Existence and Non-existence Results for a Quasilinear
Problem with Nonlinear Boundary Condition

Florica-Corina Şt. Cîrstea and Vicenţiu D. Rădulescu

Submitted by R. E. Showalter
Received December 23, 1998

We study the problem
\[-\text{div}(a(x)|u|^{p-2} \nabla u) - \lambda(1 + |x|)^{-\alpha_1} u - h(x)|u|^{r-2} u \quad \text{in} \quad \Omega \subset \mathbb{R}^N,
\]
\[a(x)|u|^{p-2} u \cdot n + b(x) |u|^{q-2} u = \theta g(x, u) \quad \text{on} \quad \Gamma,
\]
\[u \geq 0 \quad \text{in} \quad \Omega,
\]

where \(\Omega\) is an unbounded domain with smooth boundary \(\Gamma\), \(n\) denotes the unit outward normal vector on \(\Gamma\), and \(\lambda > 0\) are real parameters. We assume throughout that \(p < q < r < p^* = \frac{Np}{N-p}\), \(1 < p < N\), \(-N < \alpha_1 < q \cdot \frac{N-p}{p} - N\), while \(a, b,\) and \(h\) are positive functions. We show that there exist an open interval \(I\) and \(\lambda^* > 0\) such that the problem has no solution if \(\theta \in I\) and \(\lambda \in (0, \lambda^*)\). Furthermore, there exist an open interval \(J \subset I\) and \(\lambda_0 > 0\) such that, for any \(\theta \in J\), the above problem has at least a solution if \(\lambda \geq \lambda_0\), but it has no solution provided that \(\lambda \in (0, \lambda_0)\). Our paper extends previous results obtained by J. Chabrowski and K. Pflüger. © 2000 Academic Press

1. PRELIMINARIES

Let \(\Omega \subset \mathbb{R}^N\) be an unbounded domain with smooth boundary \(\Gamma\). We assume throughout this paper that \(p\), \(q\), \(r\), and \(\alpha_1\) are real numbers satisfying

\[1 < p < N, \quad p < q < r < p^* := \frac{pN}{N-p},
\]
\[-N < \alpha_1 < q \cdot \frac{N-p}{p} - N. \quad (1)
\]
Denote by $C^m_0(\Omega)$ the space of $C^m_0(\mathbb{R}^N)$-functions restricted to $\Omega$. We define the weighted Sobolev space $E$ as the completion of $C^m_0(\Omega)$ in the norm
\[
\|u\|_E = \left( \int_\Omega |\nabla u(x)|^p + \frac{1}{(1 + |x|)^{N-p}} |u(x)|^p \, dx \right)^{1/p}.
\]

Denote by $L^q(\Omega; w_1)$ and $L^m(\Gamma; w_2)$ the weighted Lebesgue spaces with weight functions
\[
w_i(x) = (1 + |x|)^{\alpha_i}, \quad i = 1, 2, \quad \alpha_i \in \mathbb{R}
\]
and norms defined by
\[
\|u\|_{q, w_1}^q = \int_\Omega w_1 |u(x)|^q \, dx \quad \text{and} \quad \|u\|_{m, w_2}^m = \int_\Gamma w_2 |u(x)|^m \, d\Gamma.
\]

The following embedding and trace result holds.

**Proposition 1.** Assume (1) holds. Then the embedding $E \subset L^q(\Omega; w_1)$ is compact. If
\[
p \leq m \leq p \cdot \frac{N-1}{N-p} \quad \text{and} \quad -N < \alpha_2 \leq m \cdot \frac{N-p}{p} - N + 1,
\]
then the trace operator $E \to L^m(\Gamma; w_2)$ is continuous. If the upper bounds for $m$ in (3) are strict, then the trace is compact.

This proposition is a consequence of Theorem 2 and Corollary 6 of [4].

We assume throughout that $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$ such that
\[
a(x) \geq a_0 > 0 \quad \text{for a.e.} \ x \in \Omega
\]
and
\[
\frac{c}{(1 + |x|)^{p-1}} \leq b(x) \leq \frac{C}{(1 + |x|)^{p-1}},
\]
for a.e. $x \in \Gamma$, where $c, C > 0$.

**Lemma 1.** The quantity
\[
\|u\|_F^p = \int_\Omega a(x)|\nabla u|^p \, dx + \int_\Gamma b(x)|u|^p \, d\Gamma
\]
defines an equivalent norm on $E$. 
For the proof of this result we refer to [3, Lemma 2].

Let $h : \Omega \to \mathbb{R}$ be a positive and continuous function satisfying
\[
\int_{\Omega} \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx < \infty.
\] (6)

We assume that $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following conditions:

\begin{itemize}
  \item[(g1)] $g(\cdot, 0) = 0$, $g(x, s) + g(x, -s) \geq 0$ for a.e. $x \in \Gamma$ and for any $s \in \mathbb{R}$;
  \item[(g2)] $|g(x, s)| \leq g_0(x) + g_1(x)|s|^{m-1}$, $p \leq m < p \cdot \frac{N-1}{N-p}$, where $g_i$ are nonnegative, measurable functions such that
\end{itemize}

\[ 0 \leq g_i(x) \leq C_i w_2 \quad \text{a.e.,} \quad g_0 \in L^{m/(m-1)}(\Gamma; w_2^{1/(1-m)}), \]

where $-N < \alpha < m \cdot \frac{N-1}{p} - N + 1$ and $w_2$ is defined as in (2).

Let $G$ be the primitive function of $g$ with respect to the second variable. We denote by $N_g$, $N_G$ the corresponding Nemyskii operators.

**Lemma 2.** The operators

\[
N_g : L^m(\Gamma; w_2) \to L^{m/(m-1)}(\Gamma; w_2^{1/(1-m)}), \quad N_G : L^m(\Gamma; w_2) \to L^1(\Gamma)
\]

are bounded and continuous.

**Proof.** Let $m' = m/(m-1)$ and $u \in L^m(\Gamma; w_2)$. Then, by (g2),

\[
\int_{\Gamma} |N_g(u)|^{m'} \cdot w_2^{1/(1-m)} \, d\Gamma \\
\leq 2^{m'-1} \left( \int_{\Gamma} g_0^{m'} \cdot w_2^{1/(1-m)} \, d\Gamma + \int_{\Gamma} g_1^{m'} |u|^{m} \cdot w_2^{1/(1-m)} \, d\Gamma \right) \\
\leq 2^{m'-1} \left( C + C_g \cdot \int_{\Gamma} |u|^{m} \cdot w_2 \, d\Gamma \right),
\]

which shows that $N_g$ is bounded. In a similar way we obtain

\[
\int_{\Gamma} |N_G(u)| \, d\Gamma \leq \int_{\Gamma} g_0 |u| \, d\Gamma + \int_{\Gamma} g_1 |u|^{m} \, d\Gamma \\
\leq \left( \int_{\Gamma} g_0^{m'} \cdot w_2^{1/(1-m)} \, d\Gamma \right)^{1/m'} \cdot \left( \int_{\Gamma} |u|^{m} \cdot w_2 \, d\Gamma \right)^{1/m} \\
+ C_g \cdot \int_{\Gamma} |u|^{m} \cdot w_2 \, d\Gamma
\]

and we claim that $N_G$ is bounded.
From the usual properties of Nemytskii operators we deduce the continuity of these operators.

Set

\[ X = \left\{ u \in E : \int_{\Omega} h(x)|u'| \, dx < \infty \right\} \]

endowed with the norm

\[ \|u\|_X^p = \|u\|_E^p + \left( \int_{\Omega} h(x)|u(x)|^{r} \, dx \right)^{p/r}. \]

We observe that \( X \) is a Banach space.

Consider the problem

\[
\begin{aligned}
- \text{div}(a(x)|\nabla u|^{p-2} \nabla u) &= \lambda(1 + |x|)\alpha_1|u|^{q-2}u - h(x)|u|^{r-2}u \\
&\quad\text{in } \Omega \subset \mathbb{R}^N, \\
a(x)|\nabla u|^{p-2} \nabla u \cdot n + b(x) \cdot |u|^{p-2}u &= \theta g(x, u) \\
&\quad\text{on } \Gamma, \\
u &\geq 0 \quad \text{in } \Omega.
\end{aligned}
\]

The energy functional corresponding to \((1, \theta)\) is given by \( \Phi: X \rightarrow \mathbb{R} \),

\[
\Phi(u) = \frac{1}{p} \int_{\Omega} a(x)|\nabla u|^p \, dx + \frac{1}{p} \int_{\Gamma} b(x)|u|^p \, d\Gamma - \frac{\lambda}{q} \int_{\Omega} w_1|u|^q \, dx \\
+ \frac{1}{r} \int_{\Omega} h(x)|u'| \, dx - \theta \int_{\Gamma} G(x, u) \, d\Gamma.
\]

Proposition 1 shows that the embedding \( E \subset L^q(\Omega; w_1) \) is continuous. This implies that the functional \( \Phi \) is well defined. Solutions to problem \((1, \theta)\) will be found as critical points of \( \Phi \). Therefore, a function \( u \in X \) is a solution of the problem \((1, \theta)\) provided that, for any \( v \in X \),

\[
\int_{\Omega} a|\nabla u|^{p-2} \nabla u \cdot \nabla v + \int_{\Gamma} b|u|^{p-2}uv \\
= \lambda \int_{\Omega} w_1|u|^{q-2}uv - \int_{\Omega} h|u|^{r-2}uv + \theta \int_{\Gamma} gv.
\]

2. MAIN RESULTS

**Theorem 1.** Assume hypotheses (1), (4), (5), (6), (g1), and (g2) hold. Then there exist real numbers \( \theta^*, \theta^* \), and \( \lambda^* > 0 \) such that the problem
Existence and Non-Existence Results

$U_1$ does not have a nontrivial solution, for any $\theta_* < \theta < \theta^*$ and $0 < \lambda < \lambda^*$.

**Proof.** Suppose that $u$ is a solution in $X$ of (1.1.8). Then $u$ satisfies

$$
\int_\Omega a(x)|\nabla u|^p \, dx + \int_\Gamma b(x)|u|^p \, d\Gamma - \theta \int_\Gamma g(x, u)u \, d\Gamma + \int_\Omega h(x)|u|^r \, dx = \lambda \int_\Omega w_1|u|^q \, dx.
$$

(7)

It follows from the Young inequality that

$$
\lambda \int_\Omega w_1|u|^q \, dx = \int_\Omega \frac{\lambda w_1}{h^{q/r}} \cdot h^{q/r}|u|^q \, dx
\leq \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx + \frac{q}{r} \int_\Omega h|u|^r \, dx.
$$

This combined with (7) gives

$$
\|u\|_b^p - \theta \int_\Gamma g(x, u)u \, d\Gamma \leq \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx + \frac{q}{r} \int_\Omega h|u|^r \, dx
\leq \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx.
$$

(8)

Set

$$
A = \left\{ u \in X : \int_\Gamma g(x, u)u \, d\Gamma < 0 \right\},
$$

$$
B = \left\{ u \in X : \int_\Gamma g(x, u)u \, d\Gamma > 0 \right\}
$$

(9)

$$
\theta_* = \sup_{u \in A} \frac{\|u\|_b^p}{\int_\Gamma g(x, u)u \, d\Gamma}, \quad \theta^* = \inf_{u \in B} \frac{\|u\|_b^p}{\int_\Gamma g(x, u)u \, d\Gamma}.
$$

We introduce the convention that if $A = \emptyset$ then $\theta_* = -\infty$ and if $B = \emptyset$ then $\theta^* = +\infty$.

We show that if we take $\theta_* < \theta < \theta^*$ then there exists $C_0 > 0$ such that

$$
C_0\|u\|_b^p \leq \|u\|_b^p - \theta \int_\Gamma g(x, u)u \, d\Gamma \quad \text{for all} \ u \in X.
$$

(10)
If $\theta < \theta^*$ then there exists a constant $C_1 \in (0, 1)$ such that

$$\theta \leq (1 - C_1) \theta^* \leq (1 - C_1) \frac{\|u\|_E}{\int_{\Gamma} g(x, u) u \, d\Gamma}$$

for all $u \in B$

which implies that

$$\|u\|_E - \theta \int_{\Gamma} g(x, u) u \, d\Gamma \geq C_1 \|u\|_E$$

for all $u \in B$. \quad (11)

If $\theta^* < \theta$ then there exists a constant $C_2 \in (0, 1)$ such that

$$\frac{\|u\|_E}{\int_{\Gamma} g(x, u) u \, d\Gamma} \leq (1 - C_2) \theta^* \leq \theta$$

for all $u \in A$

which yields

$$\|u\|_E - \theta \int_{\Gamma} g(x, u) u \, d\Gamma \geq C_2 \|u\|_E$$

for all $u \in A$. \quad (12)

From (11) and (12) we conclude that

$$\|u\|_E - \theta \int_{\Gamma} g(x, u) u \, d\Gamma \geq \min\{C_1, C_2\} \|u\|_E$$

for all $u \in X$

and taking $C_0 = \min\{C_1, C_2\}$ we obtain (10).

By (7), (10), and Proposition 1 we have

$$C_0 \bar{C} \left( \int_{\Omega} w_1 |u|^q \, dx \right)^{p/q} \leq C_0 \|u\|_E \leq \lambda \int_{\Omega} w_1 |u|^q \, dx, \quad (13)$$

for some constant $\bar{C} > 0$. This inequality implies

$$\left( \bar{C} \lambda^{-1} C_0 \right)^{q/(q-p)} \leq \int_{\Omega} w_1 |u|^q \, dx$$

which combined with (13) leads to the inequality

$$C_0 \bar{C} \left( \bar{C} \lambda^{-1} C_0 \right)^{p/(q-p)} \leq C_0 \|u\|_E.$$

Combining this with (8) and (10) we obtain that

$$C_0 \bar{C} \left( \bar{C} \lambda^{-1} C_0 \right)^{p/(q-p)} \leq \frac{r-q}{r} \lambda^{r/(r-q)} \int_{\Omega} w_1^{r/(r-q)} \, dx.$$
If we take

$$\lambda^* = \left( \frac{C_0C}{r} \right)^{q/(q-p)} \frac{r}{r-q} \left( \int_{\Omega} \frac{w_1^{p/(r-q)}}{h^{q/(r-q)}} \, dx \right)^{-1} \left( r-q\chi_{q-p}/(r-p) \right)$$

the result follows.

Set

$$U = \left\{ u \in X : \int_{\Gamma} G(x, u) \, d\Gamma < 0 \right\}, \quad V = \left\{ u \in X : \int_{\Gamma} G(x, u) \, d\Gamma > 0 \right\}$$

$$\theta_- = \sup_{u \in U} \frac{\|u\|^p}{p\int_{\Gamma} G(x, u) \, d\Gamma}, \quad \theta^+ = \inf_{u \in V} \frac{\|u\|^p}{p\int_{\Gamma} G(x, u) \, d\Gamma}. \quad (14)$$

If $U = \emptyset$ (resp. $V = \emptyset$) then we set $\theta_- = -\infty$ (resp. $\theta^+ = +\infty$). Proceeding in the same manner as we did for proving (10) we can show that if we take $\theta_- < \theta < \theta^+$ then there exists $c > 0$ such that

$$\frac{1}{p} \|u\|^p - \theta \int_{\Gamma} G(x, u) \, d\Gamma \geq c\|u\|^p \quad \text{for all } u \in X. \quad (15)$$

In what follows, we shall employ the following elementary inequality: for every $h > 0, k > 0, \text{ and } 0 < \beta < \gamma$ we have

$$k|u|^{\beta} - h|u|^{\gamma} \leq C_{\beta, \gamma, k} \left( \frac{k}{h} \right)^{\beta/(\gamma - \beta)} \quad (16)$$

for all $u \in \mathbb{R}$, where $C_{\beta, \gamma} > 0$ is a constant depending on $\beta$ and $\gamma$.

**Proposition 2.** If $\theta_- < \theta < \theta^+$ then the functional $\Phi$ is coercive.

**Proof.** By virtue of (16) we write the estimate

$$\int_{\Omega} \left( \frac{\lambda}{q} |u|^q w_1 - \frac{h}{2r} |u|^r \right) \, dx \leq C_{r, q} \left( \int_{\Omega} \lambda w_1 \left( \frac{\lambda w_1}{h} \right)^{q/(r-q)} \, dx \right)$$

$$= C_{r, q} \lambda^{r/(r-q)} \left( \int_{\Omega} \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx \right).$$
Using (15) it follows that
\[
\Phi(u) = \frac{1}{p} \|u\|_p^p - \theta \int_{\Gamma} G(x, u) \, d\Gamma - \int_{\Omega} \left( \frac{\lambda}{q} |u|^q w_1 - \frac{h}{2r} |u|^r \right) \, dx \\
+ \frac{1}{2r} \int_{\Omega} h |u|^r \, dx \\
\geq c \|u\|_p^p + \frac{1}{2r} \int_{\Omega} h |u|^r \, dx - C_1
\]
and the coercivity follows.

**Proposition 3.** Suppose \( \theta_- < \theta < \theta^+ \) and let \( \{u_n\} \) be a sequence in \( X \) such that \( \Phi(u_n) \) is bounded. Then there exists a subsequence of \( \{u_n\} \), relabelled again by \( \{u_n\} \), such that \( u_n \rightharpoonup u_0 \) in \( X \) and
\[
\Phi(u_0) \leq \liminf_{n \to \infty} \Phi(u_n).
\]

**Proof.** Since \( \Phi \) is coercive in \( X \) we see that the boundedness of \( \Phi(u_n) \) implies that \( \|u_n\|_0 \) and \( \int_{\Omega} h |u_n|^r \, dx \) are bounded. From Proposition 1 we have that the embedding \( E \subset L^q(\Omega; w_1) \) is compact and using the fact that \( \{u_n\} \) is bounded in \( E \) we may assume that \( u_n \rightharpoonup u_0 \) in \( E \) and \( u_n \to u_0 \) in \( L^q(\Omega; w_1) \).

Set \( F(x, u) = \frac{1}{q} |u|^q w_1 - \frac{1}{r} h |u|^r \) and \( f(x, u) = F_n(x, u) \).

A simple computation yields
\[
f_a(x, u) = (q - 1) \lambda |u|^{q-2} w_1 - (r - 1) h |u|^{r-2} \\
\leq C_{r, q} \lambda w_1 \left( \frac{\lambda w_1}{h} \right)^{(q-2)/(r-q)},
\]
where the last inequality follows from (16) and \( C_{r, q} > 0 \) is a constant depending only on \( r \) and \( q \). We now use (17) to derive the estimate for \( \Phi(u_0) - \Phi(u_n) \),
\[
\Phi(u_0) - \Phi(u_n) \\
= \frac{1}{p} \int_{\Omega} a(x) |\nabla u_0|^p \, dx + \frac{1}{p} \int_{\Gamma} b(x) |u_0|^p \, d\Gamma \\
- \frac{1}{p} \int_{\Omega} a(x) |\nabla u_n|^p \, dx - \frac{1}{p} \int_{\Gamma} b(x) |u_n|^p \, d\Gamma
\]
- \theta \int_{\Gamma} G(x, u_0) \, d\Gamma + \theta \int_{\Gamma} G(x, u_n) \, d\Gamma \\
+ \int_{\Omega} (F(x, u_n) - F(x, u_0)) \, dx \\
= \frac{1}{p} (\|u_0\|_p^p - \|u_n\|_p^p) \\
+ \theta \left( \int_{\Gamma} G(x, u_n) \, d\Gamma - \int_{\Gamma} G(x, u_0) \, d\Gamma \right) \\
+ \int_{\Omega} \left( \int_0^1 \int_0^s f_n(x, u_0 + t(u_n - u_0)) \, dt \, ds \right) \\
\times (u_n - u_0)^2 \, dx \\
\leq \frac{1}{p} (\|u_0\|_p^p - \|u_n\|_p^p) + \theta \left( \int_{\Gamma} G(x, u_n) \, d\Gamma - \int_{\Gamma} G(x, u_0) \, d\Gamma \right) \\
+ C_2 \int_{\Omega} (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{H(q-2)/(r-q)} \, dx,

where \( C_2 = \frac{1}{2} C_{r, q} \lambda^{(r-2)/(r-q)} \). We show that the last integral tends to 0 as \( n \to \infty \). Indeed, applying H"older's inequality we obtain

\[
\int_{\Omega} (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{H(q-2)/(r-q)} \, dx \leq \left( \int_{\Omega} \frac{w_1^{r/(r-q)}}{H^{(q-2)/(r-q)}} \, dx \right)^{(q-2)/q} \cdot \left( \int_{\Omega} |u_n - u_0|^q \, dx \right)^{2/q}.
\]

Since \( u_n \to u_0 \) in \( L^q(\Omega; w_1) \) we obtain

\[
\lim_{n \to \infty} \int_{\Omega} (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{H(q-2)/(r-q)} \, dx = 0. \quad (18)
\]

The compactness of the trace operator \( E \to L^m(\Gamma; w_2) \) and the continuity of the Nemytskii operator \( N_G : L^m(\Gamma; w_2) \to L^1(\Gamma) \) imply \( N_G(u_n) \to N_G(u_0) \) in \( L^1(\Gamma) \), i.e., \( \int_{\Gamma} |N_G(u_n) - N_G(u_0)| \, d\Gamma \to 0 \) as \( n \to \infty \). It follows that

\[
\lim_{n \to \infty} \int_{\Gamma} G(x, u_n) \, d\Gamma = \int_{\Gamma} G(x, u_0) \, d\Gamma. \quad (19)
\]
Since the norm in $E$ is lower semicontinuous with respect to the weak topology we deduce from (18) and (19) that

$$\Phi(u_0) \leq \liminf_{n \to \infty} \Phi(u_n).$$

**Proposition 4.** If $\theta_+ < \theta < 2^*$ and $u$ is a solution of problem $(1_{\lambda, \theta})$, then

$$C_0\|u\|^q + \frac{r - q}{r} \int_\Omega h|u|^r \, dx \leq \frac{r - q}{r} \int_\Omega \frac{w_1^r}{h^{r/(r-q)}} \, dx$$

and

$$\|u\|_b \geq K\lambda^{-1/(q-p)},$$

where $K > 0$ is a constant independent of $u$.

**Proof.** If $u$ is a solution of $(1_{\lambda, \theta})$ then

$$\|u\|^q + \theta \int_{\Gamma} g(x, u) u \, d\Gamma + \int_\Omega h|u|^r \, dx$$

$$= \lambda \int_\Omega w_1|u|^q \, dx$$

$$\leq \frac{r - q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^r}{h^{r/(r-q)}} \, dx + \frac{q}{r} \int_\Omega h|u|^r \, dx.$$  

Using (10) we obtain the first part of the assertion.

From Proposition 1 we have that there exists $C_q > 0$ such that

$$\|u\|_{L^q(\Omega; w)} \leq C_q \|u\|_q,$$

for all $u \in E$.

This inequality and (10) imply that

$$\|u\|_b \geq C_0^{-1/(q-p)} C_q^{-1/(q-p)}\lambda^{-1/(q-p)}$$

and taking $K = C_0^{-1/(q-p)} C_q^{-1/(q-p)}$ the second part follows.

**Theorem 2.** Assume hypotheses (1), (4), (5), (6), (g1), and (g2) hold. Set $\bar{\theta} = \max(\theta_+, \bar{\theta})$, $\bar{\theta} = \min(\theta^*, \bar{\theta})$, and $J = (\bar{\theta}, \bar{\theta})$. There exists $\lambda_0 > 0$ such that the following hold:

(i) the problem $(1_{\lambda, \theta})$ admits a nontrivial solution, for any $\lambda \geq \lambda_0$ and every $\theta \in J$;

(ii) the problem $(1_{\lambda, \theta})$ does not have any nontrivial solution, provided that $0 < \lambda < \lambda_0$ and $\theta \in J$. 

Proof. According to Propositions 2 and 3, $\Phi$ is coercive and lower semicontinuous. Therefore there exists $\tilde{u} \in X$ such that $\Phi(\tilde{u}) = \inf_X \Phi(u)$. To ensure that $\tilde{u} \neq 0$ we shall prove that $\inf_X \Phi < 0$. We set

$$\tilde{\lambda} := \inf \left\{ \frac{q}{p} \| u \|_0^p - q \theta \int_{\Gamma} G(x, u) \, d\Gamma + \frac{q}{r} \int_{\Omega} h|u|^r \, dx : u \in X, \int_{\Omega} w_1|u|^q \, dx = 1 \right\}.$$ 

First we check that $\tilde{\lambda} > 0$. In order to prove that we consider the constrained minimization problem

$$M := \inf \left\{ \int_{\Omega} a(x)|\nabla u|^p \, dx + \int_{\Gamma} b(x)|u|^p \, d\Gamma : u \in E, \int_{\Omega} w_1|u|^q \, dx = 1 \right\}.$$ 

Clearly, $M > 0$. Since $X$ is embedded in $E$, we have

$$\int_{\Omega} a(x)|\nabla u|^p \, dx + \int_{\Gamma} b(x)|u|^p \, d\Gamma \geq M$$

for all $u \in X$ with $\int_{\Omega} w_1|u|^q \, dx = 1$. Now, applying the Hölder inequality we find

$$1 = \int_{\Omega} w_1|u|^q \, dx = \int_{\Omega} \frac{w_1}{h^{q/r}} h^{q/r}|u|^q \, dx$$

$$\leq \left( \int_{\Omega} \frac{w_1^{1/(r-q)}}{h^{q/(r-q)}} \, dx \right)^{(r-q)/r} \cdot \left( \int_{\Omega} h|u|^r \, dx \right)^{q/r}.$$ 

Relation (15) implies that

$$\frac{q}{p} \| u \|_0^p - q \theta \int_{\Gamma} G(x, u) \, d\Gamma \geq q \epsilon \| u \|_0^p.$$ 

By virtue of (20) we have

$$\frac{q}{p} \| u \|_0^p - q \theta \int_{\Gamma} G(x, u) \, d\Gamma + \frac{q}{r} \int_{\Omega} h|u|^r \, dx$$

$$\geq q \epsilon \| u \|_0^p + \frac{q}{r} \int_{\Omega} h|u|^r \, dx$$

$$\geq q \epsilon M + \frac{q}{r} \left( \int_{\Omega} \frac{w_1^{1/(r-q)}}{h^{q/(r-q)}} \, dx \right)^{-r-q/q}.$$
for all \( u \in X \) with \( \int_\Omega w_1 |u|^q \, dx = 1 \). It follows that

\[
\tilde{\lambda} \geq qcM + \frac{q}{r} \left( \int_\Omega \frac{w_1^{(r-q)}}{h^{q/(r-q)}} \, dx \right)^{-(r-q)/q}
\]

and our claim follows.

Let \( \lambda > \tilde{\lambda} \). Then there exists a function \( u \in X \) with \( \int_\Omega w_1 |u|^q \, dx = 1 \) such that

\[
\lambda > \frac{q}{p} ||u||_p^p - q \theta \int_\Gamma G(x, u) \, d\Gamma + \frac{q}{r} \int_\Omega h|u|^r \, dx.
\]

This can be rewritten as

\[
\Phi(u) = \frac{1}{p} ||u||_p^p - \theta \int_\Gamma G(x, u) \, d\Gamma + \frac{1}{r} \int_\Omega h|u|^r \, dx - \frac{\lambda}{q} \int_\Omega w_1 |u|^q \, dx < 0
\]

and consequently \( \inf_{u \in X} \Phi(u) < 0 \). By Propositions 2 and 3 it follows that the problem \((1_{\lambda, \rho})\) has a solution.

We set

\[
\lambda_0 = \inf\{ \lambda > 0 : (1_{\lambda, \rho}) \text{ admits a solution} \}.
\]

Suppose \( \lambda_0 = 0 \). Then taking \( \lambda_1 \in (0, \lambda^*) \) (where \( \lambda^* \) is given by Theorem 1) we have that there is \( \tilde{\lambda} \) such that the problem \((1_{\tilde{\lambda}, \rho})\) admits a solution. But this is a contradiction, according to Theorem 1. Consequently, \( \lambda_0 > 0 \).

We now show that for each \( \lambda > \lambda_0 \) problem \((1_{\lambda, \rho})\) admits a solution. Indeed, for every \( \lambda > \lambda_0 \) there exists \( \rho \in (\lambda_0, \lambda) \) such that the problem \((1_{\rho, \rho})\) has a solution \( u_\rho \) which is a subsolution of problem \((1_{\lambda, \rho})\). We consider the variational problem

\[
\inf\{ \Phi(u) : u \in X \text{ and } u \geq u_\rho \}.
\]

By Propositions 2 and 3 this problem admits a solution \( \bar{u} \). This minimizer \( \bar{u} \) is a solution of problem \((1_{\lambda, \rho})\). Since the hypothesis \( g(x, s) + g(x, -s) \geq 0 \) for a.e. \( x \in \Gamma \) and for all \( s \in \mathbb{R} \) implies that \( G(x, |\bar{u}|) \geq G(x, \bar{u}) \) (that is, \( \Phi(|\bar{u}|) \leq \Phi(\bar{u}) \)) we may assume that \( \bar{u} \geq 0 \) on \( \Omega \). It remains to show that problem \((1_{\lambda, \rho})\) also has a solution. Let \( \lambda_n \to \lambda_0 \) and \( \lambda_n > \lambda_0 \) for each \( n \). Problem \((1_{\lambda_n, \rho})\) has a solution \( u_n \) for each \( n \). By Proposition 4 the sequence \( \{u_n\} \) is bounded in \( X \). Therefore we may assume that \( u_n \to u_0 \) in \( X \) and \( u_n \to u_0 \) in \( L^q(\Omega; w_1) \). We have that \( u_0 \) is a solution of \((1_{\lambda_0, \rho})\).
Since $u_n$ and $u_0$ are solutions of (1$_{\lambda_n, \theta}$) and (1$_{\lambda_0, \theta}$), respectively, we have

$$
\int_{\Omega} a(x)\left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0\right)\left(\nabla u_n - \nabla u_0\right) \, dx
$$

$$
+ \int_{\Gamma} b(x)\left(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0\right)(u_n - u_0) \, d\Gamma
$$

$$
+ \int_{\Omega} h\left(|u_n|^{r-2} u_n - |u_0|^{r-2} u_0\right)(u_n - u_0) \, dx
$$

$$
= \lambda_n \int_{\Omega} w_1(|u_n|^{q-2} u_n - |u_0|^{q-2} u_0)(u_n - u_0) \, dx
$$

$$
+ (\lambda_n - \lambda_0) \int_{\Omega} w_1|u_0|^{q-2} u_0(u_n - u_0) \, dx
$$

$$
+ \theta \int_{\Gamma} (g(x, u_n) - g(x, u_0))(u_n - u_0) \, d\Gamma
$$

$$
= J_{1,n} + J_{2,n} + J_{3,n},
$$

where

$$
J_{1,n} = \lambda_n \int_{\Omega} w_1(|u_n|^{q-2} u_n - |u_0|^{q-2} u_0)(u_n - u_0) \, dx,
$$

$$
J_{2,n} = (\lambda_n - \lambda_0) \int_{\Omega} w_1|u_0|^{q-2} u_0(u_n - u_0) \, dx,
$$

$$
J_{3,n} = \theta \int_{\Gamma} (g(x, u_n) - g(x, u_0))(u_n - u_0) \, d\Gamma.
$$

We have

$$
|J_{1,n}| \leq \sup_{n \geq 1} \lambda_n \left( \int_{\Omega} w_1 |u_n|^{q-1} |u_n - u_0| \, dx + \int_{\Omega} w_1 |u_0|^{q-1} |u_n - u_0| \, dx \right)
$$

and it follows from the Hölder inequality that

$$
|J_{1,n}| \leq \sup_{n \geq 1} \left[ \left( \int_{\Omega} w_1 |u_n|^q \, dx \right)^{(q-1)/q} \cdot \left( \int_{\Omega} w_1 |u_n - u_0|^q \, dx \right)^{1/q} 
+ \left( \int_{\Omega} w_1 |u_0|^q \, dx \right)^{(q-1)/q} \cdot \left( \int_{\Omega} w_1 |u_n - u_0|^q \, dx \right)^{1/q} \right].
$$

We easily observe that $J_{1,n} \to 0$ as $n \to \infty$. 
From the estimate
\[ |J_{2,n}| \leq |\lambda_n - \lambda_0| \left( \int_{\Omega} w_1 |u_0|^q \, dx \right)^{(q-1)/q} \cdot \left( \int_{\Omega} w_1 |u_n - u_0|^q \, dx \right)^{1/q} \]
we obtain that \( J_{2,n} \to 0 \) as \( n \to \infty \).

Using the compactness of the trace operator \( E \to L^m(\Gamma; w_2) \), the continuity of Nemytskii operator \( N_g : L^m(\Gamma; w_2) \to L^{m/(m-1)}(\Gamma; w_2^{1/(m-1)}) \), and the estimate
\[ \int_{\Gamma} \left| g(x, u_n) - g(x, u_0) \right| \cdot |u_n - u_0| \, d\Gamma \leq \left( \int_{\Gamma} \left| g(x, u_n) - g(x, u_0) \right|^{m/(m-1)} w_2^{1/(1-m)} \, d\Gamma \right)^{(m-1)/m} \cdot \left( \int_{\Gamma} w_2 |u_n - u_0|^m \, d\Gamma \right)^{1/m} \]
we see that \( J_{3,n} \to 0 \) as \( n \to \infty \).

We have so proved that
\[ \lim_{n \to \infty} \left( \int_{\Omega} a(x)(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0)(\nabla u_n - \nabla u_0) \, dx + \int_{\Gamma} b(x)(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0)(u_n - u_0) \, d\Gamma \right) = 0. \]

Now we apply the following inequality for \( \xi, \zeta \in \mathbb{R}^N \) (see [2, Lemma 4.10])
\[ |\xi - \zeta|^p \leq C(|\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta)(\xi - \zeta), \quad \text{for } p \geq 2. \]

Then we obtain
\[ \|u_n - u_0\|_b^p \geq \int_{\Omega} a(x)|\nabla u_n - \nabla u_0|^p \, dx + \int_{\Gamma} b(x)|u_n - u_0|^p \, dx \]
\[ \leq C \left( \int_{\Omega} a(x)(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0)(\nabla u_n - \nabla u_0) \, dx + \int_{\Gamma} b(x)(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0)(u_n - u_0) \, d\Gamma \right) \to 0 \]
as \( n \to \infty \)
which shows that \( \|u_n\|_b \to \|u_0\|_b \) and, by Proposition 4, \( u_0 \neq 0 \). In the case \( 1 < p < 2 \) we obtain the same conclusion, by using the corresponding
inequality (see [2, Lemma 4.10])
\[ |\xi - \zeta|^2 \leq C(|\xi|^{p-2} \xi - |\xi|^{p-2} \zeta)(|\xi|(|\xi| + |\zeta|)^{2-p}, \]
for any \( \xi, \zeta \in \mathbb{R}^N \). This concludes our proof.

REFERENCES