing constant in a scale depending only on the energy.

In order to show precisely what is the role of energy bounds, we give all details of that reasoning. Everything is based on iterative applications of Proposition 3.4 and of the following simple lemma.

Lemma 5.1 *(Flat boxes). Suppose that* $\mathscr{M}_p(\Sigma) < E$ *for some* $p > 8$. *Then, for any given number* $1 > \eta > 0$ *there exist two positive constants* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *and* $c_1 = c_1(\eta, p) > 0$ *such that whenever a triple of points* $\Delta = (x_0, x_1, x_2) \in \Sigma^3$ *satisfies*

$$\Delta \in \mathscr{S}(\eta, d), \quad d \leqslant R_0(E, p)$$

where $R_0(E,p)$ is given by (3.1)*, then we have*

$$\Sigma \cap B(x_0, 3d) \subset U_{\varepsilon d}\big(\langle x_0, x_1, x_2\rangle\big) \tag{5.4}$$

for each $\varepsilon \in (0, \varepsilon_0(\eta))$ which satisfies the balance condition

$$\varepsilon^{16+p} d^{8-p} \geqslant c_1(\eta, p) E. \tag{5.5}$$

In other words, we have

$$\beta_\Sigma(x_0, 3d) \leqslant \frac{\varepsilon}{3}$$

(and also a slightly weaker inequality $\beta_\Sigma(x_0, d) \leqslant \varepsilon$) whenever we can find an appropriate triple of points of Σ and (5.5) is satisfied. Note that the balance condition (5.5) is satisfied for $\varepsilon \approx E^{1/(p+16)} d^\kappa$, so that the 'boxes' $B(x_0, 3d) \cap U_{\varepsilon d}(\langle x_0, x_1, x_2\rangle)$ become indeed flatter and flatter as the scale $d \to 0$.

Remark 5.2. This lemma and its iterative applications in the proof of Theorem 5.4 are one of the main reasons behind our choice of definition of $\mathscr{M}_p$. The proof presented below shows that for any integrand $\mathcal{K}_s(T)$ satisfying

$$\mathcal{K}_s(T) \approx \frac{h_{\min}(T)}{(\operatorname{diam} T)^{2+s}}, \quad s > 0,$$

for which the scaling invariant exponent equals $8/(1+s)$, the appropriate balance condition replacing (5.5) would be

$$\varepsilon^{16+p} d^{8-(1+s)p} \gtrsim \text{Energy} := \int_{\Sigma^4} \mathcal{K}_s(T)^p \, d\mu.$$

For $p > 8/(1+s)$ this would yield, instead of (5.3) above, an inequality of the form $\beta_\Sigma(x,r) \lesssim r^{\kappa(s,p)}$ with $\kappa(s,p) = (p + sp - 8)/(p+16)$. However, for $p > 24/s$ we have $\kappa(s,p) > 1$, and reasoning as in the proof of Theorem 5.4 below one could show that the normal to Σ is Hölder continuous with exponent $\kappa(s,p) > 1$, i.e. constant! Because of that we do not work with the c_{MT} curvature introduced by Lerman and Whitehouse in [14]: for sufficiently large p, the only surface with finite energy would be a plane.

Proof. We argue by contradiction. Suppose that some point $x_3 \in \Sigma \cap B(x_0, 3d)$ does not belong to $U_{\varepsilon d}(P)$, $P := \langle x_0, x_1, x_2\rangle$. Fix $\varepsilon_0 = \varepsilon_0(\eta) > 0$ so small that if $\varepsilon < \varepsilon_0$, then for all tetrahedra T' with vertices $x_i' \in B(x_i, \varepsilon^2 d)$, $i = 0, 1, 2, 3$ one has

$$\operatorname{dist}\big(x_3', \langle x_0', x_1', x_2'\rangle\big) \geqslant \frac{\varepsilon d}{4} = \frac{\varepsilon}{6} \cdot \frac{3d}{2} \quad \text{and} \quad \Delta\big(T'\big) = \big(x_0', x_1', x_2'\big) \in \mathscr{S}(\eta/2, 3d/2). \tag{5.6}$$

(An exercise, similar to the proof of Lemma 4.3, shows that one can take e.g. $\varepsilon_0(\eta) = \eta^2/200$.) Now, since $\varepsilon^2 d < d \leqslant R_0(E,p)$, we have by (5.1)

$$\mathscr{H}^2\big(\Sigma \cap B\big(x_i, \varepsilon^2 d\big)\big) \geqslant \frac{\pi}{2}\big(\varepsilon^2 d\big)^2 > \varepsilon^4 d^2$$

for $i = 0, 1, 2, 3$. Invoking Lemma 2.10 with $\kappa = \varepsilon/6$ as suggested by (5.6), we obtain an estimate of the integrand,

$$\mathcal{K}(T') \geqslant \frac{1}{50^2}\left(\frac{\eta}{2}\right)^3 \frac{\varepsilon}{6} \cdot \frac{2}{3d} = \frac{1}{18 \cdot 10^4} \frac{\eta^3 \varepsilon}{d}, \quad T' = (x_0', x_1', x_2', x_3').$$

Integrating this inequality w.r.t. $T' \in \Sigma^4 \cap \mathscr{B}_{\varepsilon^2 d}(T)$, we immediately obtain

$$\begin{aligned} E > \mathscr{M}_p(\Sigma) &\geqslant \int\limits_{\Sigma^4 \cap \mathscr{B}_{\varepsilon^2 d}(T)} \mathcal{K}^p(T')\, d\mu(T') > (\varepsilon^4 d^2)^4 \left(\frac{\eta^3 \varepsilon}{18 \cdot 10^4 d}\right)^p \\ &= \eta^{3p}(18 \cdot 10^4)^{-p} \varepsilon^{16+p} d^{8-p}, \end{aligned}$$

which is a contradiction to (5.5) if we choose $c_1(\eta, p) = \eta^{-3p}(18 \cdot 10^4)^p$. □

Remark. From now on, we fix $\eta > 0$ to be the constant whose existence is asserted in Theorem 3.3, and we write

$$c_1(p) := c_1(\eta, p) \tag{5.7}$$

for that fixed value of η.

Lemma 5.3 *(Good triples of points of Σ). Let $\Sigma \in \mathscr{A}$, $p > 8$ and $\mathscr{M}_p(\Sigma) < \infty$. Suppose that $x \in \Sigma$, $y \in \Sigma$ and $0 < d = |x - y| < d_s(x)$, where $d_s(x)$ is the stopping distance from Theorem 3.3. Then there exists a point $z \in \Sigma \cap B(x, d)$ and an affine plane H passing through x such that*

(i) $\Delta = (x, y, z) \in \mathscr{S}(\eta, d)$, *where η is the constant from Theorem 3.3;*
(ii) $\pi_H(\Sigma \cap B(x, d)) \supset H \cap B(x, d \sin\varphi_0)$, *where $\varphi_0 = \frac{\pi}{4}$;*
(iii) $\sphericalangle(H, P) \leqslant \alpha_0^*$, *where $P = \langle x, y, z\rangle$ and*

$$\alpha_0^* := \frac{\pi}{2} - \arctan\frac{1}{\sqrt{2}} = 0.955 < \cdots < \frac{\pi}{3}. \tag{5.8}$$

Proof. W.l.o.g. we suppose that $x = 0 \in \mathbb{R}^3$. Applying Proposition 3.4, we find $v \in \mathbb{S}^2$ and $H = (v)^\perp$ such that (3.5), (3.6) and (3.7) do hold for $r = d = |x - y|$, H and v. In particular,

$$D := H \cap B(x, d/\sqrt{2}) \subset \pi_H(B(x, d) \cap \Sigma), \tag{5.9}$$

and by (3.6)–(3.7)

$$\frac{\pi}{4} \leqslant \sphericalangle(y - x, v) \leqslant \frac{3\pi}{4}. \tag{5.10}$$

By (5.9), for each w in the boundary circle of the disk D the segment $I(w) := I_{d/\sqrt{2},v}(w)$ (cf. Section 2.1 for the definition) contains at least one point of Σ. Choose $w_0 \in D$ such that

$w_0 - x \perp \pi_H(y-x)$ and $|w_0 - x| = d/\sqrt{2}$ and then choose any point $z \in \Sigma \cap I(w_0)$. We claim that the conditions of the lemma are satisfied by that point z and H.

Indeed, we have $z \in B(x,d)$ and $\min(|z-x|, |z-y|) \geqslant d/\sqrt{2} \geqslant \eta d$. By choice of z and w_0, we also have

$$(z-x)\cdot(y-x) = z\cdot y = \big(z-\pi_H(z)\big)\cdot\big(y-\pi_H(y)\big) = \pm\big|z-\pi_H(z)\big|\big|y-\pi_H(y)\big|.$$

Thus, $|\cos\sphericalangle(z,y)| = (|z-\pi_H(z)|/|z|)(|y-\pi_H(y)|/|y|) \leqslant (\cos\varphi_0)^2 = \frac{1}{2}$, so that $\sphericalangle(z,y) \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. This implies that $\Delta = (x,y,z)$ is (η,d)-wide, i.e. $\Delta \in \mathscr{S}(\eta,d)$.

To check (iii), one solves an exercise in elementary geometry. For that let $P := \langle x,y,z\rangle$. It is enough to check that $\frac{\pi}{2} \geqslant \sphericalangle(P,v_i) \geqslant \arctan(1/\sqrt{2})$ and then use $\sphericalangle(P,H) = \frac{\pi}{2} - \sphericalangle(P,v)$. To compute $\sphericalangle(P,v)$, let $F = H + hv$, $h = d\cos\varphi_0 = d/\sqrt{2}$ and note that the distance $\delta := \operatorname{dist}(l_1,l_2)$ between the two straight lines $l_1 := P\cap F$ and $l_2 := \{x+sv\colon s\in\mathbb{R}\} \perp F$ satisfies $\delta \geqslant h/\sqrt{2} = d/2$. This gives the desired estimate of the angle. □

Theorem 5.4 *(Existence and oscillation of the tangent plane). Assume that $\Sigma \in \mathscr{A}$ and $\mathscr{M}_p(\Sigma) < E$ for some $p > 8$. Then, for each $x \in \Sigma$ there exists a unique plane $T_x\Sigma$ (which we refer to as* tangent plane *of Σ at x) such that*

$$\operatorname{dist}\big(x', x + T_x\Sigma\big) \leqslant C(p,E)\big|x'-x\big|^{1+\kappa} \quad \textit{for all } x' \in \Sigma \cap B_{\delta_1}(x), \tag{5.11}$$

where $\kappa := (p-8)/(p+16) > 0$ and $\delta_1 = \delta_1(E,p) > 0$. Moreover, there is a constant $A = A(p)$ such that whenever $x,y \in \Sigma$ with $0 < d = |x-y| \leqslant \delta_1(E,p)$, then

$$\sphericalangle(T_x\Sigma, T_y\Sigma) \leqslant A(p)E^{1/(p+16)}d^{\kappa}. \tag{5.12}$$

Remark 5.5. In fact a possible choice for $\delta_1(E,p)$ is

$$\delta_1(E,p) := \min\left\{1, R_0(E,p), \left(\frac{\mu_0\kappa}{400}\right)^{1/\kappa}\big(c_1(p)E\big)^{-1/(p-8)}\right\}, \tag{5.13}$$

where $R_0(E,p)$ is the absolute constant given in (3.1) of Theorem 3.1, $c_1(p)$ is defined in (5.7), and $\mu_0 := \frac{1}{4}(\frac{\pi}{3} - \alpha_0^*)$.

Proof of Theorem 5.4. Let us describe first a rough idea of the proof.

To begin, we use Lemma 5.3 and select $z \in \Sigma \cap B(x,d)$ such that the triple $\Delta = (x,y,z) \in \mathscr{S}(\eta,d)$. Then, fixing $\delta_1(E,p)$ small and setting

$$d_N := d/10^{N-1}, \quad \varepsilon_N \quad \text{such that } \varepsilon_N^{16+p} d_N^{8-p} \equiv c_1(p)E \quad \text{for } N = 1,2,\ldots,$$

we shall find triples of points, $\Delta_N = (x, y_N, z_N) \in \Sigma^3$, such that $y_N, z_N \in B(x, 2d_N)$ and the angle $\gamma_N = \sphericalangle(y_N - x, z_N - x) \approx \frac{\pi}{2}$ with a small error bounded by $C\sum\varepsilon_N$ where C depends only on p and E. The crucial tool needed to select y_N, z_N is the knowledge that $\Sigma \cap B(x,d_1)$ has large projections onto some fixed plane.

Thus, an application of Lemma 5.1 shall give

$$\Sigma \cap B(x, 3d_N) \subset U_{\varepsilon_N d_N}(P_N), \quad P_N = \langle x, y_N, z_N\rangle. \tag{5.14}$$

Moreover, we shall check that the planes P_N satisfy $\measuredangle(P_{N+1}, P_N) \leqslant C\varepsilon_N$, and P_1 is close to $P_0 = \langle x, y, z\rangle$. Thus, the sequence (v_N) of normal vectors to P_N is a Cauchy sequence in $\mathbb{S}^2$. This allows us to set the (affine) tangent plane $P \equiv T_x\Sigma + x$ to be the limit plane of the P_N, and to prove that P does not depend on the choice of y_N, z_N and P_N (which is by no means unique). (It is intuitively clear that $P = \lim P_N$ should be equal to the affine tangent plane to Σ at all points where Σ a priori happens to have a well defined tangent plane.) The whole reasoning gives

$$\measuredangle(T_x\Sigma, P_0) \leqslant C\varepsilon_1 = C'd^\kappa.$$

Reversing the roles of y and x, we run a similar iterative reasoning to obtain the above inequality with x replaced by y. An application of the triangle inequality, combined with a routine examination of the constants, ends the proof.

Let us now pass to the details.

Again, we assume for the sake of convenience that $x = 0$. Set

$$d_N := \frac{d}{10^{N-1}}, \quad d = |x-y|, \quad N = 1, 2, \ldots, \tag{5.15}$$

and let ε_N be defined by

$$\varepsilon_N^{16+p} d_N^{8-p} \equiv c_1(p)E, \quad N = 1, 2, \ldots. \tag{5.16}$$

Note that

$$\varepsilon_N = \left(\frac{c_1(p)E}{d^{8-p}}\right)^{\frac{1}{16+p}} \cdot \left(10^{N-1}\right)^{\frac{8-p}{16+p}} \to 0 \quad \text{as } N \to \infty.$$

Moreover, by our choice of δ_1 in (5.13),

$$\begin{aligned}
200\sum_{N=1}^{\infty} \varepsilon_N &= 200\big(c_1(p)E\big)^{1/(p+16)} \sum_{N=1}^{\infty} d_N^\kappa, \quad \kappa := \frac{p-8}{p+16} > 0, \\
&= 200\big(c_1(p)E\big)^{1/(p+16)} \left(\sum_{N=0}^{\infty} 10^{-N\kappa}\right) d^\kappa \\
&\leqslant \frac{400}{\kappa}\big(c_1(p)E\big)^{1/(p+16)} d^\kappa \\
&\leqslant \mu_0 = \frac{1}{4}\left(\frac{\pi}{3} - \alpha_0^*\right),
\end{aligned} \tag{5.17}$$

where $\alpha_0^* \in (0, \frac{\pi}{3})$ is given by (5.8). (We have used $\sum 10^{-j\kappa} = 10^\kappa/(10^\kappa - 1) \leqslant 2/\kappa$ in the second inequality above.) In particular $\varepsilon_N \ll 1$ for all $N \in \mathbb{N}$.

Proceeding inductively, we shall define two sequences of points $y_N, z_N \in \Sigma$ which converge to $x = 0$ and satisfy the following conditions for each $N = 1, 2, \ldots$.

$$\frac{d_N}{2} \leqslant |y_N|, \qquad |z_N| \leqslant \frac{3d_N}{2}. \tag{5.18}$$

An initial plane P_0 and planes $P_N = \langle 0, y_N, z_N \rangle$ satisfy $\alpha_N := \sphericalangle(P_N, P_{N-1}) \leqslant 200\varepsilon_N$. (5.19)

The angle $\gamma_N := \sphericalangle(y_N, z_N) \in [0, \pi]$ satisfies $\left|\gamma_N - \frac{\pi}{2}\right| \leqslant 6\varepsilon_1 + 40(\varepsilon_1 + \cdots + \varepsilon_{N-1})$. (5.20)

We shall also show that there exists a fixed plane H (given by an application of Lemma 5.3 at the first step of the whole construction) through x such that, for each $N = 1, 2, \ldots$,

$$\pi_H\bigl(B(x, d_N) \cap \Sigma\bigr) \supset D_H(x, d_N/2) := B(x, d_N/2) \cap H. \tag{5.21}$$

Here is a short description of the order of arguments: we first apply Lemma 5.3 to select P_0 and then correct it slightly to have two points y_1, z_1 satisfying (5.20). This is done in Steps 1 and 2 below. Next, proceeding inductively, we first select y_{N+1}, z_{N+1} very close to the intersection of segments $[0, y_N]$ and $[0, z_N]$ with the boundary of $\partial B_{d_{N+1}}$ (Step 3). Finally, we estimate the angle α_N (Step 4) and prove that $P = \lim P_N$ does not depend on the choice of P_0 (Step 5).

Step 1. For given x and y use Lemma 5.3 to select $z \in B_d(\Sigma)$ and the plane H satisfying conditions (i)–(iii) of that lemma. (Notice that $|x - y| = d \leqslant \delta_1(E, p) \leqslant R_0(E, p) < d_s(x_0)$ by our choice (5.13) and (3.8) in Proposition 3.5, so that Lemma 5.3 is indeed applicable.)

Let $P_0 = \langle x, y, z \rangle = \langle 0, y, z \rangle$; by (iii), we have

$$\alpha_0' := \sphericalangle(P_0, H) \leqslant \alpha_0^* = \frac{\pi}{2} - \arctan\frac{1}{\sqrt{2}} < \frac{\pi}{3}. \tag{5.22}$$

Lemma 5.1 gives $\beta_\Sigma(x, d_1) \leqslant \varepsilon_1$. Set

$$F_0 := \bigl\{z' \in B(0, d_1)\colon\ \operatorname{dist}\bigl(z', P_0\bigr) \leqslant \varepsilon_1 d_1\bigr\} = U_{\varepsilon_1 d_1}(P_0) \cap B_{d_1}. \tag{5.23}$$

We know that $\Sigma \cap B(x, d_1) \subset F_0$. The goal will be to prove that one can choose y_N, z_N so that for $P_N := \langle x, y_N, z_N \rangle$

$$\Sigma \cap B(x, d_N) \subset F_N := \bigl\{z' \in B(0, d_N)\colon\ \operatorname{dist}\bigl(z', P_N\bigr) \leqslant \varepsilon_N d_N\bigr\} = U_{\varepsilon_N d_N}(P_N) \cap B_{d_N} \tag{5.24}$$

also for $N = 1, 2 \ldots$, and to provide an estimate for $\alpha_N = \sphericalangle(P_N, P_{N-1})$ showing that for large N the center planes of the sets F_N stabilize around a fixed affine plane.

Note that (5.21) for $N = 1$ follows from Lemma 5.3(ii) since $\sin\varphi_0 = 1/\sqrt{2} > 1/2$.

Step 2 (choice of P_1). We shall choose y_1, z_1 with $\gamma_1 = \sphericalangle(y_1, z_1) \approx \frac{\pi}{2}$, and we shall show that the plane $P_1 = \langle 0, y_1, z_1 \rangle$ satisfies $\alpha_1 = \sphericalangle(P_1, P_0) \leqslant 12\varepsilon_1$. To this end, select a point $x_0 \in F_0$ such that

$$h_0 := \operatorname{dist}(x_0, H) = \max_{\xi \in F_0} \operatorname{dist}(\xi, H) > 0.$$

It is clear that x_0 exists since F_0 is compact, and that $x_0 \in \partial B_{d_1}$; see Fig. 6.

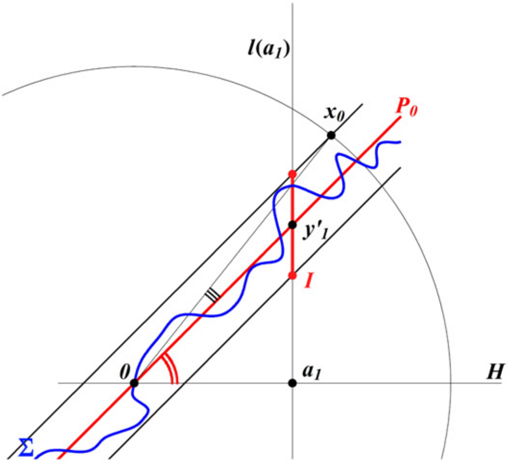

Fig. 6. The initial configuration in $B(x, d_1)$; cross-section by a plane which is perpendicular to H and P_0. A priori, at this stage we do not control the topology of Σ and we cannot even be sure that Σ is a graph over H (or P_0). The angle α_0'' is marked with a triple line.

Let $\alpha_0'' := \sphericalangle(x_0, P_0)$ denote the angle between x_0 and its orthogonal projection $\pi_{P_0}(x_0)$ onto the plane P_0. We have $\sin\alpha_0'' = \varepsilon_1 d_1/d_1 = \varepsilon_1$. Hence, $\alpha_0'' \leqslant (\pi/2)\sin\alpha_0'' < 2\varepsilon_1$.

Now, since $2\varepsilon_1 < 200\sum\varepsilon_N \leqslant \frac{1}{4}(\frac{\pi}{3} - \alpha_0^*)$ by (5.17), we can use (5.22) twice to obtain

$$h_0 = d_1 \sin\bigl(\alpha_0' + \alpha_0''\bigr) \overset{(5.22)}{<} d_1 \sin\left(\alpha_0^* + \frac{1}{4}\left(\frac{\pi}{3} - \alpha_0^*\right)\right)$$
$$= d_1 \sin\left(\frac{3}{4}\alpha_0^* + \frac{1}{4}\frac{\pi}{3}\right) \overset{(5.22)}{<} d_1 \sin\frac{\pi}{3} = d_1\frac{\sqrt{3}}{2}. \tag{5.25}$$

This implies that each straight line $l = l(w)$ which is perpendicular to H and passes through a point w in the disk

$$D_0 = D_H(0, r_0) \equiv H \cap B_{r_0}, \quad \text{where } r_0^2 + h_0^2 = d_1^2,$$

intersects the finite slab F_0 along a segment I of length $2l_0$, where $\varepsilon_1 d_1/l_0 = \cos\alpha_0'$, which gives $l_0 = (\varepsilon_1 d_1)/\cos\alpha_0' < 2\varepsilon_1 d_1$ by virtue of (5.22). Since $r_0^2 = d_1^2 - h_0^2 > d_1^2/4$ according to (5.25), we have $D := D_H(0, d_1/2) \subset D_0$ in H. Choose two points a_1, b_1 in the circle which bounds D in H so that $a_1 \perp b_1$ and $b_1 \in P_0 \cap H$. Take the lines $l(a_1)$, $l(b_1)$ passing through these points and perpendicular to H, and select

$$y_1 \in \Sigma \cap l(a_1) \cap F_0, \quad z_1 \in \Sigma \cap l(b_1) \cap F_0 \tag{5.26}$$

(such points do exist since $\Sigma \cap B(x, d_1) \subset F_0$ and the projection of $\Sigma \cap B(x, d_1)$ onto H contains D by (5.21) already verified for $N = 1$).

Note that $y_1', z_1' = b_1$ given by

$$\{y_1'\} = l(a_1) \cap P_0, \qquad \{z_1'\} = l(b_1) \cap P_0 \tag{5.27}$$

satisfy $y_1' \perp z_1'$. Let $\psi_0 := \measuredangle(y_1', y_1)$, $\theta_0 := \measuredangle(z_1', z_1)$. We have

$$\begin{aligned}\psi_0 \leqslant \tan\psi_0 &\leqslant \frac{\varepsilon_1 d_1}{(d_1/2) - l_0} \\ &\leqslant \frac{2\varepsilon_1 d_1}{d_1 - 4\varepsilon_1 d_1} \quad \text{as } l_0 \leqslant 2\varepsilon_1 d_1 \\ &\leqslant 3\varepsilon_1 \end{aligned} \tag{5.28}$$

since $\varepsilon_1 \leqslant \sum \varepsilon_N \leqslant (200)^{-1}\mu_0 \ll 1/12$ by (5.17). Similarly, we have $\tan\theta_0 \leqslant 3\varepsilon_1$, so that both angles ψ_0 and θ_0 do not exceed $3\varepsilon_1$. Therefore, $0 \leqslant \gamma_1 = \measuredangle(y_1, z_1) \leqslant \measuredangle(y_1, y_1') + \measuredangle(y_1', z_1') + \measuredangle(z_1', z_1)$ satisfies

$$\left|\gamma_1 - \frac{\pi}{2}\right| \leqslant \psi_0 + \theta_0 \leqslant 6\varepsilon_1, \tag{5.29}$$

which gives (5.20) for $N = 1$. By choice of a_1, b_1, (5.18) is satisfied for $N = 1$. Thus, the triangle $\Delta = (x, y_1, z_1)$ is (η, d)-wide, i.e. $\Delta \in \mathscr{S}(\eta, d)$ for $\eta := \min\{1/2, \pi - (\pi/2 + 6\varepsilon_1)\} = 1/2$ (by (5.17)), and $d := d_1 \leqslant R_0(E, p)$. Consequently, by virtue of (5.16) we can derive (5.24) for $N = 1$ with the help of Lemma 5.1.

Finally, normalizing $y_1', z_1' \in P_0$, we easily check that

$$\alpha_1 := \measuredangle(P_1, P_0) < 12\varepsilon_1 \quad \text{for } P_1 := \langle x, y_1, z_1\rangle, \tag{5.30}$$

which gives (5.19) for $N = 1$. Moreover, by (5.30), (5.22), and (5.17) we have

$$\measuredangle(P_1, H) \leqslant \measuredangle(P_1, P_0) + \alpha_0' \overset{(5.30),(5.22)}{<} 12\varepsilon_1 + \alpha_0^* \overset{(5.17)}{<} \frac{1}{4}\left(\frac{\pi}{3} - \alpha_0^*\right) + \alpha_0^* \overset{(5.22)}{<} \pi/3.$$

To summarize, we have now proven (5.18), (5.19), (5.20), (5.21), and (5.24) for $N = 1$.

Step 3 (induction). Suppose now that $y_1, \dots, y_N, z_1, \dots, z_N$ have already been selected so that conditions (5.18), (5.19), (5.20), (5.21), and (5.24) are satisfied for $j = 1, \dots, N$. Note that since (5.24) is satisfied for all indices up to N, we have

$$\beta_\Sigma(x, d_j) \leqslant \varepsilon_j = O\big(d_j^\kappa\big), \quad j = 1, \dots, N. \tag{5.31}$$

We shall select two new points y_{N+1}, z_{N+1} such that (5.18), (5.19), (5.20), (5.21) and (5.24) are satisfied with N replaced by $N + 1$.

Choose first two auxiliary points,

$$\big\{y_{N+1}'\big\} := [0, y_N] \cap \partial B(0, d_{N+1}), \qquad \big\{z_{N+1}'\big\} := [0, z_N] \cap \partial B(0, d_{N+1}). \tag{5.32}$$

Since $P_N = \langle 0, y_N, z_N\rangle$, we have $y_{N+1}', z_{N+1}' \in P_N \cap B_{d_{N+1}} \subset F_N$. Fix $x_N \in F_N$ such that

$$h_N := \operatorname{dist}(x_N, H) = \max_{\xi \in F_N} \operatorname{dist}(\xi, H).$$

Set $\alpha'_N := \sphericalangle(P_N, H)$, $\alpha''_N := \sphericalangle(x_N, P_N)$. We note that $x_N \in \partial B_{d_N}$ and by (5.17)

$$\alpha''_N = \arcsin \varepsilon_N < 2\varepsilon_N \leqslant 2\varepsilon_1 \overset{(5.17)}{<} \frac{1}{4}\left(\frac{\pi}{3} - \alpha_0^*\right).$$

Applying the triangle inequality and using the induction hypothesis (5.19) up to N, and (5.22), we estimate

$$\begin{aligned} \alpha'_N &= \sphericalangle(P_N, H) \\ &\leqslant \sphericalangle(P_0, H) + \sphericalangle(P_1, P_0) + \sphericalangle(P_2, P_1) + \cdots + \sphericalangle(P_N, P_{N-1}) \\ &= \alpha'_0 + \alpha_1 + \cdots + \alpha_N \\ &\leqslant \alpha_0^* + \frac{1}{4}\left(\frac{\pi}{3} - \alpha_0^*\right) \quad \text{by (5.22), (5.19), and (5.17).} \end{aligned} \tag{5.33}$$

Thus, $\alpha'_N + \alpha''_N \leqslant \alpha_0^* + \frac{1}{2}(\frac{\pi}{3} - \alpha_0^*) < \frac{\pi}{3}$ and, as in the second step, we have

$$h_N = d_N \sin(\alpha'_N + \alpha''_N) < d_N \sin\frac{\pi}{3} = d_N \frac{\sqrt{3}}{2}, \quad N \geqslant 1.$$

Hence, $d_N^2 = h_N^2 + r_N^2$ for some $r_N > d_N/2$; as previously, we conclude that each straight line $l = l(w)$ which is perpendicular to H and passes through a point w in the disk

$$D_N = D_H(0, r_N) \equiv H \cap B_{r_N},$$

intersects the finite slab F_N along a segment I of length $2l_N$, where $\varepsilon_N d_N / l_N = \cos \sphericalangle(P_N, H)$, which gives $l_N < 2\varepsilon_N d_N$ by virtue of (5.33). Moreover, by (5.21) (which, by the inductive assumption, holds for N), each segment $I(w)$ for $w \in D_H(0, d_N/2)$ vertical to H must contain at least one point of Σ.

We now choose $y_{N+1}, z_{N+1} \in F_N \cap \Sigma$ such that

$$\pi_H(y_{N+1}) = \pi_H(y'_{N+1}), \qquad \pi_H(z_{N+1}) = \pi_H(z'_{N+1}). \tag{5.34}$$

To establish the desired estimate of $\sphericalangle(P_{N+1}, P_N)$, we show first that

$$\psi_N := \sphericalangle(y_{N+1}, y_N) \leqslant 20\varepsilon_N, \tag{5.35}$$

$$\theta_N := \sphericalangle(z_{N+1}, z_N) \leqslant 20\varepsilon_N. \tag{5.36}$$

Indeed,

$$\begin{aligned} \tan \psi_N = \tan \sphericalangle(y_{N+1}, y_N) &\overset{(5.32)}{=} \tan \sphericalangle(y_{N+1}, y'_{N+1}) \\ &< \frac{\varepsilon_N d_N}{d_{N+1} - l_N} \\ &< \frac{\varepsilon_N d_N}{d_{N+1}/2} = 20\varepsilon_N, \end{aligned}$$

where the last inequality holds since $l_N < 2\varepsilon_N d_N \leqslant 2\varepsilon_1 d_N < d_N/300 < d_{N+1}/2$; remember that $2\varepsilon_1 \leqslant 2\sum_N \varepsilon_N \leqslant (100)^{-1}\mu_0 < (100)^{-1}\pi/12 < 1/300$ by (5.17).

Thus, $\psi_N \leqslant \tan\psi_N \leqslant 20\varepsilon_N$. Similarly, $\theta_N \leqslant \tan\theta_N \leqslant 20\varepsilon_N$. This proves (5.35) and (5.36). Moreover, the triangle inequality gives an estimate of the angle $\gamma_{N+1} = \sphericalangle(y_{N+1}, z_{N+1})$,

$$|\gamma_{N+1} - \gamma_N| \leqslant \theta_N + \psi_N \leqslant 40\varepsilon_N, \tag{5.37}$$

and consequently

$$\left|\gamma_{N+1} - \frac{\pi}{2}\right| \leqslant \left|\gamma_N - \frac{\pi}{2}\right| + 40\varepsilon_N.$$

By induction, this inequality implies (5.20) with N replaced by $N+1$. We also have

$$\frac{d_{N+1}}{2} \leqslant d_{N+1} - l_N \leqslant |y_{N+1}| \leqslant d_{N+1} + l_N \leqslant \frac{3d_{N+1}}{2},$$

and a similar estimate for $|z_{N+1}|$, which gives (5.18) with N replaced by $N+1$. Therefore the triangle $\Delta = (x, y_{N+1}, z_{N+1})$ is (η_1, d_{N+1})-wide, i.e. $\Delta \in \mathscr{S}(\eta_1, d_{N+1})$ for $\eta_1 := \min\{1/2, (\pi/2) - 50\sum_N \varepsilon_N\} = 1/2$ according to (5.17). Since $\eta_1 = 1/2 > \eta =$ the constant from Theorem 3.3, Lemma 5.1 is again applicable to obtain

$$\Sigma \cap B_{3d_{N+1}} \subset U_{\varepsilon_{N+1} d_{N+1}}(P_{N+1}),$$

which implies (5.24) with N replaced by $N+1$.

To check (5.21) with $N+1$ instead of N, we fix $z \in \Sigma \cap (B_{d_N} \setminus B_{d_{N+1}})$ and estimate $|\pi_H(z)|$. Since then $z \in F_N = U_{\varepsilon_N d_N}(P_N) \cap B_N$ and the angle $\alpha'_N = \sphericalangle(P_N, H)$ satisfies (5.33), we check that $\sin(\sphericalangle(z, P_N)) \leqslant \varepsilon_N d_N/|z| \leqslant \varepsilon_N d_N/d_{N+1}$ and consequently

$$\begin{aligned}
\left|\pi_H(z)\right| &= |z| \cos\sphericalangle(z, H) \\
&\geqslant |z| \cos\left(\alpha'_N + \arcsin\frac{\varepsilon_N d_N}{d_{N+1}}\right) \\
&> |z| \cos\left(\alpha'_N + 20\varepsilon_N\right) \quad \text{as } d_N = 10 d_{N+1} \\
&> |z| \cos\frac{\pi}{3} \quad \text{by (5.17) and (5.33)} \\
&\geqslant d_{N+1}/2.
\end{aligned}$$

Since by the inductive assumption ((5.21) up to index N) the projection $\pi_H(\Sigma \cap B_{d_N})$ contains the whole disk $D_H(0, d_N/2)$, we do obtain $\pi_H(\Sigma \cap B_{d_{N+1}}) \supset D_H(0, d_{N+1}/2)$.

It remains to verify (5.19) with N replaced by $N+1$, i.e., the desired inequality for the angle $\alpha_{N+1} = \sphericalangle(P_{N+1}, P_N)$.

Step 4. Estimates of α_N. We normalize the vectors spanning P_j and set $u_j := y_j/|y_j|$, $w_j := z_j/|z_j|$. We also set $M_j = |u_j \times w_j|$, noting that by (5.20) which we already have shown to hold up to $N+1$, and by (5.17) that

$$\gamma_j \in \left(\frac{5}{12}\pi, \frac{7}{12}\pi\right)$$

so that

$$1 \geqslant M_j = \sin\gamma_j \geqslant \frac{\sqrt{2}}{2} \quad \text{for all } j = 1, \dots, N+1. \tag{5.38}$$

Now, we compute the difference of unit normals to P_{N+1} and P_N,

$$\frac{u_{N+1} \times w_{N+1}}{|u_{N+1} \times w_{N+1}|} - \frac{u_N \times w_N}{|u_N \times w_N|} =: T_1 + T_2,$$

where

$$T_1 := \frac{M_N(u_{N+1} \times w_{N+1} - u_N \times w_N)}{M_N M_{N+1}},$$
$$T_2 := \frac{M_N - M_{N+1}}{M_N M_{N+1}} u_N \times w_N.$$

Since $u_N, w_N \in \mathbb{S}^2$, we can use (5.35), (5.36) (which yield the estimates of $u_{N+1} - u_N$ and $w_{N+1} - w_N$), and in addition (5.38) and (5.37), to obtain

$$\begin{aligned}
|T_1| &\overset{(5.38)}{\leqslant} \sqrt{2}|u_{N+1} \times w_{N+1} - u_N \times w_N| \\
&\leqslant \sqrt{2}\big(|u_{N+1} - u_N| + |w_{N+1} - w_N|\big) \\
&\leqslant 40\sqrt{2}\varepsilon_N < 60\varepsilon_N; \\
|T_2| &\overset{(5.38)}{\leqslant} 2|\sin\gamma_N - \sin\gamma_{N+1}| \quad \text{since sin is 1-Lipschitz} \\
&\leqslant 2|\gamma_N - \gamma_{N+1}| \overset{(5.37)}{\leqslant} 80\varepsilon_N.
\end{aligned}$$

This implies

$$\alpha_{N+1} = \sphericalangle(P_{N+1}, P_N) \leqslant 140\varepsilon_N, \tag{5.39}$$

i.e., (5.19) holds also with N replaced by $N+1$.

Finally, a computation similar to (5.33) shows that

$$\sphericalangle(H, P_{N+1}) < \pi/3. \tag{5.40}$$

To summarize, under the inductive hypothesis that (5.18), (5.19), (5.20), (5.21), and (5.24) hold up to N, we have shown that (5.18), (5.19), (5.20), (5.21), and (5.24) do hold up to $N+1$, which yields (5.18)–(5.21) and (5.24) for all $N \in \mathbb{N}$ by the induction principle.

Step 5 (existence and uniqueness of $\lim P_N$**).** The unit vectors $u_N = y_N/|y_N|$ and $w_N = z_N/|z_N|$ spanning the affine planes P_N with unit normals $\nu_N := u_N \times w_N$ satisfy $\sphericalangle(u_N, w_N) = \gamma_N \in (\frac{5}{12}\pi, \frac{7}{12}\pi)$ for all N, so that subsequences again denoted by u_N and w_N converge to unit vectors $u, w \in \mathbb{S}^1$ with $\sphericalangle(u, v) \in [\frac{5}{12}\pi, \frac{7}{12}\pi]$ spanning a limiting affine plane P with unit normal vector $\nu := u \times w$, so that we can say $P_N \to P$ as $N \to \infty$. Since all P_N contain $x = 0$ so does P. As in (5.17), summing the tail of a geometric series, we obtain by (5.19):

$$\begin{aligned}\sphericalangle(P,P_N)=\lim_{k\to\infty}\sphericalangle(P_k,P_N) &\leqslant \sum_{j=N}^{\infty}\alpha_{j+1}\leqslant 200\sum_{j=N+1}^{\infty}\varepsilon_j\\ &\overset{(5.19)}{\leqslant} \frac{400}{\kappa}\bigl(c_1(p)E\bigr)^{1/(p+16)}d_{N+1}^{\kappa}\\ &=: C_2(p)E^{1/(p+16)}d_{N+1}^{\kappa}\quad\text{for all } N=0,1,2,\ldots. \end{aligned}\tag{5.41}$$

In particular,

$$\sphericalangle(P,P_0)\leqslant C_2(p)E^{1/(p+16)}d_1^{\kappa}\equiv C_2(p)E^{1/(p+16)}d^{\kappa}.\tag{5.42}$$

However, as we cannot a priori claim that Σ is a graph over H, the choice of y_N and z_N for small values of N does not have to be unique. Suppose that for two different choices of sequences $y_N, z_N\in\Sigma$ and $y_N', z_N'\in\Sigma$ (satisfying (5.18)–(5.20), and (5.24) for all $N\in\mathbb{N}$), we obtain

$$P_N=(x,y_N,z_N)\to P,\qquad P_N'=\bigl(x,y_N',z_N'\bigr)\to P'\quad\text{as } N\to\infty,$$

but $P\neq P'$ and $\pi/2\geqslant\sphericalangle(P,P')=\vartheta>0$. Fix N so large that $\varepsilon_N<\vartheta/10$ and

$$\max\bigl(\sphericalangle(P,P_N),\sphericalangle\bigl(P',P_N'\bigr)\bigr)<\vartheta/10.$$

Since $y_N'\in P_N'$ and $d_N/2\leqslant|y_N'|\leqslant 3d_N/2$ by(5.18), we obtain $\sphericalangle(y_N',P')<\vartheta/10$. Hence, the angle between y_N' and P_N cannot be too small: $\sphericalangle(y_N',P)\geqslant\sphericalangle(P',P)-\sphericalangle(y_N',P')>9\vartheta/10$ and $\sphericalangle(y_N',P_N)\geqslant\sphericalangle(y_N',P)-\sphericalangle(P,P_N)>4\vartheta/5$. Therefore,

$$\operatorname{dist}\bigl(y_N',P_N\bigr)=\bigl|y_N'\bigr|\sin\sphericalangle\bigl(y_N',P_N\bigr)>\frac{d_N}{2}\cdot\frac{2}{\pi}\cdot\frac{4\vartheta}{5}>\frac{d_N\vartheta}{5}>2\varepsilon_N d_N,$$

which is a contradiction to

$$\Sigma\cap B(x,3d_N)\subset U_{\varepsilon_N d_N}(P_N),$$

as $|y_N'|\leqslant 3d_N/2<3d_N$. Thus, $P=\lim P_N$ is unique and does not depend on the choices of y_N, z_N.

We set $P=: x+T_x\Sigma$ to define the tangent plane $T_x\Sigma$ of Σ at the point x, and we set $n_*(x):=\nu$ to obtain a well-defined unit normal to Σ at x; the estimate (5.41) gives in fact (5.11) (justifying the term "tangent plane")

$$\begin{aligned}\operatorname{dist}\bigl(x',P\bigr)&\leqslant 2\varepsilon_N d_N+d_N\sin\sphericalangle(P,P_N)\\ &=E^{1/(p+16)}O\bigl(d_N^{1+\kappa}\bigr),\quad N\to\infty,\quad\text{for all } x'\in B(x,d_N)\cap\Sigma,\end{aligned}\tag{5.43}$$

where the constant in 'big O' above depends only on p.

Step 6 (conclusion of the proof). Reversing the roles of x and y, running the whole procedure one more time, and using (5.42) twice, we obtain

$$\sphericalangle(T_x\Sigma,T_y\Sigma)\leqslant\sphericalangle(T_x\Sigma,P_0)+\sphericalangle(P_0,T_y\Sigma)\leqslant 2C_2(p)E^{1/(p+16)}d^{\kappa}.\qquad\square\tag{5.44}$$

We state one corollary which easily follows from the last result and its proof. It tells us that it is not really important how we choose P_0; there are many choices which give a similar approximation of $T_x\Sigma$.

Corollary 5.6. *Assume that $\Sigma \in \mathscr{A}$ and $\mathscr{M}_p(\Sigma) < E$ for some $p > 8$. Let $T_x\Sigma$ and $\delta_1 = \delta_1(E, p) > 0$ be given by Theorem 5.4.*

Whenever $x, y, \zeta \in \Sigma$ with $0 < d = |x - y| \leqslant \delta_1(E, p)$, $d/2 \leqslant |x - \zeta| \leqslant d$ and $\sphericalangle(\zeta - x, y - x) \in [\pi/3, 2\pi/3]$, then $T_x\Sigma$ and the plane $P = \langle x, y, \zeta\rangle$ satisfy

$$\sphericalangle(T_x\Sigma, P) \leqslant C_3(p)E^{1/(p+16)}d^\kappa, \quad \kappa = \frac{p-8}{p+16}, \tag{5.45}$$

where the constant $C_3(p)$ depends only on p.

Proof. We use the notation introduced in the proof of Theorem 5.4. Since $\sphericalangle(T_x\Sigma, P_0) \lesssim E^{1/(p+16)}d^\kappa$ by (5.42), it is enough to show that the angle $\sphericalangle(P_0, P)$ does not exceed a constant multiple of $E^{1/(p+16)}d^\kappa$. Noting that $d/2 \leqslant |\zeta - x| \leqslant d = d_1$ and ζ belongs to the slab $U_{\varepsilon_1 d_1}(P_0)$, we easily compute this angle and finish the proof. The computational details, very similar to the proof of (5.39), are left to the reader. □

In order to deal with sequences of surfaces with equibounded energy in Section 7 we establish a local graph representation of one such surface Σ of finite $\mathscr{M}_p$-energy on a scale completely determined by the energy value $\mathscr{M}_p(\Sigma)$ and with a priori estimates on the $C^{1,\kappa}$-norm of the graph function.

Corollary 5.7. *Assume that $p > 8$, $\mathscr{M}_p(\Sigma) < E < \infty$. Then there exist two constants, $0 < a(p) < 1 < A(p) < \infty$, such that for each $x \in \Sigma$ there is a function*

$$f : T_x\Sigma \to (T_x\Sigma)^\perp \simeq \mathbb{R}$$

with the following properties:

(i) $f(0) = 0$, $\nabla f(0) = (0, 0)$,
(ii) $|\nabla f(y_1) - \nabla f(y_2)| \leqslant A(p)E^{\frac{1}{p+16}}|y_1 - y_2|^{\frac{p-8}{p+16}}$,
(iii) *If $R_1 \equiv R_1(E, p) := a(p)E^{-1/(p-8)} \leqslant R_0(E, p)$ (where $R_0(E, p)$ has been defined in (3.1) of Theorem 3.1) and if*

$$\Phi(y) := x + \big(y, f(y)\big), \quad y \in T_x\Sigma \simeq \mathbb{R}^2,$$

then

$$\Phi(D_{\frac{3}{4}R_1}) \subset \big[B(x, R_1) \cap \Sigma\big] \subset \Phi(D_{R_1}), \tag{5.46}$$

where $D_{R_1} = B(0, R_1) \cap T_x\Sigma$ is a disk in $T_x\Sigma$ around $0 \in T_x\Sigma$, and

$$\big|D\Phi(y_1) - D\Phi(y_2)\big| \leqslant A(p)E^{\frac{1}{p+16}}|y_1 - y_2|^{\frac{p-8}{p+16}}. \tag{5.47}$$

In particular, Σ is an orientable $C^{1,\kappa}$-manifold for $\kappa = (p - 8)/(p + 16)$.

Proof. Basically, we mimic here the proof of Theorem 5.1 from [33]. (In [33], we knew that the surface cannot penetrate two balls of *fixed* radius, touching Σ at every point; this is replaced here by angle estimates (5.41) and (5.42), and the existence of forbidden conical sectors, cf. Proposition 3.4.)

Fix $x \in \Sigma$. Without loss of generality suppose that $x = 0$.

Step 1 (the definition of f). We use the notation from the proof of Theorem 5.4. Recall the plane $P = x + T_x\Sigma$ (used to *define* $T_x\Sigma$) has been obtained as a limit of planes P_N satisfying (5.41); for all $x, y \in \Sigma$ with $|x - y| = d \leqslant \delta_1(E, p)$ given by (5.13) we had the angle estimate (5.44). Using (5.43), one can easily show that

$$\operatorname{dist}\big(x', P\big) \leqslant A_1(p) E^{1/(p+16)} d^{1+\kappa} \tag{5.48}$$

whenever $x' \in B(x, d) \cap \Sigma$ for some $d \leqslant \delta_1(E, p)$. We shall use this estimate and Proposition 3.4 to show that if $r \leqslant a(p)\delta_1(E, p)$ for a sufficiently small constant $a(p) \in (0, 1)$, then

$$\big(\pi_P\big(B(x, 4r/3) \cap \Sigma\big)\big) \quad \text{contains the disk } D_r := B(x, r) \cap P. \tag{5.49}$$

Indeed, otherwise there would be a point $z \in D_r$ and a segment $I = I_{h,w}(z) \perp P$ (we fix a unit vector $\mathbb{S}^2 \ni w \perp P$) of length

$$\begin{aligned} 2h &:= 2A_1(p) E^{1/(p+16)} (4r/3)^{1+\kappa} \\ &\leqslant \frac{r}{100} \quad \text{if } a(p) \text{ is small enough} \end{aligned}$$

such that $I \cap \Sigma = \emptyset$. By (5.48) all points of Σ in $B(x, d)$, $d = 4r/3$, are in fact located in the thin slab $U_h(P)$. Thus, it is easy to use Proposition 3.4, (3.6)–(3.7), and check that—no matter what is the angle between P and the vector v given by that proposition—the sets $C^{\pm}_{2r}(\varphi_0, v) \setminus B_r$ contain two open balls $B^{\pm}$ which are in two *different* components of $B(x, d) \setminus U_h(P)$. Hence,

$$B^+ \subset C^+_{2r}(\varphi_0, v) \cap U, \qquad B^- \subset C^-_{2r}(\varphi_0, v) \cap \big(\mathbb{R}^3 \setminus \overline{U}\big).$$

Now, one could use the segment I to construct a curve which contains no point of Σ but nevertheless joins a point in B^- to a point in B^+. This is a contradiction proving (5.49).

Next, using (5.44), one proves that π_P is injective on $B(x, 4r/3) \cap P$. Otherwise, there would be a point $z' \in P$, $4r/3 > |z' - x| = \rho > 0$, and a segment $I' := I_{h',w}(z')$ with

$$h' = A_1(p) E^{1/(p+16)} \rho^{1+\kappa} \leqslant \rho/100$$

such that $I' \cap \Sigma$ would contain two different points $y_1 \neq y_2$. Then, letting $P_1 = T_{y_1}\Sigma$, $v_1 = (y_1 - x)/|y_1 - x|$ and $v_2 = (y_2 - y_1)/|y_2 - y_1|$, we would use (5.44) to obtain

$$\begin{aligned} \sphericalangle(v_1, v_2) &\leqslant \sphericalangle(v_1, P) + \sphericalangle(P, P_1) + \sphericalangle(P_1, v_2) \\ &\leqslant A_2(p) E^{1/(p+16)} \rho^{\kappa} \\ &< \frac{\pi}{4} \quad \text{if } a(p) \text{ is small enough.} \end{aligned}$$

Since $v_2 \perp P$ we have on the other hand for sufficiently small $a(p)$

$$\sphericalangle(v_1, v_2) \geqslant \frac{\pi}{2} - \sphericalangle(P, v_1) \geqslant \frac{\pi}{2} - A_3(p)E^{1/(p+16)}\rho^\kappa > \frac{\pi}{4},$$

a contradiction.

For $y \in U$, where U denotes the interior in P of $\pi_P(\Sigma \cap B(x, 4r/3)$ we now define

$$f(y) = w \cdot (.\pi_P|_{\Sigma\cap B(x,4r/3)})^{-1}(y),$$

and let $\Phi(y)$ be defined by the formula given in Part (iii) of the Corollary. Note that $U \supset D_r$ by (5.49). It is clear that $f(0) = 0$ and $\nabla f(0) = (0, 0)$. The differentiability of f at other points follows from (5.48) which implies that for $\varrho \to 0$ Graph$f \cap B(x, \varrho)$ is trapped in a flat slab of height $\lesssim \varrho^{1+\kappa}$ around a *fixed* plane (depending on x but independent from ϱ).

Step 2 (bounds for $|\nabla f|$). The vector $(\nabla f(y), -1)$ is parallel to the normal direction to Σ at x when $y = \pi_P(x)$. Taking $y \in U$, we have by (5.12) of Theorem 5.4

$$\alpha(y) \equiv \sphericalangle(T_{\Phi(y)}\Sigma, T_0\Sigma) \leqslant \pi/4.$$

Since $\tan\alpha(y) = |\nabla f(y)|$, we have $|\nabla f(y)| \leqslant 1$ everywhere in D_r. Thus, f is Lipschitz with Lipschitz constant 1.

Step 3 (the oscillation of ∇f). Fix two points $y_1, y_2 \in U$ and set $a = D_1 f(y_1)$, $b = D_2 f(y_1)$, $c = D_1 f(y_2)$, $d = D_2 f(y_2)$ where D_i stands for the i-th partial derivative. The angle α between the tangent planes to Σ at $x_i = \Phi(y_i)$, $i = 1, 2$, satisfies

$$\begin{aligned} \sin^2\alpha &= \frac{(a-c)^2 + (b-d)^2 + (ad-bc)^2}{(1+a^2+b^2)(1+c^2+d^2)} \\ &\overset{\text{(Step 2)}}{\geqslant} \frac{(a-c)^2 + (b-d)^2}{4} = \frac{|Df(y_1) - Df(y_2)|^2}{4}. \end{aligned} \tag{5.50}$$

An upper bound for α is also given by (5.12). Combining the two, and noting that $|x_1 - x_2| \leqslant 2|y_1 - y_2|$, we obtain the desired estimate for $y_1, y_2 \in U$ and conclude the proof, extending f to the whole tangent plane by well-known extension theorems; see e.g. [9, Chapter 6.9]. □

Remark 5.8. Assume that some absolute small constant ε_0 is given a priori, say $\varepsilon_0 = \frac{1}{100}$. Then, shrinking $a(p)$ in the previous corollary if necessary, we have above for $y_1, y_2 \in D_{R_1}$

$$\begin{aligned} \left|\nabla f(y_1) - \nabla f(y_2)\right| &\leqslant A(p)E^{\frac{1}{p+16}}|y_1 - y_2|^{\frac{p-8}{p+16}} \leqslant A(p)E^{\frac{1}{p+16}}(2R_1)^{\frac{p-8}{p+16}} \\ &\leqslant 2A(p)E^{\frac{1}{p+16}}a(p)^{\frac{p-8}{p+16}}\left(E^{-1/(p-8)}\right)^{\frac{p-8}{p+16}} = 2A(p)a(p)^{\frac{p-8}{p+16}} < \varepsilon_0. \end{aligned}$$

Remark 5.9. It is now clear that if $\Sigma \in \mathscr{A}$ with $\mathscr{M}_p(\Sigma) < \infty$ for some $p > 8$, then $\Sigma = \partial U$ is a closed, compact surface of class $C^{1,\kappa}$. Thus, Σ is orientable and has a well defined global normal, n_Σ.

For a discussion of issues related to orientability, we refer the reader to [15] and to Dubrovin, Fomenko and Novikov's monograph, [7, Chapter 1].

6. Improved Hölder regularity of the Gauß map

In this section we prove

Theorem 6.1. *Let $\Sigma \in \mathscr{A}$; assume that $p > 8$ and $\mathscr{M}_p(\Sigma) \leqslant E < \infty$. Then Σ is an orientable manifold of class $C^{1,\lambda}(\Sigma)$ for $\lambda = 1 - \frac{8}{p}$. Moreover, the unit normal n_Σ satisfies the local estimate*

$$\left|n_\Sigma(x_1) - n_\Sigma(x_2)\right| \leqslant C(p)\left(\int\limits_{[\Sigma\cap B(x_1,10|x_1-x_2|)]^4} \mathcal{K}^p\,d\mu\right)^{1/p} |x_1 - x_2|^\lambda \tag{6.1}$$

for all $x_1, x_2 \in \Sigma$ such that $|x_1 - x_2| \leqslant \delta_2(E, p) := a_2(p)E^{-1/(p-8)}$.

Remark. Once (6.1) is established, the global estimate $|n_\Sigma(x_1) - n_\Sigma(x_2)| \leqslant \mathrm{const}|x_1 - x_2|^\lambda$ follows.

Before passing to the proof of the theorem, let us explain informally what is the main qualitative difference between the estimates in Sections 5 and 6. In Section 5, to prove that the surface is in fact $C^{1,\kappa}$, we were iteratively estimating the contribution to the energy of tetrahedra with vertices on patches that were very small when compared with the edges of those tetrahedra. A priori, this might be a tiny fraction of $\mathscr{M}_p(\Sigma)$. Now, knowing already that locally the surface is a (flat) $C^{1,\kappa}$ graph, we can use a slicing argument to gather more information from energy estimates—this time, considering not just an insignificant portion of the local energy but the whole local energy to improve the estimates of the oscillation of the normal vector.

The whole idea is, roughly speaking, similar to the proof of Theorem 1.2 in our joint paper with Marta Szumańska, see [31, Section 6]. Since the result is local, we first use Theorem 5.4 to consider only a small piece of Σ which is a (very) flat graph over some plane, and then we use the energy to improve the Hölder exponent from $\kappa = (p - 8)/(p + 16)$ to $\lambda = 1 - \frac{8}{p} > \kappa$.

Proof of Theorem 6.1. Step 1. The setting. W.l.o.g. we consider a portion of Σ which is a graph of $f\colon \mathbb{R}^2 \supset 5Q_0 \to \mathbb{R}$, where Q_0 is some fixed (small) cube centered at 0 in $\mathbb{R}^2$ and $f \in C^{1,\kappa}$ satisfies $\nabla f(0) = (0, 0)$ and has a very small Lipschitz constant, say

$$\left|f(x) - f(y)\right| \leqslant \varepsilon_0|x - y|, \quad x, y \in 5Q_0. \tag{6.2}$$

By an abuse of notation, we write $n_\Sigma(x)$ to denote the normal to Σ at the point $F(x) \in \Sigma$, where

$$F\colon \mathbb{R}^2 \supset 5Q_0 \ni x \longmapsto \big(x, f(x)\big) \in \mathbb{R}^3 \tag{6.3}$$

is the local parametrization of Σ given by the graph of f, compare with Corollary 5.7. To ensure (6.2), just use Remark 5.8.

We shall write $\mathcal{K}(x_0, x_1, x_2, x_3)$ to denote the integrand of $\mathscr{M}_p$ (without the power p) evaluated at the tetrahedron with four vertices $F(x_i) \in \Sigma$ for x_i in the domain of the parametrization F.

Since (6.2) implies that $|\nabla f| \leqslant \varepsilon_0$, we also have $|F(x) - F(y)| \leqslant (1 + \varepsilon_0)|x - y|^\kappa$; hence

$$(1 + \varepsilon_0)^2\mathscr{H}^2(U) \geqslant \mathscr{H}^2\big(\Sigma \cap F(U)\big) \geqslant \mathscr{H}^2\big(\pi_{\mathbb{R}^2}\big(\Sigma \cap F(U)\big)\big) = \mathscr{H}^2(U) \tag{6.4}$$

for every open set $U \subset 5Q_0$. For the sake of convenience, we assume in the whole proof

$$\varepsilon_0 < \frac{1}{100}. \tag{6.5}$$

It is an easy computation to check that for every two points $x, y \in 5Q_0$ we have

$$(1-2\varepsilon_0)\big|\nabla f(x) - \nabla f(y)\big| \leqslant \big|n_\Sigma(x) - n_\Sigma(y)\big| \leqslant (1+2\varepsilon_0)\big|\nabla f(x) - \nabla f(y)\big|. \tag{6.6}$$

We fix an orthonormal basis (e_1, e_2, e_3) of $\mathbb{R}^3$ so that e_1, e_2 are parallel to the sides of Q_0.

Step 2. Set, for $r \leqslant \operatorname{diam} Q_0 < 1$, and any subset $S \subset Q_0$

$$\Phi_1^*(r, S) := \max_{\substack{\|y-z\|\leqslant r\\ y,z\in Q_0\cap S}} \big|n_\Sigma(y) - n_\Sigma(z)\big|,$$

$$\Phi_2^*(r, S) := \max_{\substack{\|y-z\|\leqslant r\\ y,z\in Q_0\cap S}} \big|\nabla f(y) - \nabla f(z)\big|,$$

$$\Phi^*(r, S) := \Phi_1^*(r, S) + \Phi_2^*(r, S),$$

where $\|\cdot\|$ denotes the ℓ^∞ norm in $\mathbb{R}^2$, i.e. $\|x\| := \max(|x_1|, |x_2|)$ for $x = (x_1, x_2)$. Shrinking Q_0 if necessary, we may assume that

$$\Phi^*(\operatorname{diam} Q_0, Q_0) \leqslant \frac{1}{100} \tag{6.7}$$

(by continuity of n_Σ and of ∇f).

As in [31, Section 6], we want to prove the following

Key estimate. Assume that $u, v \in Q_0$ and let $Q(u,v) :=$ the cube centered at $(u+v)/2$ and having edge length $2|u-v|$. There exist positive numbers $\delta_2 = \delta_2(E, p) = a_2(p)E^{-1/(p-8)}$ and $C(p) > 0$ such that whenever $0 < |u-v| \leqslant \delta_2$, then

$$\big|n_\Sigma(u) - n_\Sigma(v)\big| \leqslant 40\Phi^*\left(\frac{2|u-v|}{N}, Q(u,v)\right) + C(p)E(u,v)^{1/p}|u-v|^\lambda, \tag{6.8}$$

where N is a (fixed) large natural number such that $(N/2)^\kappa > 240$ and

$$E(u,v) := \int_{[F(Q(u,v))\cap\Sigma]^4} \mathcal{K}^p \, d\mu.$$

One should view the second term on the right-hand side of (6.8) as the main one. The first one is just an error term that can be iterated away by scaling the distances down to zero.

We now postpone the proof of (6.8) for a second and show that it yields the desired result upon iteration.

Note that (6.6) and (6.5) imply $\Phi^* \leqslant 3\Phi_1^*$. Moreover, if $u, v \in B(a, R)$ and $\|u - v\| = r \leqslant R$, then $Q(u, v) \subset B(\frac{u+v}{2}, \sqrt{2}|u - v|) \subset B(\frac{u+v}{2}, 2\|u - v\|) \subset B(a, R + 2r)$. Thus, denoting

$$M_p(a, \rho) := \left(\int\limits_{[F(B(a,\rho))\cap\Sigma]^4} \mathcal{K}^p \, d\mu \right)^{1/p}, \quad a \in Q_0, \ \rho > 0,$$

and taking the supremum over $u, v \in B(a, R)$ with $|u - v| \leqslant r \leqslant R$, one checks that (6.8) implies

$$\begin{aligned} \Phi^*\big(r, B(a, R)\big) &\leqslant 120\Phi^*\big(r/n, B(a, R + 2r)\big) \\ &\quad + 3C(p)M_p(a, R + 2r)r^\lambda, \quad n \equiv N/2. \end{aligned} \tag{6.9}$$

A technique which is standard in PDE allows to get rid of the first term on the right-hand side of this inequality. Indeed, upon iteration (6.9) implies

$$\begin{aligned} \Phi^*\big(r, B(a, R)\big) &\leqslant 120^j \Phi^*\big(r/n^j, B(a, R + 2\sigma_j)\big) \\ &\quad + 3C(p)M_p(a, R + 2\sigma_j)r^\lambda \sum_{i=0}^{j-1} \left(\frac{120}{n^\lambda}\right)^i, \quad j = 1, 2, \ldots \end{aligned}$$

where

$$\sigma_j := r \sum_{i=0}^{j-1} n^{-i} \leqslant 2r.$$

As $n^\lambda = (\frac{N}{2})^\lambda > (\frac{N}{2})^\kappa > 240$, we obtain $120/n^\lambda < 1/2$ which implies $\sum_i (120/n^\lambda)^i < 2$ and hence

$$\Phi^*\big(r, B(a, R)\big) < 120^j \Phi^*\big(r/n^j, B(a, R + 4r)\big) + 6C(p)M_p(a, R + 4r)r^\lambda, \quad j = 1, 2, \ldots.$$

Now by Corollary 5.7 we have a priori $\Phi^*(r, S) \leqslant \Phi^*(r, Q_0) \leqslant Cr^\kappa$ for every set $S \subset Q_0$ and $r \leqslant \operatorname{diam} Q_0$. Thus,

$$120^j \Phi^*\big(r/n^j, B(a, R + 4r)\big) \leqslant Cr^\kappa \big(120/n^\kappa\big)^j < Cr^\kappa 2^{-j}$$

by choice of N. Passing to the limit $j \to \infty$ and setting $R = r$, we obtain

$$\Phi^*\big(r, B(a, r)\big) \leqslant 6C(p)M_p(a, 5r)r^\lambda, \tag{6.10}$$

and this oscillation estimate immediately implies the desired Hölder estimate (6.1) for the unit normal vector. In the remaining part of the proof, we just verify (6.8).

Step 3: bad and good parameters. From now on, we assume that $u \neq v \in Q_0$ are fixed. We pick the subcube $Q = Q(u, v)$ of $5Q_0$ with edges parallel to those of Q_0, so that the center of $Q(u, v)$ is at $(u + v)/2$ and the edge of $Q(u, v)$ equals $2|u - v|$. Set

$$m = (20N)^{-2}, \quad C_m = m^{-4}, \tag{6.11}$$

and consider the sets of *bad parameters* defined as follows:

$$\Sigma_0 = \{x_0 \in Q\colon\ \mathscr{H}^2(\Sigma_1(x_0)) \geqslant m|u-v|^2\}, \tag{6.12}$$

$$\Sigma_1(x_0) = \{x_1 \in Q\colon\ \mathscr{H}^2(\Sigma_2(x_0,x_1)) \geqslant m|u-v|^2\}, \tag{6.13}$$

$$\Sigma_2(x_0,x_1) = \{x_2 \in Q\colon\ \mathscr{H}^2(\Sigma_3(x_0,x_1,x_2)) \geqslant m|u-v|^2\}, \tag{6.14}$$

$$\Sigma_3(x_0,x_1,x_2) = \{z \in Q\colon\ \mathcal{K}(x_0,x_1,x_2,z) > (C_m E(u,v))^{1/p}|u-v|^{-8/p}\}. \tag{6.15}$$

A word of informal explanation to motivate the above choices: *if we already knew* that Σ is of class $C^{1,\lambda}$, $\lambda = 1-8/p$, then close to u we would have lots of tetrahedra with two perpendicular edges of the base having length $\approx |u-v|$, and the height $\lesssim |u-v|^{1+\lambda}$. For such tetrahedra our curvature integrand does not exceed, roughly, a multiple of $|u-v|^{\lambda-1} = |u-v|^{-8/p}$. Of course, there is no reason to believe a priori that it is indeed the case. But it helps, as we shall check, to look at tetrahedra that violate this naive estimate, and to try and estimate how many of them there are.

We first estimate the measure of Σ_0. Using (6.4) which gives a comparison of $d\mathscr{H}^2$ on $\Sigma \cap F(5Q_0)$ with the Lebesgue measure in $5Q_0$, we obtain

$$\begin{aligned} E(u,v) &\geqslant \int_{\Sigma_0}\int_{\Sigma_1(x_0)}\int_{\Sigma_2(x_0,x_1)}\int_{\Sigma_3(x_0,x_1,x_2)} \mathcal{K}^p(x_0,x_1,x_2,z)\,d\mathscr{H}^2_z\,d\mathscr{H}^2_{x_2}\,d\mathscr{H}^2_{x_1}\,d\mathscr{H}^2_{x_0} \\ &> C_m E(u,v) m^3 |u-v|^{-2}\mathscr{H}^2(\Sigma_0) \\ &= E(u,v) m^{-1}|u-v|^{-2}\mathscr{H}^2(\Sigma_0), \end{aligned}$$

which yields

$$\mathscr{H}^2(\Sigma_0) < m|u-v|^2 = \frac{|u-v|^2}{400N^2} \ll |Q(u,v)| = 4|u-v|^2. \tag{6.16}$$

Step 4: auxiliary good points. In a small neighbourhood of u we select $x_0 \in Q(u,v) \setminus \Sigma_0$ so that $\|x_0 - u\| \leqslant (20N)^{-1}|u-v|$. Once x_0 is chosen, we select $x_1 \in Q(u,v) \setminus \Sigma_1(x_0)$ and then $x_2 \in Q(u,v) \setminus \Sigma_2(x_0,x_1)$ so that

$$\|x_1 - x_0\| \approx \|x_2 - x_0\| \approx \frac{|u-v|}{N} \quad \text{and} \quad \sphericalangle(x_2-x_0, x_1-x_0) \approx \frac{\pi}{2}.$$

More precisely, let $Q(x_0)$ be the cube with one vertex at x_0 and two other vertices at

$$a_1 := x_0 + \frac{|u-v|}{N}e_1, \qquad a_2 := x_0 + \frac{|u-v|}{N}e_2.$$

We select $x_1, x_2 \in Q(x_0)$ such that

$$x_1 \in Q(x_0) \setminus \Sigma_1(x_0), \quad \|x_1 - a_1\| \leqslant \frac{|u-v|}{20N}, \tag{6.17}$$

$$x_2 \in Q(x_0) \setminus \Sigma_2(x_0,x_1), \quad \|x_2 - a_2\| \leqslant \frac{|u-v|}{20N}. \tag{6.18}$$

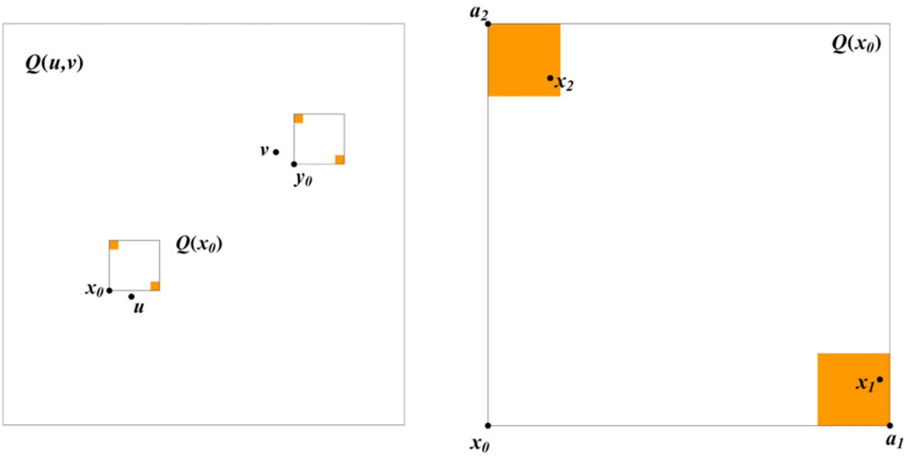

Fig. 7. The position of auxiliary good parameters in the domain of f. Left: $Q(u,v)$ and two subcubes $Q(x_0)$, $Q(y_0)$, with lower left-hand corners at x_0, y_0. Right: $Q(x_0)$ magnified. We fix $x_0 \notin \Sigma_0$, close to u, and x_1, x_2 are selected in the little shaded subcubes of $Q(x_0)$. Since the Lipschitz constant of f is small, Σ is a flat graph over $Q(u,v)$. Thus, the vectors $v_j := F(x_j) - F(x_0)$ $(j = 1, 2)$ are nearly orthogonal and have lengths very close to $|u - v|/N =$ the edge of $Q(x_0)$, see Step 5 below for the details.

(See also Fig. 7.) Since $x_0 \notin \Sigma_0$, we can use (6.12)–(6.13) to check that x_1, x_2 satisfying (6.17)–(6.18) do exist.

In a fully analogous way we select y_0, y_1, y_2 close to v—using (6.16) initially again but then by defining sets $\Sigma_1(y_0)$, $\Sigma_2(y_0, y_1)$, and $\Sigma_3(y_0, y_1, y_2)$ as in (6.13), (6.14), and (6.15). Thus, $y_0 \in Q(u,v) \setminus \Sigma_0$, $y_1 \in Q(y_0) \setminus \Sigma_1(y_0)$ and $y_2 \in Q(y_0) \setminus \Sigma_2(y_0, y_1)$, where $Q(y_0)$ is a copy of $Q(x_0)$ translated by $y_0 - x_0$, satisfy

$$\|y_0 - v\| \leqslant \frac{|u-v|}{20N}, \qquad \|y_1 - y_0\| \approx \|y_2 - y_0\| \approx \frac{|u-v|}{N}, \qquad \sphericalangle(y_2 - y_0, y_1 - y_0) \approx \frac{\pi}{2}.$$

Then we set $P_x := \langle F(x_0), F(x_1), F(x_2)\rangle$, $P_y := \langle F(y_0), F(y_1), F(y_2)\rangle$, and we let n_x, n_y denote the unit normal vectors of these two planes. By the triangle inequality,

$$\begin{aligned}
\bigl|n_\Sigma(u) - n_\Sigma(v)\bigr| \leqslant{}& \bigl|n_\Sigma(u) - n_\Sigma(x_0)\bigr| + \bigl|n_\Sigma(x_0) - n_x\bigr| \\
&+ |n_x - n_y| \\
&+ \bigl|n_y - n_\Sigma(y_0)\bigr| + \bigl|n_\Sigma(y_0) - n_\Sigma(v)\bigr|.
\end{aligned}$$

The non-obvious term is the middle one, $|n_x - n_y| \leqslant \sphericalangle(P_x, P_y)$; the remaining four terms give a small contribution which does not exceed a constant multiple of $\Phi^*(20|u-v|/N, Q(u,v))$. But due to the choice of Σ_3 the planes P_x and P_y turn out to be almost parallel: their angle is $\lesssim |u-v|^\lambda$.

Since u, v are now fixed and will not change till the end of the proof, from now we use the abbreviations

$$\Phi_i^*(r) \equiv \Phi_i^*\bigl(r, Q(u,v)\bigr), \qquad \Phi^*(r) \equiv \Phi^*\bigl(r, Q(u,v)\bigr).$$

We shall check that

$$\left|n_\Sigma(x_0) - n_x\right| \leqslant 16\Phi^*\big(2|u-v|/N\big), \tag{6.19}$$

$$\left|n_\Sigma(y_0) - n_y\right| \leqslant 16\Phi^*\big(2|u-v|/N\big), \tag{6.20}$$

$$|n_x - n_y| \leqslant K|u-v|^\lambda. \tag{6.21}$$

Combining these estimates with the obvious ones,

$$\left|n_\Sigma(u) - n_\Sigma(x_0)\right| \leqslant \Phi^*\big(|u-v|/N\big), \qquad \left|n_\Sigma(v) - n_\Sigma(y_0)\right| \leqslant \Phi^*\big(|u-v|/N\big),$$

and using monotonicity of Φ^*, one immediately obtains (6.8).

Step 5: proofs of (6.19) and (6.20). We only prove (6.19); the other proof is identical. Let

$$v_j := F(x_j) - F(x_0), \quad j = 1, 2.$$

By the fundamental theorem of calculus,

$$\begin{aligned} v_j &= \int_0^1 \nabla F\big(x_0 + t(x_j - x_0)\big)(x_j - x_0)\,dt \\ &= \nabla F(x_0)(x_j - x_0) + \int_0^1 \big(\nabla F\big(x_0 + t(x_j - x_0)\big) - \nabla F(x_0)\big)(x_j - x_0)\,dt \\ &=: w_j + \sigma_j, \quad \text{for } j = 1, 2, \end{aligned} \tag{6.22}$$

where the error terms σ_j satisfy

$$\begin{aligned} |\sigma_j| &\leqslant \left| \int_0^1 \big(\nabla F\big(x_0 + t(x_j - x_0)\big) - \nabla F(x_0)\big)\,dt\,(x_j - x_0) \right| \\ &\leqslant \Phi^*\big(2|u-v|/N\big)\operatorname{diam} Q(x_0) \\ &\leqslant 2\Phi^*\big(2|u-v|/N\big)\frac{|u-v|}{N}, \quad j = 1, 2. \end{aligned} \tag{6.23}$$

With $w_j = \nabla F(x_0) \cdot (x_j - x_0)$, $j = 1, 2$ we have

$$n_x = \frac{v_1 \times v_2}{|v_1 \times v_2|}, \quad n_\Sigma(x_0) = \frac{w_1 \times w_2}{|w_1 \times w_2|}. \tag{6.24}$$

To estimate the difference of these two vectors, we first estimate $|v_j|, |w_j|$ and the angles $\sphericalangle(v_1, v_2)$, $\sphericalangle(w_1, w_2)$. This is an elementary computation; we give some details below.[4]

[4] If you do not want to check the details of our arithmetic, please note the following: we use N *only* to fix the scale and to control the ratio of diam $Q(x_0)$ and diam $Q(u, v)$. Thus, N *does not* influence the ratio of lengths of v_1, v_2, w_1, w_2 (which are all $\approx |u-v|/N$) and the angles between these vectors (which are absolute since we assume (6.2) and (6.7)).

Therefore, the constant '16' in (6.19)–(6.20) is not really important. Any absolute constant would be fine; one would just have to adjust N to derive (6.10) from (6.8).

Using the fact that $|\nabla f|$ is bounded by $\varepsilon_0 < 1/100$ by Remark 5.8 and (6.5), $x_1 - x_0$ and $x_2 - x_0$ are close to two perpendicular sides of $Q(x_0)$, and both error terms σ_j are smaller than $|u-v|/50N$ by (6.7), one can check that

$$\frac{9}{10}\frac{|u-v|}{N} \leqslant \min\bigl(|v_j|, |w_j|\bigr) \leqslant \max\bigl(|v_j|, |w_j|\bigr) \leqslant \frac{11}{10}\frac{|u-v|}{N}$$

for $j = 1, 2$.

Note also that, cf. Fig. 7 and (6.2),

$$v_j = \frac{|u-v|}{N} e_j + \sum_{i=1}^{3} a_{ji} e_i, \quad |a_{ji}| \leqslant \frac{|u-v|\sqrt{2}}{20N},$$

which yields

$$|v_1 \cdot v_2| = \left| \frac{|u-v|}{N}(a_{12} + a_{21}) + \sum_{i=1}^{3} a_{1i} a_{2i} \right| \leqslant \frac{|u-v|^2}{6N^2}.$$

Taking the estimates of σ_j into account one more time, we obtain $|w_1 \cdot w_2| \leqslant 2|u-v|^2/(9N^2)$. Combining the inequalities for these two scalar products with the estimates of lengths of the vectors, we conclude that

$$\max\bigl(\bigl|\cos \sphericalangle(v_1, v_2)\bigr|, \bigl|\cos \sphericalangle(w_1, w_2)\bigr|\bigr) \leqslant \frac{2}{9} \cdot \left(\frac{10}{9}\right)^2.$$

Hence,

$$\min\bigl(\sin \sphericalangle(v_1, v_2), \sin \sphericalangle(w_1, w_2)\bigr) \geqslant \sqrt{1 - \left[\frac{4}{81}\left(\frac{10}{9}\right)^4\right]} > \frac{15}{16}. \tag{6.25}$$

Now,

$$A := v_1 \times v_2 - w_1 \times w_2 = |v_1 \times v_2| n_x - |w_1 \times w_2| n_\Sigma(x_0). \tag{6.26}$$

As $v_j = w_j + \sigma_j$ and $|w_j| \leqslant 11|u-v|/(10N)$, we have

$$\begin{aligned} |A| \leqslant |\sigma_1||w_2| + |\sigma_2||w_1| + |\sigma_1||\sigma_2| &\overset{(6.23)}{\leqslant} \left[2 \cdot \frac{11}{10} + \frac{1}{50}\right] 2\Phi^*\bigl(2|u-v|/N\bigr) \frac{|u-v|^2}{N^2} \\ &< 6\Phi^*\bigl(2|u-v|/N\bigr) \frac{|u-v|^2}{N^2} \end{aligned}$$

by (6.23) and (6.7). On the other hand, applying the triangle inequality, using (6.25), and the estimates $|v_j| \geqslant 9|u-v|/(10N)$ for $j = 1, 2$, we obtain first

$$|v_1 \times v_2| \overset{(6.25)}{>} \left(\frac{9}{10}\right)^2 \frac{15}{16} \frac{|u-v|^2}{N^2} > \frac{3}{4} \frac{|u-v|^2}{N^2}, \tag{6.27}$$

and then, using the second identity for A in (6.26),

$$\begin{aligned}|A| &= \big||v_1 \times v_2|\big(n_x - n_\Sigma(x_0)\big) + n_\Sigma(x_0)\big(|v_1 \times v_2| - |w_1 \times w_2|\big)\big| \\ &\geqslant |v_1 \times v_2|\big|n_x - n_\Sigma(x_0)\big| - |v_1 \times v_2 - w_1 \times w_2| \\ &\geqslant \frac{3}{4}\frac{|u-v|^2}{N^2}\big|n_x - n_\Sigma(x_0)\big| - |A|.\end{aligned}$$

Combining the lower and the upper estimate for A we obtain

$$\big|n_x - n_\Sigma(x_0)\big| \leqslant \frac{8}{3}|A|\left(\frac{|u-v|^2}{N^2}\right)^{-1} \leqslant 16\Phi^*\big(2|u-v|/N\big),$$

which yields (6.19).

Step 6: proof of (6.21). If P_x is parallel to P_y, there is nothing to prove. Let us then suppose that these planes intersect and denote their angle by γ_0. To show that $\gamma_0 \lesssim |u-v|^\lambda$, we use again the definition of bad sets. Note that for

$$G = Q(u,v) \setminus \big(\Sigma_3(x_0,x_1,x_2) \cup \Sigma_3(y_0,y_1,y_2)\big) \tag{6.28}$$

we have by (6.14)

$$\mathscr{H}^2(G) > \big|Q(u,v)\big| - 2m|u-v|^2 = \big(2|u-v|\big)^2 - 2m|u-v|^2 > |u-v|^2 \tag{6.29}$$

by choice of m. Therefore, as $\lambda - 1 = -8/p$, for all $z \in G$ we have according to (6.15) the two inequalities

$$\mathcal{K}(x_0,x_1,x_2,z) \leqslant K_0|u-v|^{\lambda-1}, \qquad \mathcal{K}(y_0,y_1,y_2,z) \leqslant K_0|u-v|^{\lambda-1}, \tag{6.30}$$

where

$$\begin{aligned}K_0 = K_0\big(p, E(u,v)\big) &:= (20N)^{8/p}E(u,v)^{1/p} \\ &\equiv C_4(p)E(u,v)^{1/p},\end{aligned}$$

as we have in fact chosen N depending only on $\kappa = (p-8)/(p+16)$.

We are now going to use formula (2.3) for $\mathcal{K}$ to estimate the distance from $F(z)$ to the planes P_x and P_y. Setting $v_j := F(x_j) - F(x_0)$ for $j = 1,2$ (as in the previous step of the proof), and $v_3 := F(z) - F(x_0)$, we obtain for the tetrahedron $T := (F(x_0), F(x_1), F(x_2), F(z))$

$$|v_3| \leqslant (1+\varepsilon_0)|z - x_0| < 2|u-v|, \qquad \operatorname{diam} T < 2|u-v|$$

by virtue of (6.5). Since the $|v_j|$ for $j = 1,2$ have been estimated before, this yields an estimate of the area of T,

$$\begin{aligned}2A(T) &= |v_1 \times v_2| + |v_2 \times v_3| + |v_1 \times v_3| + \big|(v_2 - v_1) \times (v_3 - v_2)\big| \\ &\leqslant \left(\frac{11}{10}\right)^2 \frac{|u-v|^2}{N^2} + 4\left(\frac{11}{10}\frac{|u-v|}{N}\right) \operatorname{diam} T \\ &\leqslant \frac{15|u-v|^2}{N} \quad \text{as } N > 1. \end{aligned} \tag{6.31}$$

Thus

$$\begin{aligned}\mathcal{K}(x_0, x_1, x_2, z) &= \frac{\operatorname{dist}(F(z), P_x)}{3(\operatorname{diam} T)^2} \cdot \frac{|v_1 \times v_2|}{2A(T)} \\ &\overset{(6.27)}{\geqslant} \frac{\operatorname{dist}(F(z), P_x)}{16N^2} \big(2A(T)\big)^{-1} \\ &\geqslant \frac{\operatorname{dist}(F(z), P_x)}{N^2|u-v|^2}. \end{aligned} \tag{6.32}$$

For the last inequality we have simply used (6.31) and the inequality $N > (N/2)^\kappa > 240$ which follows from our initial choice of N. Since the points y_0, y_1, y_2 have been chosen analogously to x_0, x_1, x_2, it is clear that we also have

$$\mathcal{K}(y_0, y_1, y_2, z) > \frac{\operatorname{dist}(F(z), P_y)}{N^2|u-v|^2}. \tag{6.33}$$

Combining (6.30)–(6.33), we obtain

$$\begin{aligned}\max\big(\operatorname{dist}\big(F(z), P_x\big), \operatorname{dist}\big(F(z), P_y\big)\big) &< N^2 K_0 |u-v|^{1+\lambda} \\ &\equiv C_5(p) E(u,v)^{1/p} |u-v|^{1+\lambda}, \quad z \in G. \end{aligned} \tag{6.34}$$

We shall show that the combination of (6.29) and (6.34) implies that $|n_x - n_y| \leqslant \gamma_0 = \sphericalangle(P_x, P_y)$ is estimated by a constant multiple of $|u-v|^\lambda$ thus establishing (6.21) as the only missing ingredient for the proof of the key estimate (6.8).

Indeed, consider an affine plane P which is perpendicular both to P_x and P_y. Let π_P denote the orthogonal projection onto P. By (6.34) above, we see that $\pi_P(F(G))$ is a subset of a rhombus R contained in the plane P. The height of this rhombus is equal to

$$h = 2 \cdot C_5(p) E(u,v)^{1/p} |u-v|^{1+\lambda}$$

and the (acute) angle of R is γ_0, so that the longer diagonal of R equals

$$D = \frac{h}{\sin(\gamma_0/2)} = \frac{2C_5(p)E(u,v)^{1/p}|u-v|^{1+\lambda}}{\sin(\gamma_0/2)}.$$

Therefore, the set $F(G)$ is contained in a cylinder C_0 with axis $l := P_x \cap P_y$ and radius $D/2$,

$$F(G) \subset C_0 := \big\{w\colon \operatorname{dist}(w, l) \leqslant D/2\big\}. \tag{6.35}$$

The orthogonal projection of C_0 onto the plane containing the domain of f (recall that $F(x) = (x, f(x))$ parametrizes a portion of Σ that we consider) gives us a strip S of width D. This strip must contain all good parameters $z \in G$, so that, taking (6.29) into account, we have

$$3D|u-v| > 2\sqrt{2}D|u-v| = D \operatorname{diam} Q(u,v) > \text{area of } S \cap Q(u,v) \geqslant \mathscr{H}^2(G) > |u-v|^2.$$

Hence, $D > |u-v|/3$, so that

$$\frac{2}{\pi}\frac{\gamma_0}{2} \leqslant \sin\frac{\gamma_0}{2} = \frac{2C_5(p)E(u,v)^{1/p}|u-v|^{1+\lambda}}{D} < 6C_5(p)E(u,v)^{1,p}|u-v|^{\lambda},$$

and hence

$$|n_x - n_y| \leqslant \gamma_0 < 6\pi C_5(p)E(u,v)^{1,p}|u-v|^{\lambda}$$

which establishes (6.21) and therefore concludes the whole proof. Note that we have obtained the key estimate (6.8) with $C(p) = 6\pi C_5(p)$ depending only on p, as desired. □

Applying the above result, one can sharpen Corollary 5.7 as follows.

Corollary 6.2. *Assume that $p > 8$, $\mathscr{M}_p(\Sigma) < E < \infty$. Then there exist two constants, $0 < \tilde{a}(p) < 1 < \tilde{A}(p) < \infty$, such that for each $x \in \Sigma$ there is a function*

$$f : T_x\Sigma \to (T_x\Sigma)^{\perp} \simeq \mathbb{R}$$

with the following properties:

(i) $f(0) = 0$, $\nabla f(0) = (0,0)$.
(ii) *For $\tilde{R}_1 \equiv \tilde{R}_1(E,p) := \tilde{a}(p)E^{-1/(p-8)}$ we have the estimate*

$$\left|\nabla f(y_1) - \nabla f(y_2)\right| \leqslant \tilde{A}(p)\mathscr{M}_p\big(\Sigma \cap B(x, 10\tilde{R}_1)\big)^{\frac{1}{p}}|y_1 - y_2|^{1-8/p}, \quad y_1, y_2 \in B(x, \tilde{R}_1).$$

(iii) *The map*

$$\Phi(y) := x + \big(y, f(y)\big), \quad y \in T_x\Sigma \simeq \mathbb{R}^2,$$

satisfies

$$\Phi(D_{\frac{3}{4}\tilde{R}_1}) \subset B(x, \tilde{R}_1) \cap \Sigma \subset \Phi(D_{\tilde{R}_1}), \tag{6.36}$$

where $D_{\tilde{R}_1} = B(0, \tilde{R}_1) \cap T_x\Sigma$ is a disk in $T_x\Sigma$ around $0 \in T_x\Sigma$, and

$$\left|D\Phi(y_1) - D\Phi(y_2)\right| \leqslant A(p)\mathscr{M}_p\big(\Sigma \cap B(x, 10\tilde{R}_1)\big)|y_1 - y_2|^{1-8/p},$$
$$y_1, y_2 \in B(x, \tilde{R}_1). \tag{6.37}$$

Of course, in (ii) and (iii) one can replace $\mathscr{M}_p(\Sigma \cap \cdots)$ by the total energy of the surface thus providing clear-cut a priori estimates to be used in the next section.

7. Sequences of equibounded $\mathscr{M}_p$-energy

The main issue of this final section is the proof of the following compactness theorem for admissible surfaces of equibounded energy with a uniform area bound. Notice that such an additional area bound is necessary as the example of larger and larger spheres shows. Let $S_\rho := \partial B(0,\rho)$. For any tetrahedron T (with non-coplanar vertices) we estimate

$$\mathcal{K}(T) \geqslant \frac{1}{6R(T)}, \tag{7.1}$$

where $R(T)$ denotes the radius of the circumsphere of $T = (x_0, x_1, x_2, x_3)$. There is an explicit formula,

$$\frac{1}{2R(T)} = \frac{|\langle z_3, z_1 \times z_2\rangle|}{||z_1|^2 z_2 \times z_3 + |z_2|^2 z_3 \times z_1 + |z_3|^2 z_1 \times z_2|},$$

where we have set $z_i = x_i - x_0$ for $i = 1, 2, 3$; this formula can be compared to (2.3) in order to obtain (7.1). Hence,

$$\mathscr{M}_p(S_\rho) \gtrsim \rho^{8-p} \to 0 \quad \text{as } \rho \to \infty.$$

Theorem 7.1. *Let $\Sigma_j \in \mathscr{A}$ be a sequence of admissible surfaces. Assume $0 \in \Sigma_j$ for each $j \in \mathbb{N}$ and let $E > 0$, $p < 8$ be constants such that $\mathscr{M}_p(\Sigma_j) \leqslant E$ for all $j \in \mathbb{N}$. In addition, assume that*

$$\sup \mathscr{H}^2(\Sigma_j) \leqslant H < \infty.$$

Then there is a compact $C^{1,1-\frac{8}{p}}$-manifold Σ and a subsequence $(\Sigma_{j'}) \subset (\Sigma_j)$ such that $\Sigma_{j'} \to \Sigma$ in Hausdorff distance as $j' \to \infty$ and moreover

$$\mathscr{M}_p(\Sigma) \leqslant \liminf_{j'\to\infty} \mathscr{M}_p(\Sigma_{j'}), \qquad \mathscr{H}^2(\Sigma) = \lim_{j'\to\infty} \mathscr{H}^2(\Sigma_{j'}).$$

Remark. The proof of this result will reveal that the limit surface Σ is equipped with a nice graph representation as described in Corollary 6.2, with norms and patch sizes uniformly controlled solely in terms of E and p.

Proof of Theorem 7.1. Step 1. We fix $j \in \mathbb{N}$ and look at the covering

$$\Sigma_j \subset \bigcup_{x \in \Sigma_j} B(x, R_1),$$

where now $R_1 := \tilde{R}_1(E, p) \leqslant R_0(E, p)$ is the radius defined in Corollary 6.2, and $R_0(E, p)$ appeared in (3.1) of Theorem 3.1. By means of Vitali's covering lemma we extract a subfamily of pairwise disjoint balls $B(x_k, R_1)$, $x_k \in \Sigma_j$, such that

$$\Sigma_j \subset \bigcup_k B(x, 5R_1). \tag{7.2}$$

Using Theorem 3.1 for any number N of these disjoint balls (appropriately numbered) and summing with respect to k, we infer

$$N \cdot \frac{\pi}{2} R_1^2 \leqslant \sum_{k=1}^{N} \mathscr{H}^2\big(B(x_k, R_1) \cap \Sigma_j\big) \leqslant \mathscr{H}^2(\Sigma_j) \leqslant H,$$

which means that there can be at most $\lfloor 2H/(\pi R_1^2)\rfloor$ such disjoint balls. Therefore, (7.2) leads to the estimate[5]

$$\operatorname{diam} \Sigma_j \leqslant N \operatorname{diam} B(0, 5R_1) \leqslant \frac{2H}{\pi R_1^2} \cdot 10R_1 =: \tilde{R}_0. \tag{7.3}$$

Since $0 \in \Sigma_j$ for all $j \in \mathbb{N}$, we find that the family $\{\Sigma_j\}$ is contained in the closed ball $B(0, \tilde{R}_0)$.

Step 2. Apply Blaschke's selection theorem [25] to find a compact set $\Sigma \subset B(0, \tilde{R}_0)$ and a subsequence (still labeled with j) such that

$$\Sigma_j \to \Sigma \quad \text{as } j \to \infty \tag{7.4}$$

in the Hausdorff distance. Fix $\varepsilon > 0$ small (to be specified later) and assume now that (for a further subsequence)

$$\operatorname{dist}_{\mathscr{H}}(\Sigma_j, \Sigma) < \frac{1}{2}\varepsilon R_1 \quad \text{for all } j \in \mathbb{N}, \tag{7.5}$$

where $\operatorname{dist}_{\mathscr{H}}(\cdot,\cdot)$ denotes the Hausdorff distance. Next, we form an open neighbourhood of the limit set,

$$\Sigma \subset B_{99\varepsilon R_1}(\Sigma) \subset \bigcup_{y\in\Sigma} B(y, 100\varepsilon R_1),$$

and use Vitali's lemma again to extract a subfamily[6] of disjoint balls $B(y_l, 100\varepsilon R_1)$, $y_l \in \Sigma$ for $l = 1, 2, \ldots, N$ such that

$$\Sigma \subset B_{99\varepsilon R_1}(\Sigma) \subset \bigcup_{l=1}^{N} B(y_l, 500\varepsilon R_1). \tag{7.6}$$

[5] Notice that $\tilde{R}_0$ depends on H and (via R_1) also on E and p.

[6] Since Σ is compact, we can assume w.l.o.g. that this subfamily is *finite*.

Now, each $y_l \in \Sigma$ is a limit of some $y_l^j \in \Sigma_j$, and according to (7.5) we have $|y_l - y_l^j| < \frac{1}{2}\varepsilon R_1$ for all $l = 1, \dots, N$ and all $j \in \mathbb{N}$. Therefore for each fixed $j \in \mathbb{N}$ the balls $B(y_l^j, 99\varepsilon R_1)$ are pairwise disjoint, since $|y_l^j - y_m^j \geqslant |y_l^j - y_m^j| - |y_l^j - y_l| - |y_m - y_m^j| > 200\varepsilon R_1 - 2 \cdot 12\varepsilon R_1 = 199\varepsilon R_1$. Moreover, we have

$$\Sigma_j \overset{(7.5)}{\subset} B_{\varepsilon R_1/2}(\Sigma) \subset B_{99\varepsilon R_1}(\Sigma) \subset \bigcup_{l=1}^{N} B(y_l, 500\varepsilon R_1) \overset{(7.5)}{\subset} \bigcup_{l=1}^{N} B\big(y_l^j, 501\varepsilon R_1\big) \tag{7.7}$$

for each fixed $j \in \mathbb{N}$, since $|y - y_l^j| \leqslant |y - y_l| + |y_l - y_l^j| \leqslant 501\varepsilon R_1$ by (7.5) for every $y \in B(y_l, 500\varepsilon R_1)$. Using again Theorem 3.1 for a fixed $j \in \mathbb{N}$ and summing w.r.t. to l, we deduce

$$N \cdot \frac{\pi}{2}(99\varepsilon R_1)^2 \leqslant \sum_{l=1}^{N} \mathscr{H}^2\big(B\big(y_l^j, 99\varepsilon R_1\big) \cap \Sigma_j\big) \leqslant \mathscr{H}^2(\Sigma_j) \leqslant H,$$

whence the bound $N \leqslant \lfloor 2H\pi^{-1}(99\varepsilon R_1)^{-2} \rfloor$ for the number of disjoint balls $B(y_l^j, 99\varepsilon R_1)$ for each fixed $\varepsilon > 0$.

Step 3. We consider the unit normals $n_l^j := n_{\Sigma_j}(y_l^j) \in \mathbb{S}^2$ and select subsequences finitely many times so that for all $l = 1, \dots, N$

$$n_l^j \to n_l \in \mathbb{S}^2 \quad \text{as } j \to \infty,$$

and for given small $\delta > 0$ (to be specified below)

$$\big|n_l^j - n_l\big| < \delta \quad \text{for all } j \in \mathbb{N} \text{ and all } l = 1, 2, \dots, N. \tag{7.8}$$

Now fix $\varepsilon > 0$ so small that $2000\varepsilon R_1 \leqslant R_1$ and

$$B\big(y_l^j, 2000\varepsilon R_1\big) \cap \Sigma_j \subset \Phi_l^j\big(D_{R_1}^{j,l}\big),$$

where $\Phi_l^j(y) := y_l^j + (y, f_l^j(y))$, $y \in D_{R_1}^{j,l} \subset T_{y_l^j}\Sigma_j \approx \mathbb{R}^2$ is the local graph representation of Σ_j near y_l^j on the two-dimensional disk $D_{R_1}^{j,l} = B(0, R_1) \cap T_{y_l^j}\Sigma_j$, whose existence is established in Corollary 6.2. If we choose now $\delta > 0$ sufficiently small (depending on ε) then we can arrange that

$$B(y_l, 1000\varepsilon R_1) \cap \Sigma_j \subset \tilde{\Phi}_l^j\big(D_{\frac{5}{6}R_1}^l\big),$$

where $\tilde{\Phi}_l^j(y) := y_l^j + (y, \tilde{f}_l^j(y))$ for $y \in D_{5R_1/6}^l := B(0, \frac{5}{6}R_1) \cap (n_l)^\perp$, and $\tilde{f}_l^j$ on the *fixed* disk $D_{5R_1/6}^l$ is obtained from f_l^j by slightly tilting the domain of f_l^j, i.e. by tilting the plane $T_{y_l^j}\Sigma_j$ towards the plane $(n_l)^\perp \approx \mathbb{R}^2$. (That this is indeed possible is a straightforward but a bit tedious exercise.)

The new graph functions

$$\tilde{f}_l^j \colon (n_l)^\perp \supset D_{5R_1/6}^l \longrightarrow \mathbb{R}$$

continue to be of class $C^{1,\lambda}$ for $\lambda = 1 - 8/p$ with uniform estimates for the oscillation of their gradients as in Corollary 6.2 (we use the assumption $\sup \mathscr{M}_p(\Sigma_j) \leqslant E$) so that we may apply the theorem of Arzela–Ascoli for each $l = 1, 2, \dots, N$ to obtain subsequences $\tilde{f}_l^{j'} \to f_l$ in C^1 as $j' \to \infty$. The limit functions f_l satisfy the same uniform $C^{1,\lambda}$ estimates. Thus, Σ is covered by N graphs $\Phi_l(y) = y_l + (y, f_l(y))$, $l = 1, 2, \dots, N$, by virtue of the Hausdorff convergence (7.4) and the C^1-convergence of the $\tilde{\Phi}_l^{j'}$ as $j' \to \infty$. Moreover,

$$B(y_l, 1000\varepsilon R_1) \cap \Sigma = \Phi_l\big(D^l_{5R_1/6}\big) \cap B(y_l, 1000\varepsilon R_1).$$

Now (7.6) implies that for each $y \in \Sigma$ there exists an $l \in \{1, 2, \dots, N\}$ such that the set

$$\Sigma \cap B(y, 500\varepsilon R_1) \overset{(7.6)}{\subset} B(y_l, 1000\varepsilon R_1) \cap \Sigma$$

so that

$$\Sigma \cap B(y, 500\varepsilon R_1) = \Phi_l\big(D^l_{5R_1/6}\big) \cap B(y, 500\varepsilon R_1).$$

In particular, the limit surface Σ is also a $C^{1,\lambda}$ manifold for $\lambda = 1 - 8/p$.

Step 4 (lower semicontinuity of $\mathscr{M}_p$). This follows from Fatou's lemma combined with the following properties of the integrand:

$$\mathcal{K}(T) = \lim_{i\to\infty} \mathcal{K}(T_i) \quad \text{whenever } T_i \to T \text{ and } \mathcal{K}(T) > 0, \tag{7.9}$$

$$\mathcal{K}(T) \leqslant \liminf_{i\to\infty} \mathcal{K}(T_i) \quad \text{whenever } T_i \to T \text{ and } \mathcal{K}(T) = 0. \tag{7.10}$$

The argument is standard and uses a partition of unity in a neighbourhood of Σ; we sketch it briefly. Take functions $\psi_l \in C_0^\infty(B(1000\varepsilon R_1))$, $l = 1, 2, \dots, N$, such that such that

$$\sum_{l=1}^N \psi_l \equiv 1 \quad \text{on } \subset \bigcup_{l=1}^N B(y_l, 500\varepsilon R_1). \tag{7.11}$$

This gives $\sum \psi_l \equiv 1$ on each Σ_j for j large. Inserting

$$1 = \prod_{i=0}^3 \Bigg(\sum_{l_i=1}^N \psi_{l_i}(x_i) \Bigg)$$

into the integral $\mathscr{M}_p(\Sigma_{j'}) = \int_{(\Sigma_{j'})^4} \mathcal{K}\, d\mu$ we write this integral as a sum of N^4 quadruple integrals, each of them over a product of four little patches on $\Sigma_{j'}$. Next, we use the Φ's constructed in Step 2 to parametrize these integrals; the parameters z_i (mapped to x_i) belong to *fixed* little disks D^{l_i} of radius $5R_1/6$ contained in tangent spaces to Σ. Since $\tilde{\Phi}_l^{j'} \to \Phi_l$ in C^1, it is easy to see that all products of $\psi_{l_i} \circ \Phi_{l_i}^{j'}(z_i)$, and all terms where the surface measure $d\mathscr{H}^2(x_i)$

is expressed by dz_i, converge. Combining this with (7.9)–(7.10), invoking Fatou's lemma and subadditivity of lim inf, we see that

$$\liminf \mathscr{M}_p(\Sigma_{j'}) \geqslant \text{the sum of } \liminf\text{'s of } N^4 \text{ terms} \geqslant \mathscr{M}_p(\Sigma).$$

A similar argument shows that $\mathscr{H}^2(\Sigma_{j'}) \to \mathscr{H}^2(\Sigma)$; one just replaces $\mathcal{K}$ by 1 in the above reasoning and simply passes to the limit, using the C^1 convergence of parametrizations. □

Proof of Theorems 1.6 and 1.7. This follows easily from Theorem 7.1. The two classes $\mathscr{C}_E(M_g)$ and $\mathscr{C}_A(M_g)$ of surfaces Σ which are ambiently isotopic to a fixed closed, compact, connected, smoothly embedded reference surface M_g of genus g and satisfy $\mathscr{M}_p(\Sigma) \leqslant E$, or $\mathscr{H}^2(\Sigma) \leqslant A$, respectively, are nonempty. (Just take an M_g of class C^2 to ensure, by Proposition A.1, that $\mathscr{M}_p(M_g)$ is finite; scaling M_g if necessary we can make its energy smaller than E, or its area smaller than A.) Thus, one can take a sequence Σ_j contained in $\mathscr{C}_E(M_g)$, or in $\mathscr{C}_A(M_g)$, respectively, which is minimizing for the area functional, or for $\mathscr{M}_p$. Applying Theorem 7.1, we obtain a subsequence of Σ_j which converges to some Σ in C^1. Since isotopy classes are stable under C^1-convergence, see [3], the limiting surface Σ belongs to $\mathscr{C}_E(M_g)$, or resp. to $\mathscr{C}_A(M_g)$. □

Acknowledgments

The authors would like to thank the Deutsche Forschungsgemeinschaft, Polish Ministry of Science and Higher Education, and the Centro di Ricerca Matematica Ennio De Giorgi at the Scuola Normale Superiore in Pisa, in particular Professor Mariano Giaquinta, for generously supporting this research. We are also grateful to Professor Gilad Lerman for his comments on an earlier version of this paper which helped us to improve the presentation.

Appendix A. Finiteness of energy of C^2-surfaces

As before, $T = (x_0, x_1, x_2, x_3)$ stands for a tetrahedron in $\mathbb{R}^3$. $V(T)$ is the volume of T and $A(T)$ denotes the total area of T, i.e. the sum of areas of the four triangular faces. Recall that

$$\mathcal{K}(T) = \frac{V(T)}{A(T)(\operatorname{diam} T)^2}. \tag{A.1}$$

Proposition A.1. *If $\Sigma \subset \mathbb{R}^3$ is a compact, embedded surface of class C^2, then there exists a constant $C = C(\Sigma)$ such that*

$$\mathcal{K}(T) \leqslant C \quad \textit{for each } T \in \Sigma^4.$$

This obviously implies that $\mathscr{M}_p(\Sigma) < \infty$ whenever Σ is of class C^2.

Proof. Comparing $A(T)$ with the maximum of areas of the faces, we obtain

$$\frac{1}{12}\frac{h_{\min}(T)}{(\operatorname{diam} T)^2} \leqslant \mathcal{K}(T) \leqslant \frac{1}{3}\frac{h_{\min}(T)}{(\operatorname{diam} T)^2},$$

where $h_{\min}(T)$ stands for the minimal height of T, i.e. for the minimal distance of x_i to the affine plane spanned by the other three x_j's, $i = 0, 1, 2, 3$. Since $h_{\min}(T) \leqslant \operatorname{diam} T$, it is enough to show that $\mathcal{K}(T)$ is bounded when $\operatorname{diam} T \leqslant d_0$ for some $d_0 = d_0(\Sigma)$ sufficiently small.

Thus, from now on we fix a $d_0 > 0$ such that for each $x \in \Sigma$ the intersection $\Sigma \cap B(x, 2d_0)$ coincides with a graph of a C^2-function defined of $x + T_x\Sigma$, and

$$\operatorname{dist}(y, x + T_x\Sigma) \leqslant A|y - x|^2, \quad y \in \Sigma \cap B(x, 2d_0). \tag{A.2}$$

Remark. (A.2) is the only thing we need from the C^2-property. Such an estimate holds for $C^{1,1}$-surfaces, too. If one represents such a surface locally by a function $g \in C^{1,1}$ normalized to $g(0) = 0$ and $\nabla g(0) = 0$ then the Lipschitz continuity of ∇g implies a quadratic height excess as in (A.2).

W.l.o.g. we can assume that $Ad_0 \ll 1$.

Lemma A.2. *Let $T = (x_0, x_1, x_2, x_3)$ be an arbitrary tetrahedron, with angles of the faces denoted by α_{ij}, $i, j = 0, 1, 2, 3$, $i \neq j$ so that α_{ij} is the angle at x_j on the face which is opposite to x_i. Then, two cases are possible*:

(i) *At least one of the $\alpha_{ij} \in [\frac{\pi}{9}, \frac{8\pi}{9}]$;*
(ii) *All $\alpha_{ij} \in (0, \frac{\pi}{9}) \cup (\frac{8\pi}{9}, \pi)$.*

In the latter case, eight of the α_{ij} are small, i.e. belong to $(0, \frac{\pi}{9})$ and the remaining four are large, i.e., belong to $(\frac{8\pi}{9}, \pi)$. Moreover, there is one large angle on each face and either 0 *or* 2 *such angles at each vertex of T.*

Proof. We have

$$\sum_{0 \leqslant j \leqslant 3, j \neq i} \alpha_{ij} = \pi \quad \text{for each } i = 0, 1, 2, 3, \tag{A.3}$$

$$\sum_{0 \leqslant i \leqslant 3, i \neq j} \alpha_{ij} \in (0, 2\pi) \quad \text{for each } j = 0, 1, 2, 3, \tag{A.4}$$

$$\alpha_{ij} + \alpha_{lj} > \alpha_{kj} \quad \text{for each permutation } (i, j, k, l) \text{ of } (0, 1, 2, 3). \tag{A.5}$$

(The last condition amounts to the triangle inequality for the spherical metric.)

Now, suppose that Case (i) does not hold. If there were at most three large angles, then the sum of all α_{ij} would be strictly smaller than

$$3\pi + 9 \cdot \frac{\pi}{9} = 4\pi,$$

a contradiction. Similarly, if there were at least 5 large angles, the sum of all angles of T would be strictly larger than 4π. Thus, if (i) fails, T must have precisely 4 large angles. By (A.3) and the pigeon-hole principle, there is precisely one such angle on each face. Furthermore, if there is a large angle at some vertex, then by (A.5) at least one of the remaining angles at this vertex must also be large. Since the sum of all angles at each vertex is smaller than 2π, we have precisely either 0 or 2 large angles at each vertex. $\square$

Now, fix $T \in \Sigma^4$ with $d = \operatorname{diam} T < d_0 = d_0(\Sigma)$.

1. If Case (i) of the lemma holds for T, we can assume w.l.o.g. that $x_0 = 0$, the tangent plane $T_{x_0}\Sigma = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ is horizontal, and $\sphericalangle(x_1, x_2) \in [\frac{\pi}{9}, \frac{8\pi}{9}]$. Let $P := \langle x_0, x_1, x_2\rangle$. A computation shows that there is an absolute constant c_1 such that

$$\sphericalangle(P, T_{x_0}\Sigma) \leqslant c_1 A d$$

(which is a small angle if d_0 is chosen sufficiently small). Therefore, since $\operatorname{dist}(x_3, T_{x_0}\Sigma) \leqslant A d^2$, we have

$$\operatorname{dist}(x_3, P) \leqslant c_2 A d^2,$$

which yields $\mathcal{K}(T) \leqslant c_2 A$.

2. Suppose now Case (ii) holds for T. W.l.o.g. we can assume that all angles at x_0 belong to $(0, \pi/9)$. We can also assume that all these angles exceed $c_3 A d$ for some constant c_3, since otherwise there exists a vertex and an edge of T with mutual distance $\lesssim d^2$ and we are done.

As before, we choose coordinates so that $x_0 = 0$ and $T_{x_0}\Sigma = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ is horizontal. Let π_T stands for the orthogonal projection onto $T_{x_0}\Sigma$.

For $i = 1, 2, 3$, let l_i be the straight line through x_0 and x_i. Set also $x_i' := \pi_T(x_i)$, $d_i = |x_i - x_0|$, $d_i' = |x_i' - x_0|$ and $l_i' = \pi_T(l_i)$ $(i = 1, 2, 3)$. Finally, set $h_i = |x_i - x_i'| = \operatorname{dist}(x_i, x_0 + T_{x_0}\Sigma)$. We have

$$h_i \leqslant A d_i^2, \quad d_i' \leqslant d_i \leqslant 2 d_i'.$$

Permuting the numbering of x_1, x_2, x_3, we can moreover assume that $l_1' \neq l_3'$ (if all projections of edges meeting at x_0 onto the tangent plane coincide, then $V(T) = \mathcal{K}(T) = 0$), and that the angle $\gamma := \sphericalangle(x_3' - x_0, x_1' - x_0)$ is the largest of all the angles $\sphericalangle(x_j' - x_0, x_k' - x_0)$, where $j, k = 1, 2, 3$. Set $P := \langle x_0, x_1, x_3\rangle$. Note that if β_i denotes the angle between l_i and l_i', then $\sin\beta_i \leqslant A d_i^2/d_i = A d_i \leqslant A d \ll 1$.

Let $l \subset P$ be the straight line such that $\pi_T(l) = l_2' = \pi_T(l_2)$. The crucial observation is that the angle between l and l_2' is at most $c_4 A d$ for some absolute constant c_4 (here we use the piece of information that all angles of T at x_0 are small). Using this, we estimate

$$\begin{aligned}
\operatorname{dist}(x_2, P) &\leqslant \left|x_2 - x_2'\right| + \operatorname{dist}\bigl(x_2', l\bigr) \quad \text{as } l \subset P \\
&\leqslant A d_2^2 + d_2' \sin\sphericalangle\bigl(l_2', l\bigr) \\
&\leqslant c_5 A d^2.
\end{aligned}$$

Thus, $h_{\min}(T) \leqslant c_5 A d^2$. This yields the desired estimate of $\mathcal{K}(T)$. $\square$

Remark. For Σ in C^2, the bound that we obtain for $\mathcal{K}(T)$ is of the form

$$\mathcal{K}(T) \leqslant C \cdot A,$$

where A is the maximum of the C^2-norms of functions that give a graph description of Σ in finitely many small patches.

Appendix B. Other integrands

In [12], J.C. Léger suggests an integrand that could serve as a counterpart for integral Menger curvature of one-dimensional sets, to obtain rectifiability criteria in higher dimensions. For $d = 2$, his suggestion is to use the cube of

$$\mathcal{K}_L(x_0, x_1, x_2, x_3) = \frac{\operatorname{dist}(x_3, \langle x_0, x_1, x_2\rangle)}{\prod_{j=0}^{2} |x_3 - x_j|}. \tag{B.1}$$

We are going to show that $\mathcal{K}_L$ and some of its relatives are not suitable for our purposes for a simple reason: even for a round sphere, the energy given by the L^p-norm of such an integrand would be infinite for all sufficiently large p! This surprising effect is due to the fact that $\mathcal{K}_L$ is not a symmetric function of its variables.

To be more precise, let

$$\mathcal{F}(x, y, z, \xi) := \frac{\operatorname{dist}\big(\xi, \langle x, y, z\rangle\big)}{M(|\xi - x|, |\xi - y|, |\xi - z|)^{\alpha}} \tag{B.2}$$

where $\alpha > 1$ is a parameter and $M\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree 1, monotone nondecreasing w.r.t. each of the three variables, and satisfies

$$\min(t, r, s) \leqslant M(t, r, s) \leqslant \max(t, r, s) \quad \text{for } t, r, s \geqslant 0. \tag{B.3}$$

Note that such $\mathcal{F}$ coincides with J.C. Léger's $\mathcal{K}_L$ if $M(t, r, s) = \sqrt[3]{trs}$ is the geometric mean and $\alpha = 3$.

Proposition B.1. *Whenever* $(\alpha - 1)p \geqslant 12$*, then*

$$\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathcal{F}(x, y, z, \xi)^p \, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z)\, d\mathscr{H}^2(\xi) = +\infty.$$

Proof. We follow a suggestion of K. Oleszkiewicz [22] (to whom we are grateful for a brief sketch of this proof) and consider the behaviour of $\mathcal{F}$ on such quadruples of nearby points (x, y, z, ξ) for which the plane $\langle x, y, z\rangle$ is very different from the tangent plane at ξ. It turns out that

$$\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathcal{F}(x, y, z, \xi)^p \, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z) = +\infty \quad \text{for each } \xi \in \mathbb{S}^2.$$

To check this, suppose without loss of generality that $\xi = (0, 0, 1)$. Fix a small $\varepsilon \in (0, 1)$ and $r_n = 2^{-2n}$ for $n = 1, 2, 3, \ldots$. Consider the sets $\Delta_n \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$,

$$\Delta_n := \big(B\big(a_n, \varepsilon r_n^2\big) \cap \mathbb{S}^2\big) \times \big(B\big(b_n, \varepsilon r_n^2\big) \cap \mathbb{S}^2\big) \times \big(B\big(c_n, \varepsilon r_n^2\big) \cap \mathbb{S}^2\big), \tag{B.4}$$

where

$$a_n := \left(r_n, 0, \sqrt{1-r_n^2}\right), \tag{B.5}$$

$$b_n := \left(r_n, 2r_n, \sqrt{1-5r_n^2}\right), \tag{B.6}$$

$$c_n := \left(r_n, -2r_n, \sqrt{1-5r_n^2}\right). \tag{B.7}$$

Note that for $\varepsilon \in (0,1)$ all Δ_n are pairwise disjoint. We shall show that whenever a triple of points $(x,y,z) \in \Delta_n$, then the plane $P = \langle x,y,z\rangle$ is almost perpendicular to $T_\xi \mathbb{S}^2$ (the angle differs from $\pi/2$ at most by a fixed constant multiple of ε) and

$$\operatorname{dist}(\xi, P) \geqslant r_n/2, \quad \mathcal{F}^p(x,y,z,\xi) \geqslant A \cdot r_n^{p(1-\alpha)}$$

for some constant A depending on ε, p and α but *not* on n. Let $v_n := b_n - a_n$, $w_n := c_n - a_n$ $(n = 1,2,\ldots)$. Since $\sqrt{1-x} = 1 - x/2 + O(x^2)$ as $x \to 0$, we have

$$v_n = \left(0, 2r_n, -2r_n^2 + O\left(r_n^4\right)\right), \qquad w_n = \left(0, -2r_n, -2r_n^2 + O\left(r_n^4\right)\right).$$

A computation shows that

$$u_n := v_n \times w_n = \left(-8r_n^3, 0, 0\right) + e_n, \quad |e_n| \leqslant C_1 r_n^5,$$

where C_1 is an absolute constant. Therefore,

$$\sigma_n := \frac{u_n}{|u_n|} = (-1,0,0) + f_n, \quad |f_n| \leqslant C_2 r_n^2,$$

again with some absolute constant C_2. Now, let $(x,y,z) \in \Delta_n$ and let $v_n' := y - x$, $w_n' := z - x$. By triangle inequality, we have

$$\max\left(\left|v_n - v_n'\right|, \left|w_n - w_n'\right|\right) \leqslant 2\varepsilon r_n^2,$$

so that another elementary computation shows that $\sigma_n' := (v_n' \times w_n')/|v_n' \times w_n'|$ satisfies

$$\left|\sigma_n - \sigma_n'\right| \leqslant C_3 \varepsilon, \quad n = 1,2,3,\ldots$$

for ϵ sufficiently small. Moreover,

$$\begin{aligned}
\operatorname{dist}\left(\xi, \langle x,y,z\rangle\right) &= \left|(\xi - x)\cdot \sigma_n'\right| \\
&= \left|\left((\xi - a_n) + (a_n - x)\right)\cdot\left(\sigma_n + \left(\sigma_n' - \sigma_n\right)\right)\right| \\
&\geqslant r_n - C_4 \varepsilon r_n = \frac{r_n}{2}
\end{aligned} \tag{B.8}$$

if we choose $\varepsilon = 1/2C_4$. By (B.3), we also have

$$M\left(|\xi - x|, |\xi - y|, |\xi - z|\right) \approx r_n, \quad (x,y,z) \in \Delta_n, \quad n = 1,2,3,\ldots. \tag{B.9}$$

Combining (B.8) and (B.9), we estimate

$$\begin{aligned}
&\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathcal{F}(x,y,z,\xi)^p \, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z) \\
&\quad \gtrsim \sum_{n=1}^{\infty} \iint \int_{\{(x,y,z)\in\Delta_n\}} \mathcal{F}(x,y,z,\xi)^p \, d\mathscr{H}^2(x)\, d\mathscr{H}^2(y)\, d\mathscr{H}^2(z) \\
&\quad \gtrsim \sum_{n=1}^{\infty} \left(\pi \varepsilon r_n^2\right)^6 \frac{r_n^p}{r_n^{\alpha p}} \\
&\quad \approx \sum_{n=1}^{\infty} (r_n)^{12+(1-\alpha)p} \\
&\quad = +\infty \quad \text{for } (\alpha-1)p \geqslant 12.
\end{aligned}$$

This completes the proof. □

Remark. One can check that a similar argument shows that

$$\int_U \int_U \int_U \int_U \mathcal{F}^p = +\infty \quad \text{if } (\alpha-1)p \geqslant 12$$

whenever U is a patch of a C^2 surface $\Sigma \subset \mathbb{R}^3$ such that the Gaussian curvature of Σ is strictly positive on U.

The phenomenon described in Proposition B.1 does not appear for the integrand

$$\mathcal{K}_R(x,y,z,\xi) = 1/R(x,y,z,\xi),$$

where $R(x,y,z,\xi)$ denotes the radius of a circumsphere of four points of the surface—we simply have $1/R = \text{const}$ for all quadruples of pairwise distinct points of a round sphere. However, one can easily find examples of smooth surfaces for which $1/R \to \infty$ at some points: take e.g. the graph of $f(x,y) = xy$ near 0. It contains two straight lines and for every $\delta > 0$ there are lots of triangles with all vertices on these lines, all angles (say) $\geqslant \pi/6$ and diameter $\leqslant \delta$. For each such triangle Δ one can take a sphere S which has the circumcircle of Δ as the equatorial circle. The radius of S is $\lesssim \delta$ and S intersects the graph of f at infinitely many points that are not coplanar with vertices of Δ.

References

[1] W.K. Allard, On the first variation of a varifold, Ann. of Math. (2) 95 (1972) 417–491.
[2] J.R. Banavar, O. Gonzalez, J.H. Maddocks, A. Maritan, Self-interactions of strands and sheets, J. Stat. Phys. 110 (2003) 35–50.
[3] S. Blatt, Note on continuously differentiable isotopies, preprint Nr. 34, Institut f. Mathematik, RWTH Aachen, 2009.
[4] L.M. Blumenthal, K. Menger, Studies in Geometry, Freeman and Co., San Francisco, CA, 1970.

[5] G. David, C. Kenig, T. Toro, Asymptotically optimally doubling measures and Reifenberg flat sets with vanishing constant, Comm. Pure Appl. Math. 54 (4) (2001) 385–449.
[6] G. David, S. Semmes, Analysis of and on Uniformly Rectifiable Sets, Math. Surveys Monogr., vol. 38, Amer. Math. Soc., Providence, RI, 1993.
[7] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, Modern Geometry – Methods and Applications Part II. The Geometry and Topology of Manifolds, Grad. Texts in Math., vol. 104, Springer-Verlag, New York, 1985.
[8] W. Fenchel, On total curvatures of Riemannian manifolds: I, J. Lond. Math. Soc. (2) 15 (1940) 15–22.
[9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin–New York, 2001, reprint of the 1998-ed.
[10] O. Gonzalez, J.H. Maddocks, Global curvature thickness and the ideal shape of knots, Proc. Natl. Acad. Sci. USA 96 (1999) 4769–4773.
[11] O. Gonzalez, J.H. Maddocks, F. Schuricht, H. von der Mosel, Global curvature and self-contact of nonlinearly elastic curves and rods, Calc. Var. Partial Differential Equations 14 (2002) 29–68.
[12] J.C. Léger, Menger curvature and rectifiability, Ann. of Math. (2) 149 (1999) 831–869.
[13] G. Lerman, J.T. Whitehouse, High-dimensional menger-type curvatures – part I: Geometric multipoles and multiscale inequalities, arXiv:0805.1425v1, 2008.
[14] G. Lerman, J.T. Whitehouse, High-dimensional menger-type curvatures – part II: d-Separation and a menagerie of curvatures, J. Constr. Approx. 30 (2009) 325–360.
[15] E.L. Lima, Orientability of smooth hypersurfaces and the Jordan–Brouwer separation theorem, Expo. Math. 5 (1987) 283–286.
[16] J. Malý, Absolutely continuous functions of several variables, J. Math. Anal. Appl. 231 (2) (1999) 492–508.
[17] P. Mattila, Rectifiability analytic capacity and singular integrals, in: Proc. ICM, vol. II, Berlin, 1998, in: Doc. Math. Extra, vol. II, 1998, pp. 657–664 (electronic).
[18] P. Mattila, Search for geometric criteria for removable sets of bounded analytic functions, Cubo 6 (2004) 113–132.
[19] M. Melnikov, Analytic capacity: Discrete approach and curvature of measure, Sb. Math. 186 (1995) 827–846.
[20] M. Melnikov, J. Verdera, A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz curves, Int. Math. Res. Not. 7 (1995) 325–331.
[21] K. Menger, Untersuchungen über allgemeine Metrik. Vierte Untersuchung. Zur Metrik der Kurven, Math. Ann. 103 (1930) 466–501.
[22] K. Oleszkiewicz, private communication.
[23] H. Pajot, Analytic Capacity Rectifiability Menger Curvature and the Cauchy Integral, Springer Lecture Notes, vol. 1799, Springer, Berlin–Heidelberg–New York, 2002.
[24] D. Preiss, X. Tolsa, T. Toro, On the smoothness of Hölder doubling measures, Calc. Var. Partial Differential Equations 35 (3) (2009) 339–363.
[25] G.B. Price, On the completeness of a certain metric space with an application to Blaschke's selection theorem, Bull. Amer. Math. Soc. 46 (1940) 278–280.
[26] E.R. Reifenberg, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960) 1–92.
[27] F. Schuricht, H. von der Mosel, Global curvature for rectifiable loops, Math. Z. 243 (2003) 37–77.
[28] F. Schuricht, H. von der Mosel, Euler–Lagrange equations for nonlinearly elastic rods with self-contact, Arch. Ration. Mech. Anal. 168 (2003) 35–82.
[29] F. Schuricht, H. von der Mosel, Characterization of ideal knots, Calc. Var. Partial Differential Equations 19 (2004) 281–305.
[30] L. Simon, Existence of surfaces minimizing the Willmore functional, Comm. Anal. Geom. 1 (2) (1993) 281–326.
[31] P. Strzelecki, M. Szumańska, H. von der Mosel, Regularizing and self-avoidance effects of integral Menger curvature, Ann. Sc. Norm. Super. Pisa Cl. Sci. 9 (2010) 145–187.
[32] P. Strzelecki, H. von der Mosel, On a mathematical model for thick surfaces, in: Millett Calvo, Stasiak Rawdon (Eds.), Physical and Numerical Models in Knot Theory, in: Ser. Knots Everything, vol. 36, World Scientific, Singapore, 2005, pp. 547–564.
[33] P. Strzelecki, H. von der Mosel, Global curvature for surfaces and area minimization under a thickness constraint, Calc. Var. Partial Differential Equations 25 (2006) 431–467.
[34] M. Szumańska, Integral versions of Menger curvature: Smoothing potentials for rectifiable curves, PhD thesis, Technical University of Warsaw, 2009 (in Polish).
[35] T. Toro, Surfaces with generalized second fundamental form in L^2 are Lipschitz manifolds, J. Differential Geom. 39 (1) (1994) 65–101.
[36] J. Verdera, The L^2 boundedness of the Cauchy integral and Menger curvature, in: Harmonic Analysis and Boundary Value Problems, in: Contemp. Math., vol. 277, Amer. Math. Soc., Providence, RI, 2001, pp. 139–158.