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Keller–Osserman conditions for diffusion-type operators on Riemannian manifolds

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Abstract

In this paper we obtain essentially sharp generalized Keller–Osserman conditions for wide classes of differential inequalities of the form $Lu \geqslant b(x)f(u)\ell(|\nabla u|)$ and $Lu \geqslant b(x)f(u)\ell(|\nabla u|) - g(u)h(|\nabla u|)$ on weighted Riemannian manifolds, where L is a non-linear diffusion-type operator. Prototypical examples of these operators are the p-Laplacian and the mean curvature operator. The geometry of the underlying manifold is reflected, via bounds for the modified Bakry–Emery Ricci curvature, by growth conditions for the functions b and ℓ. A weak maximum principle which extends and improves previous results valid for the φ-Laplacian is also obtained. Geometric comparison results, valid even in the case of integral bounds for the modified Bakry–Emery Ricci tensor, are presented.

Keywords: Keller–Osserman condition; Diffusion-type operators; Weak maximum principles; Weighted Riemannian manifolds; Quasi-linear elliptic inequalities

1. Introduction

Consider the Poisson-type inequality on Euclidean space $\mathbb{R}^m$

$$\Delta u \geqslant f(u), \tag{1.1}$$

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where $f \in C^0([0,+\infty))$, $f(0)=0$ and $f(t)>0$ if $t>0$. By an entire solution of (1.1) we mean a C^1 function u satisfying (1.1) on $\mathbb{R}^m$ in the sense of distributions. Let

$$F(t)=\int_0^t f(s)\,ds. \tag{1.2}$$

It is well know that if f satisfies the Keller–Osserman condition

$$\frac{1}{\sqrt{F(t)}} \in L^1(+\infty), \tag{1.3}$$

then (1.1) has no non-negative entire solutions except $u \equiv 0$. Note that in the case where $f(t)=t^q$ the integrability condition expressed by (1.3) is equivalent to $q>1$. But (1.3) is sharper than the condition on powers it is implied by. For instance (1.3) holds if $f(t)=t\log^{\beta}(1+t)$ with $\beta>2$.

As a matter of fact, if the Keller–Osserman condition fails, that is, if

$$\frac{1}{\sqrt{F(t)}} \notin L^1(+\infty), \tag{1.4}$$

then inequality (1.1) admits positive solutions. Indeed, consider the ODE problem

$$\begin{cases} \alpha''+\dfrac{m-1}{r}\alpha'=f(\alpha), \\ \alpha(0)=\alpha_o>0, \quad \alpha'(0)=0. \end{cases} \tag{1.5}$$

General theory yields the existence of a solution in a maximal interval $[0,R)$ and a first integration of (1.5) gives $\alpha'>0$ on $(0,R)$. Suppose by contradiction that $R<+\infty$. Using the maximality condition and the monotonicity of α we obtain

$$\lim_{r\to R^-}\alpha(r)=+\infty. \tag{1.6}$$

On the other hand, it follows from (1.5) that

$$\alpha'\alpha'' \leqslant f(\alpha)\alpha',$$

whence integrating over $[0,r]$, $0<r\leqslant R$, changing variables in the resulting integral, and taking square roots we obtain

$$\frac{\alpha'}{\sqrt{F(\alpha)}} \leqslant \sqrt{2}.$$

A further integration over $[0,r]$ with $0<a<r<R$ yields

$$\int_{\alpha(a)}^{\alpha(r)} \frac{dt}{\sqrt{F(t)}} \leqslant \sqrt{2}(r-a)$$

and letting $r \to R^-$ and using (1.6) we contradict (1.4). This shows that the function α is defined on $[0, +\infty)$. Setting $u(x) = \alpha(r(x))$ $(r(x) = |x|)$ gives rise to a radial positive entire solution of (1.1). Note however that any non-negative solution of (1.1) must diverge at infinity sufficiently fast. Indeed, it follows from [17], Corollary 16, that if $u \geqslant 0$ is an entire solution of (1.1) satisfying

$$u(x) = o\big(r(x)^\sigma\big) \quad \text{as } r(x) \to +\infty,$$

with $0 \leqslant \sigma < 2$, and f is non-decreasing, then $u \equiv 0$. Note that this latter conclusion can be hardly deduced from (1.4).

We also observe that differential inequalities of the type (1.1) often appear in connection with geometrical problems on complete manifolds and, in fact, R. Osserman introduced condition (1.3) in [13] in his investigation on the type of a Riemann surface. For a number of further examples we refer, for instance, to [16].

Motivated by the above considerations, from now on we will denote with $(M, \langle\, , \rangle)$ a complete, non-compact, connected Riemannian manifold of dimension $m \geqslant 2$. We fix an origin o in M and we let $r(x) = \operatorname{dist}(x, o)$ be the Riemannian distance from the chosen reference point, and we denote by B_r the geodesic ball of radius r centered at o and with ∂B_r its boundary.

Given a positive function $D(x) \in C^2(M)$ and a non-negative function $\varphi \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+)$, where, as usual $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, +\infty)$, we consider the diffusion-type operator defined on M by the formula

$$L_{D,\varphi} u = \frac{1}{D} \operatorname{div}\big(D|\nabla u|^{-1}\varphi\big(|\nabla u|\big)\nabla u\big).$$

For instance, if $D \equiv 1$ and $\varphi(t) = t^{p-1}$, $p > 1$, or $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$ we recover the usual p-Laplacian and the mean curvature operator, respectively.

If $b(x) \in C^0(M)$ and $\ell \in C^0(\mathbb{R}_0^+)$, we will be interested in solutions of the differential inequality

$$L_{D,\varphi} u \geqslant b(x) f(u) \ell\big(|\nabla u|\big). \tag{1.7}$$

By an entire classical weak solution of (1.7) we mean a C^1 function u on M which satisfies the inequality in the sense of distributions, namely,

$$-\int |\nabla u|^{-1}\varphi\big(|\nabla u|\big)\langle \nabla u, \nabla \psi\rangle D\, dV \geqslant \int b(x) f(u) \ell\big(|\nabla u|\big)\psi D\, dV \tag{1.8}$$

for every non-negative function $\psi \in C_c^\infty(M)$, where we have denoted with dV the Riemannian volume element.

Since we are dealing with a diffusion-type operator, the interplay between analysis and geometry will be taken into account by means of the modified Bakry–Emery Ricci tensor that we now introduce. Following Z. Qian [20], for $n > m$ let

$$\begin{aligned} \operatorname{Ricc}_{m,n}(L_D) &= \operatorname{Ricc}_M - \frac{1}{D}\operatorname{Hess} D + \frac{n-m-1}{n-m}\frac{1}{D^2} dD \otimes dD \\ &= \operatorname{Ricc}(L_D) - \frac{1}{n-m}\frac{1}{D^2} dD \otimes dD \end{aligned} \tag{1.9}$$

be the modified Bakry–Emery Ricci tensor, where $\mathrm{Ricc}(L_D)$ is the usual Bakry–Emery Ricci tensor, Ricc_M is the Ricci tensor of $(M, \langle\,,\rangle)$ (see D. Bakry and P. Emery [2]), and where, to simplify notation, we have denoted with L_D the operator $L_{D,\varphi}$ for $\varphi(t) = t$.

We introduce some more terminology.

Definition 1.1. Let g be a real valued function defined on $\mathbb{R}^+$. We say that g is C-increasing on $\mathbb{R}^+$ if there exists a constant $C \geqslant 1$ such that

$$\sup_{s\in(0,t]} g(s) \leqslant Cg(t) \quad \forall t \in \mathbb{R}^+. \tag{1.10}$$

It is easily verified that the above condition is equivalent to

$$\inf_{s\in[t,+\infty)} g(s) \geqslant \frac{1}{C} g(t) \quad \forall t \in \mathbb{R}^+,$$

and both formulations will be used in the sequel. Clearly, (1.10) is satisfied with $C = 1$ if g is non-decreasing on $\mathbb{R}^+$. In general, the validity of (1.10) allows a controlled oscillatory behavior such as, for instance, that of $g(t) = t^2(2 + \sin t)$.

In order to state our next result, we introduce the following set of assumptions.

- (Φ_0) $\varphi' > 0$ on $\mathbb{R}^+$.
- (F_1) $f \in C(\mathbb{R})$, $f(0) = 0$, $f(t) > 0$ if $t > 0$ and f is C-increasing on $\mathbb{R}^+$.
- (L_1) $\ell \in C^0(\mathbb{R}_0^+)$, $\ell(t) > 0$ on $\mathbb{R}^+$.
- (L_2) ℓ is C-increasing on $\mathbb{R}^+$.
- $(\varphi\ell)$ $\liminf_{t\to 0^+} \frac{\varphi(t)}{\ell(t)} = 0$, $\frac{t\varphi'(t)}{\ell(t)} \in L^1(0^+) \setminus L^1(+\infty)$.
- (θ) There exists $\theta \in \mathbb{R}$ such that the functions

$$t \to \frac{\varphi'(t)}{\ell(t)} t^\theta \quad \text{and} \quad t \to \frac{\varphi(t)}{\ell(t)} t^{\theta-1}$$

  are C-increasing on $\mathbb{R}^+$.

Clearly the last two conditions relate the operator $L_{D,\varphi}$ to the gradient term ℓ, and, in general, they are not independent. As we shall see below, in favorable circumstances (θ) implies $(\varphi\ell)$. This is the case, for instance, in the next Theorem A when $\theta < 1$. For a better understanding of these two assumptions, we examine the special but important case where $\ell(t) = t^q$, $q \geqslant 0$. First we consider the case of the p-Laplacian, so that $\varphi(t) = t^{p-1}$, $p > 1$. Then, given $\theta \in \mathbb{R}$, $(\varphi\ell)$ and (θ) are simultaneously satisfied provided

$$p > q + 1 \quad \text{and} \quad \theta \geqslant q - p + 2.$$

If we consider $\varphi(t) = te^{t^2}$ (which, when $D \equiv 1$, gives rise to the operator associated to the exponentially harmonic functions, see [5] and [6]), then $(\varphi\ell)$ and (θ) are both satisfied provided

$$q < 1 \quad \text{and} \quad q \leqslant \theta.$$

If $\varphi = \frac{t}{\sqrt{1+t^2}}$, which, for $D \equiv 1$, corresponds to the “mean curvature operator”, then $(\varphi\ell)$ does not hold for any $q \geqslant 0$. However, a variant of our arguments will allow us to analyze this situation, see Section 4 below.

Because of (L_1) and $(\varphi\ell)$ we may define a C^1-diffeomorphism $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by the formula

$$K(t) = \int_0^t \frac{s\varphi'(s)}{\ell(s)}\,ds. \tag{1.11}$$

Since K is increasing on $\mathbb{R}_0^+$ so is its inverse K^{-1}. Moreover, when $\ell \equiv 1$ then

$$K'(t) = \widetilde{H}'(t),$$

where

$$\widetilde{H}(t) = t\varphi(t) - \int_0^t \varphi(s)\,ds$$

is the pre-Legendre transform of $t \to \int_0^t \varphi(s)\,ds$.

Having defined F as in (1.2) we are ready to introduce our first generalized Keller–Osserman condition:

$$\frac{1}{K^{-1}(F(t))} \in L^1(+\infty). \tag{KO}$$

It is clear that, in the case of the Laplace–Beltrami operator (or more generally, of the p-Laplacian) and for $\ell \equiv 1$, (KO) is equivalent to the classical Keller–Osserman condition (1.3). After this preparation we are ready to state

Theorem A. *Let $(M, \langle\,,\rangle)$ be a complete manifold satisfying*

$$\mathrm{Ricc}_{n,m}(L_D) \geqslant H^2\left(1 + r^2\right)^{\beta/2}, \tag{1.12}$$

for some $n > m$, $H > 0$ and $\beta \geqslant -2$. Let also $b(x) \in C^0(M)$ be a non-negative function such that

$$b(x) \geqslant \frac{C}{r(x)^\mu} \quad \text{if } r(x) \gg 1, \tag{1.13}$$

for some $C > 0$ and $\mu \geqslant 0$. Assume that (Φ_0), (F_1), (L_1), (L_2), $(\varphi\ell)$, (θ) and (KO) *hold, and suppose that*

$$\begin{cases} \theta < 1 - \beta/2 - \mu \quad \text{or} \quad \theta = 1 - \beta/2 - \mu < 1 & \text{if } \mu > 0, \\ \theta < 1 - \beta/2 & \text{if } \mu = 0. \end{cases} \tag{$\theta\beta\mu$}$$

Then any entire classical weak solution u of the differential inequality (1.7) *is either non-positive or constant. Furthermore, if $u \geqslant 0$ and $\ell(0) > 0$, then $u \equiv 0$.*

We remark that letting $\beta < -2$ in (1.12) yields the same estimates valid for $\beta = -2$, which roughly correspond to the Euclidean behavior. Correspondingly, the conclusion of Theorem A is not improved by such a strengthening of the assumption on the modified Bakry–Emery Ricci curvature.

To better appreciate the result and the role played by geometry, we state the following consequence for the p-Laplace operator Δ_p.

Corollary A1. *Let* $(M, \langle\,,\rangle)$ *and* $b(x)$ *be as in the statement of Theorem* A *and satisfying* (1.12) *with* $D \equiv 1$ (*so that* $\mathrm{Ricc}_{n,m} = \mathrm{Ricc}$) *and* (1.13). *Let* f *satisfy* (F_1) *and let* $\ell(t) = t^q$, *for some* $q \geqslant 0$. *Assume that* p *and* μ *satisfy*

$$p > q + 1, \qquad 0 \leqslant \mu \leqslant p - q, \qquad \beta \leqslant 2(p - q - \mu - 1).$$

If

$$\frac{1}{F(t)^{1/(p-q)}} \in L^1(+\infty), \tag{KO}$$

then any entire classical weak solution u *of the differential inequality*

$$\Delta_p u \geqslant b(x) f(u) |\nabla u|^q$$

is either non-positive or constant.

Note that if $p = 2$ and $q = \mu = 0$, then the maximum amount of negative curvature allowed is obtained by choosing $\beta = 2$. In particular, the result covers the cases of Euclidean and hyperbolic space. We observe in passing that the choice $\beta = 2$ is borderline for the stochastic completeness of the underlying manifold.

To include in our analysis the case of the mean curvature operator we state the following consequence of Theorem 4.4.

Corollary A2. *Let* $(M, \langle\,,\rangle)$ *and* $b(x)$ *be as in the statement of Theorem* A *and satisfying* (1.12) *with* $D \equiv 1$ *and* (1.13). *Let* f *satisfy* (F_1) *and let* $\ell(t) = t^q$, *for some* $q \geqslant 0$. *Assume* $\mu \geqslant 0$ *and that*

$$0 \leqslant q < -\frac{\beta}{2} - \mu.$$

If

$$\frac{1}{F(t)^{1/(1-q)}} \in L^1(+\infty), \tag{$\widehat{\mathrm{KO}}$}$$

then any non-negative, entire classical weak solution u *of the differential inequality*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \geqslant b(x) f(u) |\nabla u|^q$$

is constant.

Note that, contrary to Corollary A1, the case of hyperbolic space, which corresponds to $\beta = 0$, is not covered by Corollary A2. On the other hand, if $\beta = -2$, which, as already mentioned, roughly corresponds to a Euclidean behavior, the conditions on the parameters become

$$\mu \geqslant 0, \qquad 0 \leqslant q < 1 - \mu,$$

and they are clearly compatible. This is one of the instances where the interaction between geometry and differential operators comes into play.

As briefly remarked at the beginning of this introduction, the failure of the Keller–Osserman condition may yield existence of non-constant non-negative entire solutions. The next result shows that such solutions, if they exist, have to go to infinity sufficiently fast depending on the geometry of M and, of course, of the relevant parameters in the differential inequality satisfied. To state our result we introduce the following set of assumptions.

(Φ_1) (i) $\varphi(0) = 0$; (ii) $\varphi(t) \leqslant At^\delta$ on $\mathbb{R}^+$, for some $A, \delta > 0$.
(F_0) $f \in C^0(\mathbb{R}_0^+)$.
(L_3) $\ell \in C^0(\mathbb{R}_0^+)$, $\ell(t) \geqslant Ct^\chi$ on $\mathbb{R}^+$, for some $C > 0$, $\chi \geqslant 0$.
(b_1) $b \in C^0(M)$, $b(x) > 0$ on M, $b(x) \geqslant \frac{C}{r(x)^\mu}$ if $r(x) \gg 1$, for some $C > 0$, $\mu \in \mathbb{R}$.

Theorem B. *Let $(M, \langle\,,\rangle)$ be a complete Riemannian manifold, and assume that conditions (Φ_1), (F_0), (L_3) and (b_1) hold. Given $\sigma \geqslant 0$, let $\eta = \mu - (1 + \delta - \chi)(1 - \sigma)$ and suppose that*

$$\sigma \geqslant \eta, \qquad 0 \leqslant \chi < \delta.$$

Let u be a non-constant entire classical weak solution of

$$L_{D,\varphi} u \geqslant b(x) f(u) \ell\big(|\nabla u|\big), \tag{1.7}$$

and suppose that either

$$\sigma > 0, \quad \liminf_{t \to +\infty} f(t) > 0 \quad \text{and}$$
$$u_+(x) = \max\big\{u(x), 0\big\} = o\big(r(x)^\sigma\big) \quad \text{as } r(x) \to +\infty, \tag{1.14}$$

or

$$\sigma = 0 \quad \text{and} \quad u^* = \sup_M u < +\infty. \tag{1.15}$$

Assume further that either

$$\liminf_{r \to +\infty} \frac{\log \int_{B_r} D(x)\, dV(x)}{r^{\sigma - \eta}} < +\infty \quad \text{if } \sigma - \eta > 0 \tag{1.16}$$

or

$$\liminf_{r \to +\infty} \frac{\log \int_{B_r} D(x)\, dV(x)}{\log r} < +\infty \quad \text{if } \sigma - \eta = 0. \tag{1.17}$$

Then $u^* < +\infty$ and $f(u^*) \leqslant 0$. In particular, if we also assume that $f(t) > 0$ for $t > 0$, and that $u(x_o) > 0$ for some $x \in M$, then u is constant on M, and if in addition $f(0) = 0$ and $\ell(0) > 0$, then $u \equiv 0$ on M.

Observe that the growth condition (1.14) is sharp. Indeed, we consider the case of the p-Laplace operator on Euclidean space, for which $D \equiv 1$ and $\delta = p - 1$, and suppose that $\chi = \mu = 0$ and $\sigma = \eta$. Since $\eta = p(\sigma - 1)$, the latter condition amounts to $\sigma = p'$, the Hölder conjugate exponent of p. Since condition (1.17), which now reads

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{\log r} < +\infty,$$

is clearly satisfied, all assumptions of Theorem B hold. On the other hand, a simple computation shows that the function $u(x) = \frac{1}{p'} r(x)^{p'}$ is a classical entire weak solution of $\Delta_p u = m$, for which (1.14) barely fails to be met.

We also stress that while in Theorem A the main geometric assumption is the radial lower bound on the modified Bakry–Emery Ricci curvature expressed by (1.12), in Theorem B we consider either (1.16) or (1.17), which we interpret as follows. Let $dV_D = D\,dV$ be the measure with density $D(x)$, so that, for every measurable set Ω,

$$\operatorname{vol}_D(\Omega) = \int_\Omega D(x)\,dV,$$

and consider the weighted Riemannian manifold $(M, \langle\,,\rangle, dV_D)$. With this notation, we may rewrite, for instance (1.16), in the form

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol}_D B_r}{r^{\sigma - \eta}} < \infty \quad \text{if } \sigma > \eta, \tag{1.18}$$

and interpret it as a control from above on the growth of the weighted volume of geodesic balls with respect to Riemannian distance function. This is a mild requirement, which is implied, via a version of the Bishop–Gromov volume comparison theorem for weighted manifolds, by a lower bound on the modified Bakry–Emery Ricci curvature in the radial direction. Indeed, as we shall see in Section 2 below, the latter yields an upper estimate on $L_D r$ which in turn gives the volume comparison estimate. In fact, we shall prove there that an L^p-condition on the modified Bakry–Emery Ricci curvature implies a control from above on the weighted volume of geodesic balls.

On the contrary, as in the classical case of Riemannian geometry, volume growth restrictions do not provide in general a control on $L_D r$. This in turn prevents the possibility of constructing radial super-solutions of (the equation corresponding to) (1.7), that could be used, as in the proof of Theorem A, as suitable barriers to study the existence problem via comparison techniques. This technical difficulty forces us to devise a new approach in the proof of Theorem B, based on a generalization of the weak maximum principle introduced by the authors in [22,16] (see Section 5).

In Section 6 we implement our techniques to analyze differential inequalities of the type

$$L_{D,\varphi} u \geqslant b(x) f(u) \ell\big(|\nabla u|\big) - g(u) h\big(|\nabla u|\big), \tag{1.19}$$

where g and h are continuous functions. Our first task is to find an appropriate form of the Keller–Osserman condition. To this end, we let

$$\rho \in C^0(\mathbb{R}_0^+), \quad \rho(t) \geqslant 0 \quad \text{on } \mathbb{R}_0^+, \tag{ρ}$$

and define the function $\widehat{F}(t) = \widehat{F}_{\rho,\omega}$ depending on the real parameter ω by the formula

$$\widehat{F}_{\rho,\omega}(t) = \int_0^t f(s) e^{(2-\omega)\int_0^s \rho(z)\,dz}\,ds. \tag{1.20}$$

Note that $\widehat{F}$ is well defined because of our assumptions. We assume that $t\varphi'/\ell \in L^1(0^+) \setminus L^1(+\infty)$, define K as in (1.11) and let $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be its inverse. The new version of the Keller–Osserman condition that we shall consider is

$$\frac{e^{\int_0^t \rho(z)\,dz}}{K^{-1}(\widehat{F}(t))} \in L^1(+\infty). \tag{ρKO}$$

Of course, when $\rho \equiv 0$ we recover condition (KO) introduced above. As we shall see in Section 5, the two conditions are in fact equivalent if $\rho \in L^1$ under some mild additional conditions.

We prove

Theorem C. *Let $(M, \langle\,,\rangle)$ be a complete manifold satisfying*

$$\mathrm{Ricc}_{n,m}(L_D) \geqslant H^2(1+r^2)^{\beta/2}, \tag{1.12}$$

for some $n > m$, $H > 0$ and $\beta \geqslant -2$. Assume that (Φ_0), (F_1), (L_1), (L_2), $(\varphi\ell)$, (θ) and (b_1) hold with $\mu \geqslant 0$, $\theta \leqslant 1$ and

$$\begin{cases} \theta < 1 - \beta/2 - \mu & \text{if } \theta \leqslant 1,\ \mu > 0, \\ \theta = 1 - \beta/2 - \mu & \text{if } \theta < 1,\ \mu > 0, \\ \theta < 1 - \beta/2 & \text{if } \theta \leqslant 1,\ \mu = 0. \end{cases} \tag{$\theta\beta\mu'$}$$

Suppose also that

(h) $h \in C^0(\mathbb{R}_0^+)$, $0 \leqslant h(t) \leqslant Ct^2\varphi'(t)$ on $\mathbb{R}_0^+$, for some $C > 0$,
(g) $g \in C^0(\mathbb{R}_0^+)$, $g(t) \leqslant C\rho(t)$ on $\mathbb{R}_0^+$, for some $C > 0$,

and ρ satisfying (ρ). If $(\rho\mathrm{KO})$ holds with $\omega = \theta$ in the definition of $\widehat{F}$, then any entire classical weak solution u of the differential inequality (1.19) *either non-positive or constant. Moreover, if $u \geqslant 0$ and $\ell(0) > 0$ then $u \equiv 0$.*

As already observed, $(\varphi\ell)$ is not satisfied by the mean curvature operator; however, a version of Theorem C can be given to handle this case, see Section 6 below.

As mentioned earlier, in some circumstances $(\rho\mathrm{KO})$ is equivalent to (KO). This is the case, for instance, in the next

Corollary C1. *Let* $(M, \langle\,,\rangle)$ *be as in Theorem* C. *Assume that* (g), (F_1), (L_1), (L_2), $(\varphi\ell)$, (θ) *and* (b_1) *hold with* $\varphi(t) = t^{p-1}$. *Suppose also that*

$$g_+(t) = \max\{0, g(t)\} \in L^1(+\infty).$$

If (KO) *holds, then any entire classical weak solution u of*

$$\Delta_p u \geqslant b(x) f(u)\ell(|\nabla u|) - g(u)|\nabla u|^p$$

is either non-positive or constant. Moreover, if $u \geqslant 0$ *and* $\ell(0) > 0$ *then* $u \equiv 0$.

We conclude this introduction by observing that in the literature have recently appeared other methods to obtain Liouville-type results for differential inequalities such as (1.7) or (1.19). Among them we mention the important technique developed by E. Mitidieri and S.I. Pohozaev, see, e.g., [11], which proves to be very effective when the ambient space is $\mathbb{R}^m$. Their method, which involves the use of cut-off functions in a non-local way, may be adapted to a curved ambient space, but is not suitable to deal with situations where the volume of balls grows super-polynomially.

The paper is organized as follows:

1 Introduction.
2 Comparison results.
3 Proof of Theorem A and related results.
4 A further version of Theorem A.
5 The weak maximum principle and non-existence of solutions with controlled growth.
6 Proof of Theorem C.

In the sequel C will always denote a positive constant which may vary from line to line.

2. Comparison results

In this section we consider the diffusion operator

$$L_D u = \frac{1}{D}\operatorname{div}(D\nabla u), \quad D \in C^2(M),\ D > 0, \tag{2.1}$$

and denote by $r(x)$ the distance from a fixed origin o in an m-dimensional complete Riemannian manifold $(M, \langle\,,\rangle)$. The Riemannian metric and the weight D give rise to a metric measure space, with measure $D\,dV$, dV denoting the usual Riemannian volume element. For ease of notation in the sequel we will drop the index D and write $L_D = L$.

The purpose of this section is to collect the estimates for Lr and for the weighted volume of Riemannian balls, that will be used in the sequel. The estimates are derived assuming an upper bound for a family of modified Ricci tensors, which account for the mutual interactions of the geometry and the weight function.

Although most of the material is available in the literature (see, e.g., D. Bakry and P. Emery [2], Bakry [1], A.G. Setti [24], Z. Qian [20], Bakry and Qian [3], J. Lott [10], X.-D. Li [8]), we are going to present a quick derivation of the estimates for completeness and the convenience of the reader.

We note that our method is somewhat different from that of most of the above authors. In addition we will be able to derive weighted volume estimates under integral type conditions on the modified Bakry–Emery Ricci curvature, which extend to this setting results of S. Gallot [7], P. Petersen and G. Wei [14], and S. Pigola, M. Rigoli and A.G. Setti [18].

For $n > m$ we let $\mathrm{Ricc}(L)$ and $\mathrm{Ricc}_{n,m}(L)$ denote the Bakry–Emery and the modified Bakry–Emery Ricci tensors defined in (1.9).

The starting point of our considerations is the following version of the Bochner–Weitzenböck formula for the diffusion operator L.

Lemma 2.1. *Let $u \in C^3(M)$, then*

$$\frac{1}{2}L\big(|\nabla u|^2\big) = |\mathrm{Hess}\, u|^2 + \langle \nabla Lu, \nabla u\rangle + \mathrm{Ricc}(L)(\nabla u, \nabla u). \tag{2.2}$$

Proof. It follows from the definition of L and the usual Bochner–Weitzenböck formula that

$$\begin{aligned} L\big(|\nabla u|^2\big) &= \Delta\big(|\nabla u|^2\big) + D^{-1}\big\langle \nabla D, \nabla|\nabla u|^2\big\rangle \\ &= 2|\mathrm{Hess}\, u|^2 + 2\langle \nabla \Delta u, \nabla u\rangle + 2\,\mathrm{Ricc}(\nabla u, \nabla u) + D^{-1}\big\langle \nabla D, \nabla|\nabla u|^2\big\rangle. \end{aligned}$$

Now computations show that

$$D^{-1}\big\langle \nabla D, \nabla|\nabla u|^2\big\rangle = 2D^{-1}\,\mathrm{Hess}\, u(\nabla u, \nabla D)$$

and

$$\begin{aligned} \langle \nabla \Delta u, \nabla u\rangle &= \big\langle \nabla\big(Lu - D^{-1}\langle \nabla D, \nabla u\rangle\big), \nabla u\big\rangle \\ &= \big\langle \nabla(Lu), \nabla u\big\rangle + D^{-2}\langle \nabla u, \nabla D\rangle^2 \\ &\quad - D^{-1}\,\mathrm{Hess}\, u(\nabla u, \nabla D) - D^{-1}\,\mathrm{Hess}\, D(\nabla u, \nabla u), \end{aligned}$$

so that substituting yields the required conclusion. □

Lemma 2.2. *Let $(M, \langle\, ,\rangle)$ be a complete Riemannian manifold of dimension m. Let $r(x)$ be the Riemannian distance function from a fixed reference point o, and denote with $\mathrm{cut}(o)$ the cut locus of o. Then for every $n > m$ and $x \notin \{o\} \cup \mathrm{cut}(o)$*

$$\frac{1}{n-1}(Lr)^2 + \langle \nabla Lr, \nabla r\rangle + \mathrm{Ricc}_{n,m}(\nabla r, \nabla r) \leqslant 0. \tag{2.3}$$

Proof. We use $u = r(x)$ in the generalized Bochner–Weitzenböck formula (2.2). Since $\mathrm{Hess}\, r(\nabla r, X) = 0$ for every vector field X, by taking an orthonormal frame in the orthogonal complement of ∇r, and using the Cauchy–Schwarz inequality we see that

$$|\mathrm{Hess}\, r|^2 \geqslant \frac{1}{m-1}(\Delta r)^2.$$

Using the elementary inequality

$$(a-b)^2 \geqslant \frac{1}{1+\epsilon}a^2 - \frac{1}{\epsilon}b^2, \quad a, b \in \mathbb{R},\ \epsilon > 0,$$

we estimate

$$(\Delta u)^2 = \big(Lu - D^{-1}\langle \nabla D, \nabla u\rangle\big)^2 \geqslant \frac{1}{1+\epsilon}(Lu)^2 - \frac{1}{\epsilon}D^{-2}\langle \nabla D, \nabla u\rangle^2.$$

Now, the required conclusion follows substituting into (2.2), using $|\nabla r| = 1$, choosing ϵ in such a way that $(1+\epsilon)(m-1) = n-1$, and recalling the definition of $\mathrm{Ricc}_{n,m}$. □

We are now ready to prove the weighted Laplacian comparison theorem. Versions of this results have been obtained by Setti [24], for the case where $n = m+1$ and later by Qian [20] in the general case where $n > m$ (see also [3] which deals with the case where the drift term is not even assumed to be a gradient). We present a proof modeled on the proof of the Laplacian comparison theorem described in [16].

Proposition 2.3. *Let $(M, \langle\,,\rangle)$ be a complete Riemannian manifold of dimension m. Let $r(x)$ be the Riemannian distance function from a fixed reference point o, and denote with* cut(o) *the cut locus of o. Assume that*

$$\mathrm{Ricc}_{n,m}(\nabla r, \nabla r) \geqslant -(n-1)G(r) \tag{2.4}$$

for some $G \in C^0([0,+\infty))$, let $h \in C^2([0,+\infty))$ be a solution of the problem

$$\begin{cases} h'' - Gh \geqslant 0, \\ h(0) = 0, \quad h'(0) = 1, \end{cases} \tag{2.5}$$

and let $(0, R)$, $R \leqslant +\infty$, be the maximal interval where $h(r) > 0$. Then for every $x \in M$ we have $r(x) \leqslant R$, and the inequality

$$Lr(x) \leqslant (n-1)\frac{h'(r(x))}{h(r(x))} \tag{2.6}$$

holds pointwise in $M \setminus (\mathrm{cut}(o) \cup \{o\})$ and weakly on M.

Proof. Next let $x \in M \setminus (\mathrm{cut}(o) \cup \{o\})$, let $\gamma : [0, r(x)] \to M$ be the unique minimizing geodesic parametrized by arc length joining o to x, and set $\psi(s) = (Lr) \circ \gamma(s)$. It follows from (2.3) and $\dot\gamma = \nabla r$ that

$$\begin{aligned} \frac{d}{ds}(Lr \circ \gamma)(s) &= \langle \nabla Lr, \nabla r\rangle \circ \gamma \\ &\leqslant -\frac{1}{n-1}(Lr \circ \gamma)(s)^2 + (n-1)G(s) \end{aligned} \tag{2.7}$$

on $(0, r(x))$. Moreover,

$$(Lr \circ \gamma)(s) = \frac{m-1}{s} + O(1) \quad \text{as } s \to 0^+, \tag{2.8}$$

which follows from the fact that

$$(Lr \circ \gamma)(s) = \big(\Delta r + D^{-1}\langle \nabla D, \nabla r\rangle\big) \circ \gamma(s)$$

and the second summand is bounded as $s \to 0^+$, while, by standard estimates,

$$\Delta r(x) = \frac{m-1}{r(x)} + o(1).$$

Because of (2.8), we may set

$$g(s) = s^{\frac{m-1}{n-1}} \exp\left(\int_0^s \left[\frac{(Lr \circ \gamma)(t)}{n-1} - \frac{m-1}{n-1}\frac{1}{t}\right] dt\right), \tag{2.9}$$

so that g is defined in $[0, r(x)]$, $g(s) > 0$ in $(0, r(x))$, and it satisfies

$$(n-1)\frac{g'}{g} = Lr \circ \gamma, \qquad g(0) = 0, \qquad g(s) = s^{\frac{m-1}{n-1}}\big(1 + o(1)\big) \quad \text{as } s \to 0^+. \tag{2.10}$$

It follows from this and (2.7) that g satisfies the problem

$$\begin{cases} g'' \leqslant Gg, \\ g(0) = 0, \quad g'(s) = s^{\frac{m-n}{n-1}}\big(1 + o(1)\big) \quad \text{as } s \to 0^+. \end{cases} \tag{2.11}$$

Recalling that, by assumption h satisfies (2.5), we now proceed as in the standard Sturm comparison theorems, and consider the function

$$z(s) = h'(s)g(s) - h(s)g'(s).$$

Then

$$z'(s) = gh\left(\frac{h''}{h} - \frac{g''}{g}\right) \geqslant 0$$

in the interval $(0, \tau)$, $\tau = \min\{r(x), R\}$, where g is defined and h is positive. Also, it follows from the asymptotic behavior of g and h that

$$h'(s)g(s) \asymp s^{\frac{m-1}{n-1}}, \qquad h(s)g'(s) \asymp \frac{m-1}{n-1} s^{\frac{m-1}{n-1}},$$

so that

$$z(s) \to 0^+ \quad \text{as } s \to 0^+.$$

We conclude that $z(s) \geqslant 0$ and therefore

$$\frac{g'(s)}{g(s)} \leqslant \frac{h'(s)}{h(s)}$$

in the interval $(0, \tau)$.

Integrating between ϵ and s, $0 < \epsilon < s < \tau$, yields

$$g(s) \leqslant \frac{g(\epsilon)}{h(\epsilon)} h(s),$$

showing that h must be positive in $(0, \tau)$, and therefore $r(x) \leqslant R$. Since this holds for every $x \in M$ we deduce that if $R < +\infty$ then M is compact and $\operatorname{diam}(M) \leqslant 2R$. Moreover, in $(0, r(x))$ we have

$$(L_r)\big(\gamma\big(r(x)\big)\big) = (n-1)\frac{g'}{g}\big(r(x)\big) \leqslant (n-1)\frac{h'}{h}\big(r(x)\big).$$

This shows that the inequality (2.6) holds pointwise in $M \setminus (\operatorname{cut}(o) \cup \{o\})$. The weak inequality now follows from standard arguments (see, e.g., [16], Lemma 2.2, [18], Lemma 2.5). □

As in the standard Riemannian case, the estimate for Lr allows to obtain weighted volume comparison estimates (see [24,20,3,8]).

Theorem 2.4. *Let $(M, \langle\, ,\rangle)$ be as in the previous proposition, and assume that the modified Bakry–Emery Ricci tensor* $\mathrm{Ricc}_{n,m}$ *satisfies* (2.4) *for some $G \in C^0([0, +\infty))$. Let $h \in C^2([0, +\infty))$ be a solution of the problem* (2.5)*, and let $(0, R)$ be the maximal interval where h is positive. Then, the functions*

$$r \mapsto \frac{\operatorname{vol}_D \partial B_r(o)}{h(r)^{n-1}} \tag{2.12}$$

and

$$r \mapsto \frac{\operatorname{vol}_D B_r(o)}{\int_0^r h(t)^{n-1}\,dt} \tag{2.13}$$

are non-increasing a.e., respectively non-increasing in $(0, R)$. In particular, for every $0 < r_o < R$, there exists a constant C depending on D and on the geometry of M in $B_{r_o}(o)$ such that

$$\operatorname{vol}_D\big(B_r(o)\big) \leqslant C \begin{cases} r^m & \text{if } 0 \leqslant r \leqslant r_o, \\ \int_0^r h(t)^{n-1}\,dt & \text{if } r_o \leqslant r. \end{cases} \tag{2.14}$$

Proof. By Proposition 2.3, inequality (2.6) holds weakly on M, so for every $0 \leqslant \varphi \in Lip_c(M)$, we have

$$-\int \langle \nabla r, \nabla \varphi \rangle D(x)\,dV \leqslant (n-1)\int \varphi \frac{h'(r(x))}{h(r(x))} D(x)\,dV. \tag{2.15}$$

For any $\varepsilon > 0$, consider the radial cut-off function

$$\varphi_\varepsilon(x) = \rho_\varepsilon\big(r(x)\big)h\big(r(x)\big)^{-n+1}, \tag{2.16}$$

where ρ_ε is the piecewise linear function

$$\rho_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0, r), \\ \frac{t-r}{\varepsilon} & \text{if } t \in [r, r+\varepsilon), \\ 1 & \text{if } t \in [r+\varepsilon, R-\varepsilon), \\ \frac{R-t}{\varepsilon} & \text{if } t \in [R-\varepsilon, R), \\ 0 & \text{if } t \in [R, \infty). \end{cases} \tag{2.17}$$

Note that

$$\nabla\varphi_\varepsilon = \left\{-\frac{\chi_{R-\varepsilon,R}}{\varepsilon} + \frac{\chi_{r,r+\varepsilon}}{\varepsilon} - (n-1)\frac{h'(r(x))}{h(r(x))}\rho_\varepsilon\right\}h\big(r(x)\big)^{-n+1}\nabla r,$$

for a.e. $x \in M$, where $\chi_{s,t}$ is the characteristic function of the annulus $B_t(o) \setminus B_s(o)$. Therefore, using φ_ε into (2.15) and simplifying, we get

$$\frac{1}{\varepsilon}\int\limits_{B_R(o)\setminus B_{R-\varepsilon}(o)} h\big(r(x)\big)^{-n+1} \leqslant \frac{1}{\varepsilon}\int\limits_{B_{r+\varepsilon}(o)\setminus B_r(o)} h\big(r(x)\big)^{-n+1}.$$

Using the co-area formula we deduce that

$$\frac{1}{\varepsilon}\int\limits_{R-\varepsilon}^{R} \operatorname{vol}\partial B_t(o)h(t)^{-n+1} \leqslant \frac{1}{\varepsilon}\int\limits_{r}^{r+\varepsilon} \operatorname{vol}\partial B_t(o)h(t)^{-n+1}$$

and, letting $\varepsilon \searrow 0$,

$$\frac{\operatorname{vol}_D \partial B_R(o)}{h(R)^{m-1}} \leqslant \frac{\operatorname{vol}_D \partial B_r(o)}{h(r)^{m-1}} \tag{2.18}$$

for a.e. $0 < r < R$. The second statement follows from the first and the co-area formula, since, as noted by M. Gromov (see [4]), for general real valued functions $f(t) \geqslant 0$, $g(t) > 0$,

$$\text{if } t \to \frac{f(t)}{g(t)} \text{is decreasing, then } t \to \frac{\int_0^t f}{\int_0^t g} \text{ is decreasing.} \qquad \square$$

We next consider the situation where the modified Bakry–Emery Ricci curvature satisfies some L^p-integrability conditions and extends results obtained in [18] for the Riemannian volume which in turn slightly generalize previous results by P. Petersen and G. Wei [14] (see also [7] and [9]).

Since we will be interested in the case the underlying manifold is non-compact, we assume that G is a non-negative, continuous function on $[0, +\infty)$ and that $h(t) \in C^2([0, +\infty))$ is the solution of the problem

$$\begin{cases} h''(t) - G(t)h(t) = 0, \\ h(0) = 0, \quad h'(0) = 1. \end{cases}$$

The assumption that $G \geqslant 0$ implies that $h' \geqslant 1$ on $[0, +\infty)$ and therefore $h > 0$ on $(0, +\infty)$. For ease of notation, in the course of the arguments that follow we set

$$A_{G,n}(r) = h(r)^{n-1} \quad \text{and} \quad V_{G,n}(r) = \int_0^r h(t)^{n-1}\, dt \tag{2.19}$$

so that $A_{G,n}(r)$ and $V_{G,n}(r)$ are multiples of the measures of the sphere and of the ball of radius r centered at the pole in the n-dimensional model manifold M_G with radial Ricci curvature equal to $-(n-1)G$.

Using an exhaustion of $E_o = M \setminus \mathrm{cut}(o)$ by means of starlike domains one shows (see, e.g., [18], p. 35) that for every non-negative test function $\varphi \in Lip_c(M)$,

$$-\int_M \langle \nabla r, \nabla \varphi \rangle D\, dV \leqslant \int_{E_o} \varphi L r D\, dV. \tag{2.20}$$

We outline the argument for the convenience of the reader. Let Ω_n be such an exhaustion of E_o, so that, if ν_n denotes the outward unit normal to $\partial\Omega_n$, then $\langle \nu_n, \nabla r\rangle \geqslant 0$. Integrating by parts shows that

$$\begin{aligned} -\int_M \langle \nabla r, \nabla \varphi \rangle D\, dV &= -\lim_n \int_{\Omega_n} \langle \nabla r, \nabla \varphi \rangle D\, dV \\ &= \lim_n \left\{ \int_{\Omega_n} \varphi \left[\Delta r + \frac{1}{D} \langle \nabla D, \nabla r \rangle \right] D\, dV - \int_{\partial\Omega_n} \varphi \langle \nabla r, \nu_n \rangle D\, d\sigma \right\} \\ &\leqslant \lim_n \int_{\Omega_n} \varphi L_D r D\, dV = \int_{E_o} \varphi L_D r D\, dV, \end{aligned}$$

where the inequality follows from $\langle \nabla r, \nu_n \rangle \geqslant 0$, and the limit on the last line exists because, by Proposition 2.3, Lr is bounded above by some positive integrable function g on the relatively compact set $E_o \cap \operatorname{supp}\varphi$ (namely, if $\mathrm{Ricc}_{m,n} \geqslant -(n-1)H^2$ on $E_o \cap \operatorname{supp}\varphi$ for some $H > 0$, we can choose $g = H \coth(Hr)$).

Applying the above inequality to the test function

$$\varphi_\epsilon(x) = \rho_\epsilon\big(r(x)\big) h\big(r(x)\big)^{-n+1},$$

already considered in (2.16), arguing as in the proof of Theorem 2.4, and using the fact that $A_{G,n}(r) = h(r)^{n-1}$ is non-decreasing, we deduce that for a.e. $0 < r < R$

$$\frac{\mathrm{vol}_D\, \partial B_R}{A_{G,n}(R)} - \frac{\mathrm{vol}_D\, \partial B_r}{A_{G,n}(r)} \leqslant \frac{1}{A_{G,n}(r)} \int_{B_R \setminus B_r} \psi D\, dV, \tag{2.21}$$

where we have set

$$\psi(x)=\begin{cases}\max\{0, Lr(x)-(n-1)\frac{h'(r(x))}{h(r(x))}\} & \text{if } x\in E_o,\\ 0 & \text{if } x\in \operatorname{cut}(o).\end{cases} \tag{2.22}$$

Note by virtue of the asymptotic behavior of Lr and h'/h as $r(x)\to 0$, ψ vanishes in a neighborhood of o. Moreover, if $\mathrm{Ricc}_{n,m}(\nabla r,\nabla r)\geqslant -(n-1)G(r(x))$, then, by the weighted Laplacian comparison theorem, $\psi(x)\equiv 0$, and we recover the fact that the function

$$r\to \frac{\mathrm{vol}_D\,\partial B_r}{A_{G,n}(r)} \tag{2.23}$$

is non-increasing for a.e. r.

Using the co-area formula, inserting (2.21), and applying Hölder inequality with exponents $2p$ and $2p/(2p-1)$ to the right-hand side of the resulting inequality we conclude that

$$\begin{aligned}\frac{d}{dR}\left(\frac{\mathrm{vol}_D\,B_R(o)}{V_{G,n}(R)}\right)&=\frac{V_{G,n}(R)\,\mathrm{vol}_D\,\partial B_R-A_{G,n}(R)\,\mathrm{vol}_D\,B_R}{V_{G,n}(R)^2}\\ &=V_G(R)^{-2}\int_0^R\big(A_{G,n}(r)\,\mathrm{vol}\,\partial B_R-A_{G,n}(R)\,\mathrm{vol}_D\,\partial B_r\big)\,dr\\ &\leqslant\frac{RA_{G,n}(R)}{V_{G,n}(R)^{1+1/2p}}\left(\frac{\mathrm{vol}_D\,B_R}{V_{G,n}(R)}\right)^{1-1/2p}\left(\int_{B_R}\psi^{2p}D\,dV\right)^{1/2p}.\end{aligned} \tag{2.24}$$

Now we define

$$\begin{aligned}\rho(x)&=-\min\big\{0,\mathrm{Ricc}_{n,m}(\nabla r,\nabla r)+(n-1)G\big(r(x)\big)\big\}\\ &=\big[\mathrm{Ricc}_{n,m}(\nabla r,\nabla r)+(n-1)G\big(r(x)\big)\big]_-.\end{aligned} \tag{2.25}$$

We will need to estimate the integral on the right-hand side of (2.24) in terms of ρ. This is achieved in the following lemma, which is a minor modification of [14], Lemma 2.2, and [18], Lemma 2.19.

Lemma 2.5. *For every $p>n/2$ there exists a constant $C=C(n,p)$ such that for every R*

$$\int_{B_R}\psi^{2p}D\,dV\leqslant C\int_{B_R}\rho^pD\,dV,$$

with $\rho(x)$ defined in (2.25).

Proof. Integrating in polar geodesic coordinates we have

$$\int_{B_R}fD\,dV=\int_{S^{m-1}}d\theta\int_0^{\min\{R,c(\theta)\}}f(t\theta)(D\omega)(t\theta)\,dt,$$

where ω is the volume density with respect to Lebesgue measure $dt\,d\theta$, and $c(\theta)$ is the distance from o to the cut locus along the ray $t \to t\theta$. It follows that it suffices to prove that for every $\theta \in S^{m-1}$

$$\int_0^{\min\{R,c(\theta)\}} \psi^{2p}(t\theta)(D\omega)(t\theta)\,dt \leqslant C \int_0^{\min\{R,c(\theta)\}} \rho^p(t\theta)(D\omega)(t\theta)\,dt. \tag{2.26}$$

An easy computation which uses (2.7) yields

$$\frac{\partial}{\partial t}\left\{Lr-(n-1)\frac{h'}{h}\right\} \leqslant -\frac{(Lr)^2}{n-1} - \mathrm{Ricc}_{n,m}(\nabla r,\nabla r) - (n-1)\left\{\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right\}.$$

Thus, recalling the definitions of ψ and ρ, we deduce that the locally Lipschitz function ψ satisfies the differential inequality

$$\psi' + \frac{\psi^2}{n-1} + 2\frac{h'}{h}\psi \leqslant \rho,$$

on the set where $\rho > 0$ and a.e. on $(0,+\infty)$. Multiplying through by $\psi^{2p-2}D\omega$, and integrating we obtain

$$\int_0^r \left(\psi'\psi^{2p-2} + \frac{1}{n-1}\psi^{2p} + 2\frac{h'}{h}\psi^{2p-1}\right)D\omega \leqslant \int_0^r \rho\psi^{2p-2}D\omega. \tag{2.27}$$

On the other hand, integrating by parts, and recalling that

$$(D\omega)^{-1}\partial(D\omega)/\partial t = Lr \leqslant \psi + (n-1)\frac{h'}{h}$$

and that $\psi(t\theta) = 0$ if $t \geqslant c(\theta)$, yield

$$\begin{aligned}\int_0^r \psi'\psi^{2p-2}\omega &= \frac{1}{2p-1}\psi(r)^{2p-1}(D\omega)(r\theta) - \frac{1}{2p-1}\int_0^r \psi^{2p-1}Lr\,D\omega \\ &\geqslant -\frac{1}{2p-1}\int_0^r \psi^{2p-1}\left(\psi + (n-1)\frac{h'}{h}\right)D\omega.\end{aligned}$$

Substituting this into (2.27), and using Hölder inequality we obtain

$$\left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p}D\omega + \left(2 - \frac{n-1}{2p-1}\right)\int_0^r \psi^{2p-1}\frac{h'}{h}D\omega$$

$$\leqslant \int_0^r \rho \psi^{2p-2} D\omega$$

$$\leqslant \left(\int_0^r \rho^p D\omega \right)^{1/p} \left(\int_0^r \psi^{2p} D\omega \right)^{(p-1)/p},$$

and, since the coefficient of the first integral on the left-hand side is positive, by the assumption on p, while the second summand is non-negative, rearranging and simplifying we conclude that (2.26) holds with

$$C(n,p) = \left(\frac{1}{n-1} - \frac{1}{2p-1} \right)^{-p}. \qquad \square$$

We are now ready to state the announced weighted volume comparison theorem under assumptions on the L^p-norm of the modified Bakry–Emery Ricci curvature.

Theorem 2.6. *Keeping the notation introduced above, let $p > n/2$ and let*

$$f(t) = \frac{C_{n,p}^{1/2p} t A_{G,n}(t)}{V_{G,n}(t)^{1+1/2p}} \left(\int_{B_t} \rho^p D\, dV \right)^{1/2p}, \tag{2.28}$$

where $C_{n,p}$ is the constant in Lemma 2.5*. Then for every $0 < r < R$,*

$$\left(\frac{\mathrm{vol}_D B_R(o)}{V_{G,n}(R)} \right)^{1/2p} - \left(\frac{\mathrm{vol}_D B_r(o)}{V_{G,n}(r)} \right)^{1/2p} \leqslant \frac{1}{2p} \int_r^R f(t)\, dt. \tag{2.29}$$

Moreover for every $r_o > 0$ there exists a constant C_{r_o} such that, for every $R \geqslant r_o$

$$\frac{\mathrm{vol}_D B_R(o)}{V_{G,n}(R)} \leqslant \left(C_{r_o} + \frac{1}{2p} \int_{r_o}^R f(t)\, dt \right)^{2p}, \tag{2.30}$$

and

$$\frac{\mathrm{vol}_D \partial B_R(o)}{A_{G,n}(R)} \leqslant \left(C_{r_o} + \frac{1}{2p} \int_{r_o}^R f(t)\, dt \right)^{2p} + \frac{R}{V_{G,n}(R)^{1/2p}} \left(\int_{B_R} \rho^p \right)^{1/2p} \left(C_{r_o} + \frac{1}{2p} \int_{r_o}^R f(t)\, dt \right)^{2p-1}. \tag{2.31}$$

Proof. Set

$$y(r)=\frac{\mathrm{vol}_D\, B_r(o)}{V_{G,n}(r)}.$$

According to (2.24), Lemma 2.5 and (2.28) we have

$$\begin{cases} y'(t)\leqslant f(t)y(t)^{1-1/2p}, \\ y(t)\sim c_m t^{m-n} \quad \text{as } t\to 0^+, \quad y(t)>0 \quad \text{if } t>0, \end{cases}$$

whence, integrating between r and R we obtain

$$y(R)^{1/2p}-y(r)^{1/2p}\leqslant \frac{1}{2p}\int_r^R f(t)\,dt,$$

that is, (2.29), and (2.30) follows at one with $C_{r_o}=(\frac{\mathrm{vol}_D\, B_{r_o}(o)}{V_{G,n}(r_o)})^{1/2p}$. On the other hand, according to (2.24) and Lemma 2.5,

$$\frac{\mathrm{vol}_D\,\partial B_R}{A_{G,n}(R)}\leqslant \frac{\mathrm{vol}_D\, B_R}{V_{G,n}(R)}+\frac{R}{V_{G,n}(R)^{1/2p}}\left(\int_{B_t}\rho^p D\,dV\right)^{1/2p}\left(\frac{\mathrm{vol}_D\, B_R}{V_{G,n}(R)}\right)^{1-1/2p}$$

and the conclusion follows inserting (2.30). □

Keeping the notation introduced above, assume, for instance, that $G=B^2\geqslant 0$, so that

$$A_{G,n}(t)=\begin{cases} t^{n-1} & \text{if } B=0, \\ (B^{-1}\sinh Bt)^{n-1} & \text{if } B>0 \end{cases}$$

and suppose that

$$\rho=\left[\mathrm{Ricc}_{n,m}+(n-1)B^2\right]_-\in L^p(M,D\,dV),$$

for some $p>n/2$. Then, arguing as in the proof of [18], Corollary 2.21, we deduce that for every r_o sufficiently small there exist constants C_1 and C_2, depending on r_o, B, m, p and on the $L^p(M,D\,dV)$-norm of ρ, such that, for every $R\geqslant r_o$,

$$\mathrm{vol}_D\, B_R\leqslant C_1\begin{cases} R^{2p} & \text{if } B=0, \\ e^{(n-1)BR} & \text{if } B>0 \end{cases}$$

and

$$\mathrm{vol}_D\,\partial B_R\leqslant C_2\begin{cases} R^{2p-1} & \text{if } B=0, \\ e^{(n-1)BR} & \text{if } B>0. \end{cases}$$

3. Proof of Theorem A and further results

The aim of this section is to give a proof of a somewhat stronger form of Theorem A (see Theorem 3.5 below), together with a version of the result valid when (KO) fails.

The idea of proof of Theorem A is to construct a function $v(x)$ defined on an annular region $B_{\bar{R}} \setminus B_{r_o}$, with $0 < r_o < \bar{R}$ sufficiently large, with the following properties: for fixed $r_o < r_1 < \bar{R}$ and $0 < \epsilon < \eta$

$$\begin{cases} v(x) = \epsilon & \text{on } \partial B_{r_o}, \\ \epsilon \leqslant v(x) \leqslant \eta & \text{on } B_{r_1} \setminus B_{r_o}, \\ v(x) \to +\infty & \text{as } r(x) \to +\infty, \end{cases} \tag{3.1}$$

and v is a weak super-solution on $B_{\bar{R}} \setminus B_{r_o}$ of

$$L_{D,\varphi} w = b(x) f(w) \ell\big(|\nabla w|\big). \tag{3.2}$$

This is achieved by taking v of the form

$$v(x) = \alpha\big(r(x)\big), \tag{3.3}$$

where α is a suitable super-solution of the radialized inequality (3.2), whose construction depends in a crucial way on the validity of the Keller–Osserman condition (KO).

The conclusion is then reached comparing v with the solution of (1.7). To this end, we will extend a comparison technique first introduced in [15].

Finally, in Theorem 3.6 below we will consider the case where the Keller–Osserman condition fails, that is,

$$\frac{1}{K^{-1}(F(t))} \notin L^1(+\infty). \tag{3.4}$$

Its proof is based on a modification of the previous arguments and uses (3.4) in a way which is, in some sense, dual to the use of (KO) in the proof of Theorem A.

We begin with the following simple

Lemma 3.1. *Assume that f, ℓ and φ satisfy the assumptions (F_1), (L_1) and $(\varphi\ell)_2$, and let $\sigma > 0$. Then* (KO) *holds if and only if*

$$\frac{1}{K^{-1}(\sigma F(s))} \in L^1(+\infty). \tag{KOσ}$$

Proof. We consider first the case $0 < \sigma \leqslant 1$. Since K^{-1} is non-decreasing,

$$\int^{+\infty} \frac{ds}{K^{-1}(F(s))} \leqslant \int^{+\infty} \frac{1}{K^{-1}(\sigma F(s))}.$$

On the other hand, if $C \geqslant 1$ is such that $\sup_{s \leqslant t} f(s) \leqslant C f(t)$, then, for every $0 < \sigma \leqslant 1$, $f(C\sigma^{-1}t) \geqslant C^{-1} f(t)$ and

$$F\left(\frac{Ct}{\sigma}\right)=\int_0^{\frac{Ct}{\sigma}} f(z)\,dz=\frac{C}{\sigma}\int_0^t f\left(\frac{C\xi}{\sigma}\right)d\xi \geqslant \frac{1}{\sigma}\int_0^t f(\xi)\,d\xi=\frac{1}{\sigma}F(t),$$

so, using the monotonicity of K^{-1}, we obtain

$$\int^{+\infty}\frac{ds}{K^{-1}(\sigma F(s))}=\frac{C}{\sigma}\int^{+\infty}\frac{dt}{K^{-1}(\sigma F(\frac{Ct}{\sigma}))}\leqslant\frac{C}{\sigma}\int^{+\infty}\frac{dt}{K^{-1}(F(t))},$$

showing that (KO) and (KOσ) are equivalent in the case $\sigma \leqslant 1$.

Consider now the case $\sigma > 1$, and set $f_\sigma = \sigma f$, $F_\sigma = \sigma F$. Since (KOσ) is precisely (KO) for F_σ, and since $\sigma^{-1} \leqslant 1$, by what we have just proved it is equivalent to

$$\frac{1}{K^{-1}(\sigma^{-1}F_\sigma(s))}=\frac{1}{K^{-1}(F(s))}\in L^1(+\infty),$$

as required. □

We note for future use that the conclusion of the lemma depends only on the monotonicity of K^{-1} and the C-monotonicity of f.

Before proceeding toward our main result we would like to explore the mutual connections between (θ) and $(\varphi\ell)$. To simplify the writing, with the statement "$(\theta)_1$ holds" we will mean that the first half of condition (θ) is valid.

Proposition 3.2. *Assume that conditions (Φ_0) and (L_1) hold. Then $(\theta)_1$ with $\theta < 2$ implies $(\varphi\ell)_2$, and $(\theta)_2$ with $\theta < 1$ implies $(\varphi\ell)_1$. As a consequence, (θ) with $\theta < 1$ implies $(\varphi\ell)$.*

Proof. Assume $(\theta)_1$, that is, the function $t \to \frac{\varphi'(t)}{\ell(t)}t^\theta$ is C-increasing on $\mathbb{R}^+$. By definition there exists $C \geqslant 1$ such that

$$0 < s^\theta\frac{\varphi'(st)}{\ell(st)} \leqslant C\frac{\varphi'(t)}{\ell(t)} \quad \forall t \in \mathbb{R}^+,\ s \in (0,1],$$

or, equivalently,

$$s^\theta\frac{\varphi'(st)}{\ell(st)} \geqslant C^{-1}\frac{\varphi'(t)}{\ell(t)} \quad \forall t \in \mathbb{R}^+,\ s \in [1,+\infty). \tag{3.5}$$

Letting $t = 1$, we deduce that if $\theta < 2$ then $\frac{s\varphi'(s)}{\ell(s)} \in L^1(0^+) \setminus L^1(+\infty)$, which is $(\varphi\ell)_2$.

In an entirely similar way, if $(\theta)_2$ holds, that is,

$$\frac{\varphi(st)}{\ell(st)}(st)^{\theta-1} \leqslant C\frac{\varphi(t)}{\ell(t)}(t)^{\theta-1} \quad \forall t \in \mathbb{R}^+,\ s \in (0,1],$$

and $\theta < 1$, then $s^{\theta-1}\frac{\varphi(s)}{\ell(s)} \in L^\infty((0,1))$, and

$$\lim_{s\to 0^+} \frac{\varphi(s)}{\ell(s)} = 0,$$

which implies $(\varphi\ell)_1$. □

Remark 3.1. Note that the above argument also shows that if $(\theta)_2$ holds with $\theta < 2$ then $\frac{\varphi(t)}{\ell(t)} \in L^1(0^+) \setminus L^1(+\infty)$.

Proposition 3.3. *Assume that conditions (Φ_0) and (L_1) hold, and let F be a positive function defined on $\mathbb{R}_0^+$. If $(\theta)_1$ holds with $\theta < 2$, then there exists a constant $B \geqslant 1$ such that, for every $\sigma \leqslant 1$ we have*

$$\frac{\sigma^{1/(2-\theta)}}{K^{-1}(\sigma F(t))} \leqslant \frac{B}{K^{-1}(F(t))} \quad \text{on } \mathbb{R}^+. \tag{3.6}$$

Proof. Observe first of all that according to Proposition 3.2, $(\theta)_1$ with $\theta < 2$ implies $(\varphi\ell)_2$, so that K^{-1} is well defined on $\mathbb{R}_0^+$.

Changing variables in the definition of K, and using (3.5) above, for every $\lambda \geqslant 1$ and $t \in \mathbb{R}^+$, we have

$$\begin{aligned} K(\lambda t) &= \int_0^{\lambda t} s\frac{\varphi'(s)}{\ell(s)}\,ds = \lambda^2 \int_0^t s\frac{\varphi'(\lambda s)}{\ell(\lambda s)}\,ds \\ &\geqslant C^{-1}\lambda^{2-\theta} \int_0^t s\frac{\varphi'(s)}{\ell(s)}\,ds = C^{-1}\lambda^{2-\theta}K(t), \end{aligned}$$

where $C \geqslant 1$ is the constant in $(\theta)_1$. Applying K^{-1} to both sides of the above inequality, and setting $t = K^{-1}(\sigma F(s))$ we deduce that

$$\lambda K^{-1}\big(\sigma F(s)\big) \geqslant K^{-1}\big(\lambda^{2-\theta}\sigma C^{-1}F(s)\big),$$

whence, setting $\lambda = (C/\sigma)^{1/(2-\theta)} \geqslant 1$, the required conclusion follows with $B = C^{1/(2-\theta)}$. □

Remark 3.2. We note for future use that the estimate holds for any positive function F on $\mathbb{R}^+$, without any monotonicity property, and it depends only on the fact that the integrand $\psi(s) = s\varphi'(s)/\ell(s)$ in the definition of K satisfies the C-monotonicity property

$$\psi(\lambda s) \geqslant C^{-1}\lambda^{1-\theta}\psi(s) \quad \forall s \in \mathbb{R}^+,\ \forall \lambda \geqslant 1.$$

In order to state the next proposition we introduce the following assumption

(b) $\tilde{b}(t) \in C^1(\mathbb{R}_0^+)$, $\tilde{b}(t) > 0$, $\tilde{b}'(t) \leqslant 0$ for $t \gg 1$, and $\tilde{b}^\lambda \notin L^1(+\infty)$ for some $\lambda > 0$.

Proposition 3.4. *Assume that conditions (Φ_0), (F_1), (L_1), (L_2) $(\varphi\ell)_1$, (θ),* (KO) *hold, and let $\tilde{b}$ be a function satisfying assumption* (b)*, $A > 0$, and $\beta \in [-2, +\infty)$. If λ and θ are the constants specified in* (b) *and (θ), assume also that*

$$\lambda(2-\theta) \geqslant 1 \quad \textit{and either}$$
$$\text{(i) } t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1}\int_1^t \tilde{b}(s)^\lambda\,ds \leqslant C \quad \textit{for } t \geqslant t_0, \quad \textit{or}$$
$$\text{(ii) } t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1} \leqslant C \quad \textit{for } t \geqslant t_0 \textit{ and } \theta < 1. \tag{3.7}$$

Then there exists $T > 0$ *sufficiently large such that, for every* $T \leqslant t_0 < t_1$ *and* $0 < \epsilon < \eta$, *there exist* $\bar{T} > t_1$ *and a* C^2 *function* $\alpha : [t_0, \bar{T}) \to [\epsilon, +\infty)$ *which is a solution of the problem*

$$\begin{cases} \varphi'(\alpha')\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant \tilde{b}(t)f(\alpha)\ell(\alpha) \quad \textit{on } [t_0, \bar{T}), \\ \alpha' > 0 \quad \textit{on } [t_0, \bar{T}), \quad \alpha(t_0) = \epsilon, \quad \alpha(t) \to +\infty \quad \textit{as } t \to \bar{T}^- \end{cases} \tag{3.8}$$

and satisfies

$$\epsilon \leqslant \alpha \leqslant \eta \quad \textit{on } [t_0, t_1]. \tag{3.9}$$

Proof. Note first of all, that the first condition in (3.7) forces $\theta < 2$, and $(\varphi\ell)_2$ follows from $(\theta)_1$.

We choose $T > 0$ large enough that, by (b), $\tilde{b}(t) > 0$ and $\tilde{b}'(t) \leqslant 0$ on $[T, +\infty)$. Since (b) and (3.7) are invariant under scaling of $\tilde{b}$, we may assume without loss of generality that $\tilde{b} \leqslant 1$ on $[T, \infty)$.

Let t_0, t_1, ϵ, η be as in the statement of the proposition, and, for a given $\sigma \in (0, 1]$, set

$$C_\sigma = \int_\epsilon^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))}, \tag{3.10}$$

which is well defined in view of (KO) and Lemma 3.1. Since $\tilde{b}(t) \notin L^1(+\infty)$, there exists $T_\sigma > t_0$ such that

$$C_\sigma = \int_{t_0}^{T_\sigma} \tilde{b}(s)^\lambda\,ds.$$

We note that, by monotone convergence, $C_\sigma \to +\infty$ as $\sigma \to 0^+$, and we may therefore choose $\sigma > 0$ small enough that $T_\sigma > t_1$. We let $\alpha : [t_0, T_\sigma) \to [\epsilon, +\infty)$ be implicitly defined by the equation

$$\int_t^{T_\sigma} \tilde{b}(s)^\lambda\,ds = \int_{\alpha(t)}^{\infty} \frac{ds}{K^{-1}(\sigma F(s))}, \tag{3.11}$$

so that, by definition,

$$\alpha(t_0) = \epsilon, \qquad \alpha(t) \to +\infty \quad \text{as } t \to T_\sigma^-.$$

Differentiating (3.11) yields

$$\alpha'(t) = \tilde{b}(t)^\lambda K^{-1}\big(\sigma F\big(\alpha(t)\big)\big), \tag{3.12}$$

so that $\alpha' > 0$ on $[t_0, T_\sigma)$, and

$$\sigma F(\alpha) = K\big(\alpha'/\tilde{b}^\lambda\big).$$

Differentiating once more, using the definition of K and (3.12), we obtain

$$\sigma f(\alpha)\alpha' = K'\big(\alpha'/\tilde{b}^\lambda\big)\big(\alpha'/\tilde{b}^\lambda\big)' = \frac{\alpha'}{\tilde{b}^\lambda}\frac{\varphi'(\alpha'/\tilde{b}^\lambda)}{\ell(\alpha'/\tilde{b}^\lambda)}\left(\frac{\alpha'}{\tilde{b}^\lambda}\right)'. \tag{3.13}$$

Since $f(t) > 0$ on $(0, \infty)$, $\alpha' > 0$ and $\tilde{b}' \leqslant 0$, we have $(\alpha'/\tilde{b}^\lambda)' \geqslant 0$ and $\alpha'/\tilde{b}^\lambda$ is non-decreasing. Moreover,

$$\left(\frac{\alpha'}{\tilde{b}^\lambda}\right)' = \big(\alpha''/\tilde{b}^\lambda\big) - \lambda\big(\alpha'\tilde{b}'/\tilde{b}^{\lambda+1}\big) \geqslant \big(\alpha''/\tilde{b}^\lambda\big).$$

Inserting this into (3.13), using the fact that $\tilde{b}^{-\lambda} \geqslant 1$ and $(\theta)_1$ (in the form of (3.5)), and rearranging we obtain

$$\varphi'\big(\alpha'\big)\alpha'' \leqslant \big\{C\sigma\tilde{b}^{\lambda(2-\theta)}\big\}\tilde{b}f(\alpha)\ell\big(\alpha'\big) \quad \text{on } [t_0, T_\sigma). \tag{3.14}$$

In order to estimate the term $At^{\beta/2}\varphi(\alpha')$ we rewrite (3.13) in the form

$$\varphi\big(\alpha'/\tilde{b}^\lambda\big)\big(\alpha'/\tilde{b}^\lambda\big)' = \sigma\tilde{b}^\lambda f(\alpha)\ell\big(\alpha'/\tilde{b}^\lambda\big) \quad \text{on } [t_0, T_\sigma),$$

integrate between t_0 and $t \in (t_0, T_\sigma)$, use the fact that α and $\alpha/\tilde{b}^\lambda$ are increasing, and f and ℓ are C-increasing to deduce that

$$\varphi\big(\alpha'/\tilde{b}^\lambda\big) \leqslant \varphi\big(\alpha'/\tilde{b}^\lambda\big)(t_0) + C\sigma f(\alpha)\ell\big(\alpha'/\tilde{b}^\lambda\big)\int_{t_0}^{t}\tilde{b}(s)^\lambda\,ds,$$

for some constant $C \geqslant 1$. On the other hand, since $t^{\theta-1}\varphi(t)/\ell(t)$ is C-increasing and $\tilde{b} \leqslant 1$, we have

$$\begin{aligned}
\frac{\varphi(\alpha')}{\ell(\alpha')} &\leqslant C\tilde{b}^{\lambda(1-\theta)}\frac{\varphi(\alpha'/\tilde{b}^\lambda)}{\ell(\alpha'/\tilde{b}^\lambda)} \\
&\leqslant C\tilde{b}^{\lambda(1-\theta)}\left[\frac{\varphi(\alpha'/\tilde{b}^\lambda)(t_0)}{\ell(\alpha'/\tilde{b}^\lambda)} + \sigma f(\alpha)\int_{t_0}^{t}\tilde{b}(s)^\lambda\right] \\
&\leqslant C\tilde{b}^{\lambda(1-\theta)-1}\left[\frac{\varphi(\alpha'/\tilde{b}^\lambda)(t_0)}{f(\epsilon)\ell(\alpha'/\tilde{b}^\lambda)(t_0)} + \sigma\int_{t_0}^{t}\tilde{b}(s)^\lambda\right]\tilde{b}f(\alpha),
\end{aligned} \tag{3.15}$$

where the second inequality follows from the fact that α and $\alpha'/\tilde{b}^{\lambda}$ are increasing, and f and ℓ are C-increasing.

Using (3.14) and (3.15), and recalling that, by (3.12), $(\alpha'/\tilde{b}^{\lambda})(t_0) = K^{-1}(\sigma F(\epsilon))$, we obtain

$$\varphi'(\alpha')\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant N_\sigma(t)\tilde{b} f(\alpha)\ell(\alpha'), \tag{3.16}$$

where

$$N_\sigma(t) = C\sigma\tilde{b}^{\lambda(2-\theta)-1} + ACt^{\beta/2}\tilde{b}^{\lambda(1-\theta)-1}\frac{\varphi(K^{-1}(\sigma F(\epsilon)))}{\ell(K^{-1}(\sigma F(\epsilon)))f(\epsilon)} + AC\sigma t^{\beta/2}\tilde{b}^{\lambda(1-\theta)-1}\int_{t_0}^{t}\tilde{b}(s)^{\lambda} = (I)(t) + (II)(t) + (III)(t). \tag{3.17}$$

Since $\tilde{b} \leqslant 1$, and $\lambda(2-\theta) - 1 \geqslant 0$ by (3.7), we see that

$$(I)(t) \to 0 \quad \text{uniformly on } [t_0, +\infty) \text{ as } \sigma \to 0.$$

As for (II), according to (3.7)

$$t^{\beta/2}\tilde{b}^{\lambda(1-\theta)-1} \leqslant C \quad \text{on } [t_0, +\infty),$$

so that, using $(\phi\ell)_1$, we deduce that

$$\liminf_{\sigma\to 0^+}\frac{\varphi(\widehat{K}^{-1}(\sigma F(\epsilon)))}{f(\epsilon)\ell(\widehat{K}^{-1}(\sigma F(\epsilon)))} = 0.$$

Thus

$$(II)(t) \to 0 \quad \text{uniformly on } [t_0, +\infty) \text{ along a sequence } \sigma_k \to 0.$$

It remains to analyze (III). Clearly, if (3.7)(i) holds, then $(III)(t) \to 0$ uniformly on $[t_0, +\infty)$ as $\sigma \to 0$. Assume therefore that (3.7)(ii) holds, so that

$$(III)(t) \leqslant AC\sigma\int_{t_0}^{t}\tilde{b}(s)^{\lambda}\,ds. \tag{3.18}$$

By the definition of $\alpha(t)$, Proposition 3.3, and (KO)

$$\int_{t_0}^{t}\tilde{b}(s)^{\lambda}\,ds = \int_{\epsilon}^{\alpha(t)}\frac{ds}{K^{-1}(\sigma F(s))} \leqslant B\sigma^{-1/(2-\theta)}\int_{\epsilon}^{+\infty}\frac{ds}{K^{-1}(F(s))} \leqslant C\sigma^{-1/(2-\theta)}$$

in $[t_0, T_\sigma)$. Since $\theta < 1$ we conclude that

$$(III)(t) \leqslant C\sigma^{1-1/(2-\theta)} \to 0 \quad \text{uniformly in } [t_0, T_\sigma) \text{ as } \sigma \to 0.$$

Putting together the above estimates, we conclude that we can choose σ small enough that $N_\sigma(t) \leqslant 1$, showing that $\alpha(t)$ satisfies the differential inequality in (3.8).

In order to complete the proof we only need to prove that $\epsilon \leqslant \alpha(t) \leqslant \eta$ for $t_0 \leqslant t \leqslant t_1$. Again from the definition of α we have

$$\int_{t_0}^{t_1} \tilde{b}(s)^\lambda \, ds = \int_\epsilon^{\alpha(t_1)} \frac{ds}{K^{-1}(\sigma F(s))},$$

so if we choose $\sigma \in (0, 1]$ small enough to have

$$\int_{t_0}^{t_1} \tilde{b}(s)^\lambda \, ds \leqslant \int_\epsilon^{\eta} \frac{ds}{K^{-1}(\sigma F(s))},$$

then clearly $\alpha(t_1) \leqslant \eta$, and, since α is increasing, this finishes the proof. □

We are now ready to prove

Theorem 3.5. *Let $(M, \langle\, ,\rangle)$ be a complete Riemannian manifold satisfying*

$$\mathrm{Ricc}_{n,m}(L_D) \geqslant H^2\big(1 + r^2\big)^{\beta/2}, \tag{1.12}$$

for some $n > m$, $H > 0$ and $\beta \geqslant -2$ and assume that (Φ_0), (F_1), (L_1), (L_2), $(\varphi\ell)_1$, and (θ) hold. Let $b(x) \in C^0(M)$, $b(x) \geqslant 0$ on M and suppose that

$$b(x) \geqslant \tilde{b}\big(r(x)\big) \quad \textit{for } r(x) \gg 1, \tag{3.19}$$

where $\tilde{b}$ satisfies assumption (b) *and* (3.7). *If the Keller–Osserman condition*

$$\frac{1}{K^{-1}(F(t))} \in L^1(+\infty) \tag{KO}$$

holds then any entire classical weak solution u of the differential inequality

$$L_{D,\varphi} u \geqslant b(x) f(u) \ell\big(|\nabla u|\big) \tag{1.7}$$

is either non-positive or constant. Furthermore, if $u \geqslant 0$, and $\ell(0) > 0$, then u vanishes identically.

Proof. If $u \leqslant 0$ then there is nothing to prove. We argue by contradiction and assume that u is non-constant and positive somewhere. We choose $T > 0$ sufficiently large that (3.19) holds in $M \setminus B_T$ and for every $r_o \geqslant T$ we have

$$0 < u_o^* = \sup_{B_{r_o}} u \leqslant u^* = \sup_M u.$$

We consider first the case where $u^* < +\infty$. We claim that $u_o^* < u^*$. Otherwise there would exist $x_o \in \overline{B}_{r_o}$ such that $u(x_o) = u^*$, and by (1.7) and assumptions (F_1) and (ℓ_1),

$$L_{D,\varphi} u \geqslant 0$$

in the connected component Ω_o of $\{u \geqslant 0\}$ containing x_o. By the strong maximum principle [19], u would then be constant and positive on Ω_o. Since $u = 0$ on $\partial\Omega_o$ this would imply that $\Omega_o = M$ and u is a positive constant on M, contradicting our assumption.

Next, we choose $\eta > 0$ small enough that $u_o^* + 2\eta < u^*$ and $\tilde{x} \notin \overline{B}_{r_o}$ satisfying $u(\tilde{x}) > u^* - \eta$. We let $t_0 = r_o$ and $t_1 = r(\tilde{x})$. Because of (1.12), Proposition 2.3 and [18], Proposition 2.11, there exists $A = A(T) > 0$ such that

$$L_D r \leqslant A r^{\beta/2} \quad \text{on } M \setminus B_T.$$

According to Proposition 3.4 there exist $\bar{T} > t_1$ and a C^2 function $\alpha : [t_0, \bar{T}) \to [\epsilon, +\infty)$ which satisfies

$$\begin{cases} \varphi'(\alpha')\alpha'' + A t^{\beta/2}\varphi(\alpha') \leqslant (2C)^{-1}\tilde{b}(t) f(\alpha)\ell(\alpha) & \text{on } [t_0, \bar{T}), \\ \alpha' > 0 \quad \text{on } [t_0, \bar{T}), \quad \alpha(t_0) = \epsilon, \quad \alpha(t) \to +\infty & \text{as } t \to \bar{T}^- \end{cases}$$

and

$$\epsilon \leqslant \alpha \leqslant \eta \quad \text{on } [t_0, t_1],$$

where C is the constant in the definition of C-monotonicity of f.

It follows that the radial function defined on $B_{\bar{R}} \setminus B_{r_o}$ by $v(x) = \alpha(r(x))$ satisfies the differential inequality

$$L_{D,\varphi} v \leqslant (2C)^{-1} b(x)\big[f(\alpha)\ell(\alpha')\big]\big(r(x)\big) \tag{3.20}$$

pointwise in $(B_{\bar{R}} \setminus \overline{B}_{r_o}) \setminus \mathrm{cut}(o)$ and weakly in $B_{\bar{R}} \setminus \overline{B}_{r_o}$. Furthermore v satisfies (3.1), and

$$u(\tilde{x}) - v(\tilde{x}) > u^* - 2\eta.$$

Since

$$u(x) - v(x) \leqslant u_o^* - \epsilon < u^* - 2\eta - \epsilon \quad \text{on } \partial B_{r_o}$$

and

$$u(x) - v(x) \to -\infty \quad \text{as } x \to \partial B_{\bar{R}},$$

we deduce that the function $u - v$ attains a positive maximum μ in $B_{\bar{R}} \setminus \overline{B}_{r_o}$. We denote by Γ_μ a connected component of the set

$$\{x \in B_{\bar{R}} \setminus \overline{B}_{r_o}\colon u(x) - v(x) = \mu\}$$

and note that Γ_μ is compact.

We claim that for every $y \in \Gamma_\mu$ we have

$$u(y) > v(y), \qquad |\nabla u(y)| = |\alpha'(r(y))|. \tag{3.21}$$

Indeed, this is obvious if y is not in the cut locus $\mathrm{cut}(o)$ of o, for then $\nabla u(y) = \nabla v(y) = \alpha'(r(y))\nabla r(y)$. On the other hand, if $y \in \mathrm{cut}(o)$, let γ be a unit speed minimizing geodesic joining o to y, let $o_\epsilon = \gamma(\epsilon)$ and let $r_\epsilon(x) = d(x, o_\epsilon)$. By the triangle inequality,

$$r(x) \leqslant r_\epsilon(x) + \epsilon \quad \forall x \in M,$$

with equality if and only if x lies on the portion of the geodesic γ between o_ϵ and y (recall that γ ceases to be minimizing past y). Define $v_\epsilon(x) = \alpha(\epsilon + r_\epsilon(x))$, then, since α is strictly increasing,

$$v_\epsilon(x) \geqslant v(x)$$

with equality if and only if x lies on the portion of γ between o_ϵ and y. We conclude that $\forall x \in B_R \setminus \overline{B}_{r_o}$,

$$(u - v_\epsilon)(y) = (u - v)(y) \geqslant (u - v)(x) \geqslant (u - v_\epsilon)(x),$$

and $u - v_\epsilon$ attains a maximum at y. Since y is not on the cut locus of o_ϵ, v_ϵ is smooth there, and

$$|\nabla u(y)| = |\nabla v_\epsilon(y)| = \alpha'(\epsilon + r_\epsilon(y))|\nabla r_\epsilon(y)| = \alpha'(r(y)),$$

as claimed.

Since f is C-increasing,

$$b(y) f(u(\xi))\ell(|\nabla u|(y)) \geqslant \frac{1}{C} b(y) f(v(y))\ell(\alpha'(r(y)))$$

and by continuity the inequality

$$b(x) f(u)\ell(|\nabla u|) \geqslant \frac{1}{2C} b(x) f(v(x))\ell(\alpha'(r(x)))$$

holds in a neighborhood of y. It follows from this and the differential inequalities satisfied by u and v that

$$L_{D,\varphi} u \geqslant L_{D,\varphi} v \tag{3.22}$$

weakly in a sufficiently small neighborhood $\mathcal{U}$ of Γ_μ. Now fix $y \in \Gamma_\mu$ and for $\zeta \in (0,\mu)$ let $\Omega_{y,\zeta}$ be the connected component containing y of the set

$$\{x \in B_{\bar{R}} \setminus \overline{B}_{r_o}\colon u(x) > v(x) + \zeta\}.$$

By choosing ζ sufficiently close to μ we may arrange that $\overline{\Omega}_{y,\zeta} \subset \mathcal{U}$, and, since $u = v + \zeta$ on $\partial\Omega_{y,\zeta}$, (3.22) and the weak comparison principle (see, e.g., [16], Proposition 6.1) implies that $u \leqslant v + \zeta$ on $\Omega_{y,\zeta}$, contradicting the fact that $y \in \Omega_{y,\zeta}$.

The case where $u^* = +\infty$ is easier, and left to the reader. □

Remark 3.3. Theorem A is a special case of Theorem 3.5 with the choice $\tilde{b}(r) = C/r^\mu$ for $r \gg 1$. Assume first that $\mu > 0$. Choosing $\lambda = 1/\mu$, it follows that

$$t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1} = O\big(t^{\theta-1+\beta/2+\mu}\big) \quad \text{and} \quad \int_1^t \tilde{b}(s)^\lambda\,ds = O(\log t).$$

Then $(\theta\beta\mu)$ (and $\beta \geqslant -2$) implies first that $\lambda(2-\theta) - 1 \geqslant \mu^{-1}(1+\beta/2) \geqslant 0$, and then that either (i) or (ii) in (3.7) holds. Thus Theorem 3.5 applies. On the other hand, if $\mu = 0$ and $\theta < 1 - \beta/2$, then $\theta < 1 - \beta/2 - \mu_o$ for sufficiently small $\mu_o > 0$, and the conclusion follows from the previous case.

The next example shows that the validity of the generalized Keller–Osserman condition (KO) is indeed necessary for Theorem 3.5 to hold. Since (KO) is independent of geometry, we consider the most convenient setting where $(M, \langle\,,\rangle)$ is $\mathbb{R}^m$ with its canonical flat metric. We further simplify our analysis by considering the differential inequality

$$\Delta_p u \geqslant f(u)\ell\big(|\nabla u|\big), \tag{3.23}$$

for the p-Laplacian Δ_p, where f is increasing and satisfies $f(0) = 0$, $f(t) > 0$ for $t > 0$, ℓ is non-decreasing and satisfies (L_1), and $(\varphi\ell)$ and (θ) hold. We let $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined as in (1.11), and assume that

$$\frac{1}{K^{-1}(F(t))} \notin L^1(+\infty). \tag{$\neg$KO}$$

Define implicitly the function w on $\mathbb{R}_0^+$ by setting

$$t = \int_1^{w(t)} \frac{ds}{K^{-1}(F(s))}. \tag{3.24}$$

Note that w is well defined, $w(0) = 1$, and ($\neg$KO) implies that $w(t) \to +\infty$ as $t \to \infty$. Differentiating (3.24) yields

$$w' = K^{-1}\big(F\big(w(t)\big)\big) > 0, \tag{3.25}$$

and a further differentiation gives

$$\left(w'\right)^{p-2}w'' = \frac{1}{p-1}f(w)\ell\left(|\nabla w|\right). \tag{3.26}$$

We fix $\bar{t} > 0$ to be specified later, and let $u_1(x)$ be the radial function defined on $\mathbb{R}^m \setminus B_{\bar{t}}$ by the formula

$$u_1(x) = w\left(|x|\right).$$

Using (3.25) and (3.26) we conclude that u_1 satisfies

$$\Delta_p u_1 = (p-1)\left(w'\right)^{p-2}w'' + \frac{m-1}{|x|}\left(w'\right)^{p-1} \geqslant f(u_1)\ell\left(|\nabla u_1|\right) \tag{3.27}$$

on $\mathbb{R}^m \setminus \overline{B}_{\bar{t}}$.

Next we fix constants β_o, $\Lambda > 0$, and, denoting with p' the conjugate exponent of p, we let

$$\beta(t) = \frac{\Lambda}{p'}t^{p'} + \beta_o.$$

Noting that $\beta'(0) = 0$, we deduce that the function

$$u_2(x) = \beta\left(|x|\right)$$

is C^1 on $\mathbb{R}^m$, and an easy calculation shows that

$$\Delta_p u_2 = \Lambda^{p-1}\operatorname{div}\left(|x|x\right) = m\Lambda^{p-1}. \tag{3.28}$$

Since $\beta' \geqslant 0$, and f and ℓ are monotonic, it follows that, if

$$m\Lambda^{p-1} \geqslant f\left(\beta(\bar{t})\right)\ell\left(\beta'(\bar{t})\right), \tag{3.29}$$

then

$$\Delta_p u_2 \geqslant f(u_2)\ell\left(|\nabla u_2|\right) \quad \text{on } B_{\bar{t}}. \tag{3.30}$$

The point now is to join u_1 and u_2 in such a way that the resulting function u is a classical C^1 weak sub-solution of

$$\Delta_p u = f(u)\ell\left(|\nabla u|\right).$$

This is achieved provided we may choose the parameters $\bar{t}$, Λ, β_o, in such a way that (3.29) and

$$\begin{cases} \beta(\bar{t}) = w(\bar{t}), \\ \beta'(\bar{t}) = w'(\bar{t}) \end{cases} \tag{3.31}$$

are satisfied. Towards this end, we define

$$\bar{t} = \int_1^{\lambda} \frac{ds}{K^{-1}(F(s))} > 0, \tag{3.32}$$

where $1 < \lambda \leqslant 2$. Note that, by definition, $w(\bar{t}) = \lambda$, and, by the monotonicity of K^{-1} and F

$$\frac{\lambda - 1}{K^{-1}(F(2))} \leqslant \bar{t} \leqslant \frac{\lambda - 1}{K^{-1}(F(1))}, \tag{3.33}$$

so that, in particular, $\bar{t} \to 0$ as $\lambda \to 1^+$. Putting together (3.29) and (3.31) and recalling the relevant definitions we need to show the following system of inequalities

$$\begin{cases} \text{(i) } K^{-1}\big(F(\lambda)\big)\bar{t}/p' + \beta_o = \lambda, \\ \text{(ii) } \Lambda \bar{t}^{p'-1} = K^{-1}\big(F(\lambda)\big), \\ \text{(iii) } m\Lambda^{p-1} \geqslant f(\lambda)\ell\big(K^{-1}\big(F(\lambda)\big)\big). \end{cases} \tag{3.34}$$

Since, by (3.33),

$$K^{-1}\big(F(\lambda)\big)\frac{\bar{t}}{p'} \leqslant \frac{1}{p'}\frac{K^{-1}(F(2))}{K^{-1}(F(1))}(\lambda - 1)$$

for λ sufficiently close to 1 the first summand on the left-hand side of (i) is strictly less that 1, and therefore we may choose $\beta_o > 0$ in such a way that (i) holds. Next we let Λ be defined by (ii), and note, that

$$\Lambda = K^{-1}\big(F(\lambda)\big)\bar{t}^{1-p'} \geqslant K^{-1}\big(F(1)\big) \to +\infty \quad \text{as } \lambda \to 1^+.$$

Therefore, since

$$f(\lambda)\ell\big(K^{-1}\big(F(\Lambda)\big)\big) \leqslant f(2)\ell\big(K^{-1}\big(F(2)\big)\big),$$

if λ is close enough to 1 then (iii) is also satisfied.

Summing up, if λ is sufficiently close to 1, the function

$$u(x) = \begin{cases} u_1(x) & \text{on } \mathbb{R}^m \setminus B_{\bar{t}}, \\ u_2(x) & \text{on } B_{\bar{t}} \end{cases} \tag{3.35}$$

is a classical weak solution of (3.23).

We remark that we may easily arrange that assumptions $(\varphi\ell)$ and (θ) are also satisfied. Indeed, if we choose, for instance, $\ell(t) = t^q$ with $q \geqslant 0$, then, as already noted in the Introduction, (ℓ) holds for every $p > 1 + q$ and (θ) is verified for every $\theta \in \mathbb{R}$ such that $p \geqslant 2 + q - \theta$.

We also stress that the solution u of (3.23) just constructed is positive and diverges at infinity. Indeed the method used in the proof of Theorem 3.5 may be adapted to yield non-existence of non-constant, non-negative bounded solutions even when $(\neg\text{KO})$ holds. This is the content of the next

Theorem 3.6. *Maintain notation and assumptions of Theorem* 3.5*, except for* (KO) *which is replaced by* (¬KO)*. Then any non-negative, bounded, entire classical weak solution u of the differential inequality* (1.7) *is constant. Furthermore, if $\ell(0) > 0$, then u is identically zero.*

The proof of the theorem follows the lines of that of Theorem 3.5 once we prove the following

Proposition 3.7. *In the assumptions of Proposition* 3.4*, with* (KO) *replaced by* (¬KO)*, there exists $T > 0$ large enough that for every $T \leqslant t_0 < t_1$, and $0 < \epsilon < \eta$, there exists a C^2 function $\alpha : [t_0, +\infty) \to [\epsilon, +\infty)$ which solves the problem*

$$\begin{cases} \varphi'(\alpha')\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant \tilde{b}(t) f(\alpha)\ell(\alpha) & \text{on } [t_0, \bar{T}), \\ \alpha' > 0 \quad \text{on } [t_0, \bar{T}), \quad \alpha(t_0) = \epsilon, \quad \alpha(t) \to +\infty \quad \text{as } t \to +\infty \end{cases} \tag{3.36}$$

and satisfies

$$\epsilon \leqslant \alpha \leqslant \eta \quad \text{on } [t_0, t_1]. \tag{3.37}$$

Proof. The argument is similar to that of Proposition 3.4. The main difference is in the definition of α which now proceeds as follows. We fix $T > 0$ large enough that (b) holds on $[\bar{T}, +\infty)$. For t_0, t_1, ϵ, η as in the statement, and $\sigma \in (0, 1]$ we implicitly define $\alpha : [t_0, +\infty) \to [\epsilon, +\infty)$ by setting

$$\int_{t_0}^{t} \tilde{b}(s)^{\lambda}\, ds = \int_{\epsilon}^{\alpha(t)} \frac{ds}{K^{-1}(\sigma F(s))},$$

so that $\alpha(t_0) = \epsilon$, and, by (b) and (¬KO), $\alpha(t) \to +\infty$ as $t \to +\infty$. The rest of the proof proceeds as in Proposition 3.4. □

Summarizing, the differential inequality (1.7) may admit non-constant, non-negative entire classical weak solutions only if (¬KO) holds, and possible solutions are necessarily unbounded. We shall address this case in Section 5.

4. A further version of Theorem A

As mentioned in the Introduction, condition $(\varphi\ell)$ fails, for instance, when φ is of the form

$$\varphi(t) = \frac{t}{\sqrt{1 + t^2}}$$

which, when $D(x) \equiv 1$, corresponds to the mean curvature operator. Because of the importance of this operator, in Geometry as well as in Analysis, it is desirable to have a version of Theorem A valid when $(\varphi\ell)_2$ fails. To deal with this situation we consider an alternative form of the Keller–Osserman condition, and correspondingly, modify our set of assumptions. We therefore replace assumption $(\varphi\ell)_2$ with

(Φ_2) There exists $C>0$ such that $\varphi(t)\geqslant Ct\varphi'(t)$ on $\mathbb{R}^+$.
$(\varphi\ell)_3$ $\frac{\varphi(t)}{\ell(t)}\in L^1(0^+)\setminus L^1(+\infty)$.

As noted in Remark 3.1, $(\varphi\ell)_3$ is implied by $(\theta)_2$ with $\theta<2$.

It is easy to verify that in the case of the mean curvature operator,

$$t\varphi'(t)=\frac{t}{(1+t^2)^{3/2}}\leqslant\varphi(t)\quad\text{and}\quad\varphi(t)\sim\begin{cases}t & \text{as } t\to 0^+,\\ 1 & \text{as } t\to+\infty,\end{cases}$$

so that (Φ_2) holds, and $(\varphi\ell)_3$ is satisfied provided $t\ell^{-1}\in L^1(0^+)$ and $\ell^{-1}\notin L^1(+\infty)$. By contrast, the choice

$$\varphi(t)=te^{t^2},$$

corresponding to the operator of exponentially harmonic functions, does not satisfy (Φ_2).

According to $(\varphi\ell)_3$, we may define a function $\widehat{K}$ by

$$\widehat{K}(t)=\int_0^t\frac{\varphi(s)}{\ell(s)}\,ds \tag{4.1}$$

which is well defined on $\mathbb{R}_0^+$, tends to $+\infty$ as $t\to+\infty$ and therefore gives rise to a C^1-diffeomorphism of $\mathbb{R}_0^+$ onto itself.

The variant of the generalized Keller–Osserman condition mentioned above is then

$$\frac{1}{\widehat{K}^{-1}(F(t))}\in L^1(+\infty). \tag{$\widehat{\mathrm{K}}$O}$$

Analogues of Lemma 3.1, Propositions 3.3 and 3.4 are also valid in this setting.

Lemma 4.1. *Assume that f, ℓ and φ satisfy the assumptions (F_1), (L_1) and $(\varphi\ell)_3$, and let $\sigma>0$. Then ($\widehat{\mathrm{K}}$O) holds if and only if*

$$\frac{1}{\widehat{K}^{-1}(\sigma F(s))}\in L^1(+\infty). \tag{$\widehat{\mathrm{K}}\mathrm{O}\sigma$}$$

Indeed, the proof of Lemma 3.1 depends only on the monotonicity of K and the C-monotonicity of f, and can be repeated without change replacing K with $\widehat{K}$.

Similarly, using Remark 3.2, one establishes the following

Proposition 4.2. *Assume that conditions (Φ_0) and (L_1) hold, and let F be a positive function defined on $\mathbb{R}_0^+$. If $(\theta)_2$ holds with $\theta<2$, then there exists a constant $B>1$ such that, for every $\sigma\leqslant 1$ we have*

$$\frac{\sigma^{1/(2-\theta)}}{\widehat{K}^{-1}(\sigma F(t))}\leqslant\frac{B}{\widehat{K}^{-1}(F(t))}\quad\text{on } \mathbb{R}^+. \tag{4.2}$$

Finally, we have

Proposition 4.3. *Assume that* (Φ_0), (Φ_2), (F_1), (L_1), (L_2), $(\varphi\ell)_1$, $(\theta)_2$ *and* $(\widehat{\mathrm{K}}\mathrm{O})$ *hold, let* $\tilde{b}$ *be a function satisfying assumption* (b)*, and let* $A > 0$*, and* $\beta \in [-2, +\infty)$*. If* λ *and* θ *are the constants specified in* (b) *and* (θ)*, assume also that*

$$\lambda(2-\theta) \geqslant 1 \quad \text{and either}$$
$$\text{(i) } t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1}\int_1^t \tilde{b}(s)^\lambda\,ds \leqslant C \quad \text{for } t \geqslant t_0, \quad \text{or}$$
$$\text{(ii) } t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1} \leqslant C \quad \text{for } t \geqslant t_0 \text{ and } \theta < 1. \tag{3.7}$$

Then there exists $T > 0$ *sufficiently large such that, for every* $T \leqslant t_0 < t_1$ *and* $0 < \epsilon < \eta$*, there exist* $\bar{T} > t_1$ *and a* C^2 *function* $\alpha : [t_0, \bar{T}) \to [\epsilon, +\infty)$ *which is a solution of the problem*

$$\begin{cases} \varphi'(\alpha')\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant \tilde{b}(t)f(\alpha)\ell(\alpha) & \text{on } [t_0, \bar{T}),\\ \alpha' > 0 \quad \text{on } [t_0, \bar{T}), \quad \alpha(t_0) = \epsilon, \quad \alpha(t) \to +\infty & \text{as } t \to \bar{T}^- \end{cases} \tag{3.8}$$

and satisfies

$$\epsilon \leqslant \alpha \leqslant \eta \quad \text{on } [t_0, t_1]. \tag{3.9}$$

Proof. The proof is a small variation of that of Proposition 3.4, using $\widehat{K}$ instead of K in the definition of α.

Note first of all that (3.7) forces $\theta < 2$, so that $(\varphi\ell)_3$ is automatically satisfied.

Arguing as in Proposition 3.4, one deduces that $\alpha' > 0$ and α satisfies

$$\sigma f(\alpha)\alpha' = \frac{\varphi(\alpha'/\tilde{b}^\lambda)}{\ell(\alpha'/\tilde{b}^\lambda)}\big(\alpha'/\tilde{b}^\lambda\big)', \tag{4.3}$$

so, again, $\alpha'/\tilde{b}^\lambda$ is increasing on $[t_0, T_\sigma)$. From this, using the fact that $t^{\theta-1}\varphi(t)/\ell(t)$ is C-increasing (assumption $(\theta)_2$), $\varphi(t) \geqslant Ct\varphi'(t)$ (assumption (Φ_2)), and $\tilde{b}(t)^{-\lambda} > 1$, we obtain

$$\varphi'(\alpha')\alpha'' \leqslant \big(C\sigma\tilde{b}^{\lambda(2-\theta)-1}\big)bf(\alpha)\ell(\alpha') \tag{4.4}$$

on $[t_0, T_\sigma)$, for some constant $C > 0$. On the other hand, applying (Φ_2) to (4.3), rearranging, integrating over $[t_0, t]$, and using (F_1), (L_2) and the fact that α and $\alpha'/\tilde{b}^\lambda$ are increasing, we deduce that

$$\varphi\big(\alpha'/\tilde{b}^\lambda\big) \leqslant \varphi\big(\alpha'/\tilde{b}^\lambda\big)(t_0) + C\sigma f(\alpha)\ell\big(\alpha'/\tilde{b}^\lambda\big)\int_{t_0}^t \tilde{b}(s)^\lambda\,ds.$$

Finally, using (F_1), (L_2), the fact that α and $\alpha'/\tilde{b}^\lambda$ are non-decreasing, $\alpha(t_0) = \epsilon$ and $(\theta)_2$ we obtain

$$\frac{\varphi(\alpha')}{\ell(\alpha')} \leqslant C\tilde{b}^{\lambda(1-\theta)-1}\left[\frac{\varphi(\alpha'/\tilde{b}^{\lambda})(t_0)}{f(\epsilon)\ell(\alpha'/\tilde{b}^{\lambda})(t_0)} + \sigma\int_{t_0}^{t}\tilde{b}(s)^{\lambda}\right]\tilde{b}f(\alpha). \tag{4.5}$$

Combining (4.4) and (4.5) we conclude that

$$\varphi'(\alpha')'\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant N_\sigma \tilde{b} f(\alpha)\ell(\alpha') \tag{4.6}$$

with $N_\sigma(t)$ defined as in (3.17).

The proof now proceeds exactly as in the case of Proposition 3.4. □

We then have the following version of Theorem 3.5:

Theorem 4.4. *Let $(M, \langle\,,\rangle)$ be a complete Riemannian manifold satisfying*

$$\mathrm{Ricc}_{n,m}(L_D) \geqslant H^2(1+r^2)^{\beta/2}, \tag{1.12}$$

for some $n > m$, $H > 0$ and $\beta \geqslant -2$ and assume that (Φ_0), (Φ_2), (F_1), (L_1), (L_2), $(\varphi\ell)_1$, $(\varphi\ell)_2$ and $(\theta)_2$ hold. Let $b(x) \in C^0(M)$, $b(x) \geqslant 0$ on M and suppose that

$$b(x) \geqslant \tilde{b}(r(x)) \quad \textit{for } r(x) \gg 1, \tag{3.19}$$

where $\tilde{b}$ satisfies assumption (b) *and* (3.7). *If the modified Keller–Osserman condition*

$$\frac{1}{\widehat{K}^{-1}(F(t))} \in L^1(+\infty) \tag{$\widehat{\mathrm{K}}$O}$$

holds then any entire classical weak solution u of the differential inequality

$$L_{D,\varphi}u \geqslant b(x)f(u)\ell(|\nabla u|) \tag{1.7}$$

is either non-positive or constant. Furthermore, if $u \geqslant 0$, and $\ell(0) > 0$, then u vanishes identically.

According to Remark 3.3, Theorem 4.4 holds if we assume that $\tilde{b}(t) = C/t^{\mu}$ for $t \gg 1$ where $\mu \geqslant 0$ and

$$\begin{cases} \theta < 1-\beta/2-\mu \quad \text{or} \quad \theta = 1-\beta/2-\mu < 1 & \text{if } \mu > 0, \\ \theta < 1-\beta/2 & \text{if } \mu = 0. \end{cases} \tag{$\theta\beta\mu$}$$

We note that in the model case of the mean curvature operator with

$$\ell(t) = t^q, \quad q \geqslant 0,$$

then assumptions (Φ_0), (Φ_2), $(\varphi\ell)_1$ and $(\theta)_2$ hold provided

$$(0 \leqslant)q < 1, \qquad \theta \geqslant 1+q$$

and the above restrictions are compatible with $(\theta\beta\mu)$.

5. The weak maximum principle and non-existence of solutions with controlled growth

As shown in Section 3 above, the failure of the Keller–Osserman condition allows to deduce existence of solutions of the differential inequality (1.7). The solutions thus constructed diverge at infinity. This is no accident. Indeed, Theorem B shows that under rather mild conditions on the coefficients and on the geometry of the manifold, if solutions exist, they must be unbounded, and in fact, must go to infinity sufficiently fast.

The proof of Theorem B depends on the following weak maximum principle for the diffusion operator $L_{D,\varphi}$ which improves on the weak maximum principle for the φ-Laplacian already considered in [21,23,22,16]. It is worth pointing out that, besides allowing the presence of a term depending on the gradient of u, we are able to deal with C^1 functions, removing the requirement that $u \in C^2(M)$ and that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ be C^1.

In order to formulate our version of the weak maximum principle, we note that if X is a C^1 vector field, and v a positive continuous function on an open set Ω, then the following two statements hold:

(i) $\inf_\Omega v^{-1} \operatorname{div} X \leqslant C_o$,
(ii) if $\operatorname{div} X \geqslant Cv$ on Ω for some constant C, then $C \leqslant C_o$.

Since (ii) is meaningful in distributional sense, we may take it as the weak definition of (i), and apply it to the case where X is only C^0 (L^∞_{loc} would suffice), and v is only assumed to be non-negative and continuous. Indeed, it is precisely the implication stated in (ii) that will allow us to prove Theorem B.

In view of applications to the case of the diffusion operator $L_{D,\varphi}$, it may also be useful to observe that, if the weight function $D(x)$ is assumed to be C^1 (indeed, $W^{1,1}_{loc}$ is enough if X is assumed to be merely in L^∞_{loc}), then the weak inequality

$$D(x)^{-1} \operatorname{div} X \geqslant Cv$$

is in fact equivalent to the inequality

$$\operatorname{div} X \geqslant CD(x)v.$$

Theorem 5.1. *Let $(M, \langle\,,\rangle)$ be a complete Riemannian manifold, let $D(x) \in C^0(M)$ be a positive weight on M, and let φ satisfy (Φ_1). Given σ, μ, $\chi \in \mathbb{R}$, let*

$$\eta = \mu + (\sigma - 1)(1 + \delta - \chi),$$

and assume that

$$\sigma \geqslant 0, \quad \sigma - \eta \geqslant 0, \quad \textit{and} \quad 0 \leqslant \chi < \delta.$$

Let $u \in C^1(M)$ be a non-constant function such that

$$\hat{u} = \limsup_{r(x)\to+\infty} \frac{u(x)}{r(x)^\sigma} < +\infty, \tag{5.1}$$

and suppose that either

$$\liminf_{r\to+\infty}\frac{\log \mathrm{vol}_D B_r}{r^{\sigma-\eta}} = d_0 < +\infty \quad \text{if } \sigma-\eta>0 \tag{5.2}$$

or

$$\liminf_{r\to+\infty}\frac{\log \mathrm{vol}_D B_r}{\log r} = d_0 < +\infty \quad \text{if } \sigma-\eta=0. \tag{5.3}$$

Suppose that $\gamma\in\mathbb{R}$ is such that the superset $\Omega_\gamma=\{x\in M\colon\ u(x)>\gamma\}$ is not empty, and that the weak inequality

$$\mathrm{div}\big(D(x)|\nabla u|^{-1}\varphi\big(|\nabla u|\big)\nabla u\big) \geqslant K\big(1+r(x)\big)^{-\mu}|\nabla u|^{\chi}D(x) \tag{5.4}$$

holds on Ω_γ. Then the constant K satisfies

$$K \leqslant C(\sigma,\delta,\eta,\chi,d_0)\max\{\hat{u},0\}^{\delta-\chi}, \tag{5.5}$$

where $C=C(\sigma,\delta,\eta,\chi,\delta_0)$ is given by

$$C=\begin{cases} 0 & \text{if } \sigma=0,\\ Ad_0(\sigma-\eta)^{1+\delta-\chi} & \text{if } \sigma>0,\ \eta<0,\\ Ad_0\sigma^{\delta-\chi}(\sigma-\eta) & \text{if } \sigma>0,\ \eta\geqslant 0,\end{cases} \tag{5.6}$$

if $\sigma-\eta>0$ and by

$$C=\begin{cases} 0 & \text{if } \sigma=0 \text{ or } \sigma>0,\ \delta(\sigma-1)+d_0-1\leqslant 0,\\ A\sigma^{\delta-\chi}[\delta(\sigma-1)+d_0-1] & \text{if } \sigma>0,\ \delta(\sigma-1)+d_0-1>0\end{cases} \tag{5.7}$$

if $\sigma-\eta=0$.

Remark 5.1. According to what observed before the statement, if u in C^2, the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is C^1 and $\chi=0$, then the conclusion of the theorem is that

$$\inf_{\Omega_\gamma}\big(1+r(x)\big)^{\mu}L_{D,\varphi}u \leqslant C(\sigma,\delta,\eta,\chi,\delta_0)\max\{\hat{u},0\}^{\delta},$$

and we recover an improved version of Theorem 4.1 in [16].

Proof of Theorem 5.1. The proof is an adaptation of that of Theorem 4.1 in [16]. Clearly we may assume that $K>0$, for otherwise there is nothing to prove.

Note also that since u is assumed to be non-constant, then it cannot be constant on any connected component E_o of Ω_γ. Indeed, if u were constant in E_o, then $\emptyset\neq\partial E_o\subseteq\partial\Omega_\gamma$. Since, by continuity, $u=\gamma$ on $\partial\Omega_\gamma$, we would conclude that $u\equiv\gamma$ on $E_o\subset\Omega_\gamma$, contradicting the fact that $u>\gamma$ on Ω_γ.

Next, because both the assumptions and the conclusions of the theorem are left unchanged by adding a constant to u, arguing as in the proof of Theorem 4.1 in [16] shows that given $b>\max\{\hat{u},0\}$, we may assume that

$$\text{(i) } \frac{u}{(1+r)^\sigma} < b \quad \text{and} \quad \text{(ii) } u(x_o) > 0 \quad \text{for some } x_o \in \Omega_\gamma. \tag{5.8}$$

Further, we observe that if (5.5) follows from (5.4) for some γ then the conclusion holds for any $\gamma' \leqslant \gamma$. Thus, by increasing γ if necessary, we may also suppose that $\gamma > 0$.

We fix $\theta \in (1/2, 1)$ and choose $R_0 > 0$ large enough that $|\nabla u| \not\equiv 0$ on the nonempty set $B_{R_0} \cap \Omega_\gamma$. Given $R > R_0$, let $\psi \in C^\infty(M)$ be a cut-off function such that

$$0 \leqslant \psi \leqslant 1, \qquad \psi \equiv 1 \quad \text{on } B_{\theta R}, \qquad \psi \equiv 0 \quad \text{on } M \setminus B_R, \qquad |\nabla \psi| \leqslant \frac{C}{R(1-\theta)}, \tag{5.9}$$

for some absolute constant $C > 0$. Let also $\lambda \in C^1(\mathbb{R})$ and $F(v, r) \in C^1(\mathbb{R}^2)$ be such that

$$0 \leqslant \lambda \leqslant 1, \qquad \lambda = 0 \quad \text{on } (-\infty, \gamma], \qquad \lambda > 0,\ \lambda' \geqslant 0 \quad \text{on } (\gamma, +\infty), \tag{5.10}$$

and

$$F(v, r) > 0, \qquad \frac{\partial F}{\partial v}(v, r) < 0 \tag{5.11}$$

on $[0, +\infty) \times [0, +\infty)$, where v is given by

$$v = \alpha(1+r)^\sigma - u, \tag{5.12}$$

and α is a constant greater than b, so that $v > 0$ on Ω_γ. Indeed, according to (5.8), and the assumption that $\gamma \geqslant 0$, so that $u > 0$ on Ω_γ, we have

$$(\alpha - b)(1+r)^\sigma \leqslant v \leqslant \alpha(1+r)^\sigma \quad \text{on } \Omega_\gamma. \tag{5.13}$$

By definition of the weak inequality (5.4), for every non-negative test function $0 \leqslant \rho \in H_0^1(\Omega_\gamma)$,

$$-\int_{\Omega_\gamma} \langle \nabla \rho, |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \rangle D(x)\, dx \geqslant K \int_{\Omega_\gamma} \rho (1+r)^{-\mu} |\nabla u|^\chi D(x)\, dx.$$

We use as test function the function $\rho = \psi^{1+\delta} \lambda(u) F(v, r)$ which is non-negative, Lipschitz, compactly supported in M and vanishes on $M \setminus (\Omega_\gamma \cap B_R(o))$. Inserting the expression for $\nabla \rho$ in the above integral inequality, using the conditions $\lambda' > 0$, $F(v, r) > 0$, $\partial F / \partial v < 0$, and $|\nabla u| \leqslant A^{-1/\delta} \varphi(|\nabla u|)^{1/\delta}$, which in turn follows from the structural condition $\varphi(t) \leqslant A t^\delta$, after some computations we obtain

$$(1+\delta) \int \psi^\delta \lambda(u) F(v, r) \varphi(|\nabla u|) |\nabla \psi| D(x)\, dx \geqslant \int \psi^{1+\delta} \lambda(u) \left| \frac{\partial F}{\partial v} \right| B(u, r) D(x)\, dx, \tag{5.14}$$

where

$$\begin{aligned}B(u,r) &= A^{-1/\delta}\varphi\big(|\nabla u|\big)^{1+1/\delta}\\ &\quad + KA^{-\chi/\delta}\frac{F(v,r)}{|\partial F/\partial v|}(1+r)^{-\mu}\varphi\big(|\nabla u|\big)^{\chi/\delta}\\ &\quad + \left(\frac{\partial F/\partial r}{|\partial F/\partial v|} - \alpha\sigma(1+r)^{\sigma-1}\right)|\nabla u|^{-1}\varphi\big(|\nabla u|\big)\langle\nabla r, \nabla u\rangle. \end{aligned} \tag{5.15}$$

Now one needs to consider several cases separately. We treat in detail only the case where M satisfies the volume growth condition (5.2), $\sigma > 0$, and $\eta < 0$.

In this case we let

$$F(v,r) = \exp\big[-qv(1+r)^{-\eta}\big],$$

where $q > 0$ is a constant that will be specified later. An elementary computation which uses the estimate for v given in (5.13) shows that

$$0 \geqslant \frac{\frac{\partial F}{\partial r}(v,r)}{|\frac{\partial F}{\partial v}(v,r)|} - \alpha\sigma(1+r)^{\sigma-1} \geqslant -\alpha(\sigma-\eta)(1+r)^{\sigma-1} \tag{5.16}$$

and

$$\frac{F(v,r)}{|\frac{\partial F}{\partial v}(v,r)|} = \frac{1}{q}(1+r)^{\eta}. \tag{5.17}$$

Inserting (5.16) and (5.17) into (5.15), and using the Cauchy–Schwarz inequality we deduce that

$$\begin{aligned}B(u,r) \geqslant \varphi\big(|\nabla u|\big)^{\chi/\delta}\bigg\{&\frac{1}{A^{1/\delta}}\varphi\big(|\nabla u|\big)^{\frac{\delta+1-\chi}{\delta}} + \frac{K}{qA^{\chi/\delta}}(1+r)^{(1+\delta-\chi)(\sigma-1)}\\ &- \alpha(\sigma-\eta)(1+r)^{\sigma-1}\varphi\big(|\nabla u|\big)^{\frac{\delta-\chi}{\delta}}\bigg\}. \end{aligned} \tag{5.18}$$

In order to estimate the right-hand side of (5.18) we use the following calculus result (see [16], Lemma 4.2): let ν, ρ, β, ω be positive constants, and let f be the function defined on $[0,+\infty)$ by $f(s) = \omega s^{1+\nu} + \rho - \beta s^{\nu}$. Then the inequality $f(s) \geqslant \Lambda s^{1+\nu}$ holds on $[0,+\infty)$ provided

$$\Lambda \leqslant \omega - \frac{\nu\beta^{1+1/\nu}}{(1+\nu)^{1+1/\nu}\rho^{1/\nu}}. \tag{5.19}$$

Applying this result with $\nu = \delta - \chi$ and $s = \varphi(|\nabla u|)^{1/\delta}$, and recalling the definition of η we deduce that the estimate

$$B(u,r) \geqslant \Lambda\varphi\big(|\nabla u|\big)^{1+1/\delta} \tag{5.20}$$

holds provided

$$\Lambda \leqslant \frac{1}{A^{1/\delta}} - \frac{\nu q^{1/\nu}A^{\chi/\delta\nu}[\alpha(\sigma-\eta)]^{1+1/\nu}}{(1+\nu)^{1+1/\nu}K^{1/\nu}}. \tag{5.21}$$

In particular, given $\tau \in (0,1)$ if we let

$$\Lambda = \frac{1-\tau}{A^{1/\delta}} \quad \text{and} \quad q = \frac{\tau^{\nu}(1+\nu)^{1+\nu}}{\nu^{\nu} A[\alpha(\sigma-\eta)]^{1+\nu}} K, \tag{5.22}$$

then Λ is positive, and satisfies (5.21) with equality.

Inserting (5.20) and the expression for $\partial F/\partial v$ into (5.14), we deduce that

$$\frac{q\Lambda}{1+\delta} \int_{\Omega_\gamma \cap B_R} \psi^{1+\delta} \lambda(u) F(v,r)(1+r)^{-\eta} \varphi\big(|\nabla u|\big)^{1+1/\delta} D(x)\,dx \leqslant \int_{\Omega_\gamma \cap B_R} \psi^{\delta} \lambda(u) F(v,r) |\nabla \psi| \varphi\big(|\nabla u|\big) D(x)\,dx.$$

Now the proof proceeds as in [16]: applying Hölder inequality with conjugate exponents $1+\delta$ and $1+1/\delta$ to the integral on the right-hand side, and simplifying we obtain

$$\left(\frac{q\Lambda}{1+\delta}\right)^{1+\delta} \int_{\Omega_\gamma \cap B_R} \psi^{1+\delta} \lambda(u) F(v,r)(1+r)^{-\eta} \varphi\big(|\nabla u|\big)^{1+1/\delta} D(x) \leqslant \int_{\Omega_\gamma \cap B_R} \lambda(u) F(v,r)(1+r)^{\eta\delta} |\nabla \psi|^{1+\delta} D(x). \tag{5.23}$$

By the volume growth assumption (5.2), for every $d > d_0$, there exists a diverging sequence $R_k \uparrow +\infty$ with $R_1 > 2R_0$ such that

$$\log \operatorname{vol} B_{R_k} \leqslant d R_k^{\sigma-\eta}. \tag{5.24}$$

Since $\theta R_k > R_k/2 > R_0$, we may let $R = R_k$ in (5.23), and use the support properties of ψ, the estimate for $|\nabla \psi|$, and the fact that $\lambda \leqslant 1$, $\eta < 0$ to show that

$$E = \left(\frac{q\Lambda}{1+\delta}\right)^{1+\delta} \int_{\Omega_\gamma \cap B_{R_0}} \lambda(u) F(v,r) \varphi\big(|\nabla u|\big)^{1+1/\delta} D(x) \leqslant C^{1+\delta}(1+\theta R_k)^{\eta\delta} \big[(1-\theta)R_k\big]^{-(1+\delta)} \int_{\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})} F(v,r) D(x). \tag{5.25}$$

Now, since $|\nabla u| \not\equiv 0$ on $\Omega_\gamma \cap B_{R_0}$, then $E > 0$. On the other hand, using the bound (5.13) for v, and the expression of F we get

$$F(v,r) \leqslant \exp\big(-q(\alpha-b)(1+\theta R_k)^{\sigma-\eta}\big)$$

on $\Omega_\gamma \cap (B_{R_k} \setminus B_{\theta R_k})$, so inserting this into the right-hand side of (5.25) we conclude that

$$0 < E \leqslant C R_k^{\delta\eta-(1+\delta)} \exp\bigl(d R_k^{\sigma-\eta} - q(\alpha - b)(1+\theta R_k)^{\sigma-\eta}\bigr), \tag{5.26}$$

where C is a constant independent of k. In order for this inequality to hold for every k, we must have

$$d \geqslant (\alpha - b)q\theta^{\sigma-\eta},$$

whence, letting θ tend to 1,

$$d \geqslant (\alpha - b)q.$$

We set $\alpha = tb$, insert the definition (5.22) of q in the above inequality, solve with respect to K, and then let τ tend to 1 to obtain

$$K \leqslant Adb^{\nu}(\sigma - \eta)^{1+\nu} \frac{\nu^{\nu}}{(1+\nu)^{1+\nu}} \frac{t^{1+\nu}}{t-1}.$$

The conclusion is then obtained minimizing with respect to $t > 1$, letting $d \to d_0$ and $b \to \max\{\hat{u}, 0\}$ and recalling that $\nu = \delta - \chi$.

The other cases are treated adapting the arguments carried out in the proof of [16], Theorem 4.1, cases II and III, and of Theorem 4.3 for the case of polynomial volume growth. □

Proof of Theorem B. We begin by showing that if under the assumptions of the theorem, u is necessarily bounded above. Indeed, assume by contradiction that $u^* = +\infty$, so that, by (1.14), $\sigma > 0$, and there exist γ_o and $C > 0$ such that $f(t) > C$ for $t \geqslant \gamma$. Keeping into account the assumptions on b and ℓ, we deduce that u satisfies the differential inequality

$$\operatorname{div}\bigl(D(x)|\nabla u|^{-1}\varphi\bigl(|\nabla u|\bigr)\nabla u\bigr) \geqslant K\bigl(1 + r(x)\bigr)^{-\mu}|\nabla u|^{\chi} D(x)$$

weakly on Ω_{γ_o}, with a constant $K > 0$. On the other hand, because of growth assumption on u, the constant $\hat{u}$ in the statement of Theorem 5.1 is equal to zero, and this shows that $K = 0$, and the contradiction shows that $u^* < +\infty$ is bounded above.

Assume now that $f(u^*) > 0$. Since $f(t) > 0$ for $t > 0$, by continuity there exists γ_o such that $f(u) \geqslant C > 0$ on Ω_{γ_o}, and a contradiction is reached as above.

The final statement follows immediately from this and from the assumptions. □

6. Proof of Theorem C

The aim of this section is to prove Theorem C in the Introduction together with a version covering the case of the mean curvature operator. Before proceeding, we analyze the Keller–Osserman condition

$$\frac{e^{\int_0^t \rho(z)\,dz}}{K^{-1}(\widehat{F}(t))} \in L^1(+\infty), \tag{ρKO}$$

where $\rho \in C^0(\mathbb{R}_0^+)$ is non-negative on $\mathbb{R}_0^+$ and $\widehat{F}(t) = F_{\rho,\omega}$ is defined in (1.20), namely,

$$F_{\rho,\omega}(t) = \int_0^t f(s) e^{(2-\omega)\int_0^s \rho(z)\,dz}\,ds. \tag{1.20}$$

Lemma 6.1. *Assume that* (F_1)*,* (L_1) *and the first part of* $(\theta)_1$ *with* $\theta < 2$ *hold, and let* $\omega = \theta$ *and* $\sigma \in \mathbb{R}^+$*. Then* $(\rho\mathrm{KO})$ *is equivalent to*

$$\frac{e^{\int_0^t \rho(z)\,dz}}{K^{-1}(\sigma \widehat{F}(t))} \in L^1(+\infty). \tag{$\rho\mathrm{KO}_\sigma$}$$

Proof. Assume first that $\sigma \leqslant 1$. Since K^{-1} is non-decreasing,

$$\frac{1}{K^{-1}(\widehat{F}(t))} \leqslant \frac{1}{K^{-1}(\sigma \widehat{F}(t))}$$

and $(\rho\mathrm{KO}_\sigma)$ implies $(\rho\mathrm{KO})$. On the other hand, according to Proposition 3.3 and Remark 3.2 there exists a constant $B \geqslant 1$ such that

$$\frac{\sigma^{1/(2-\theta)}}{K^{-1}(\sigma \widehat{F}(t))} \leqslant \frac{B}{K^{-1}(\widehat{F}(t))} \quad \text{on } \mathbb{R}^+,$$

and $(\rho\mathrm{KO})$ implies $(\rho\mathrm{KO}_\sigma)$. Thus the stated equivalence holds when $\sigma \leqslant 1$. Then the case $\sigma \geqslant 1$ follows as in Lemma 3.1. □

We observe that in favorable circumstances (KO) and $(\rho\mathrm{KO})$ are indeed equivalent. For instance we have

Proposition 6.2. *Assume that* (F_1)*,* (L_1)*,* $(\varphi\ell)_2$ *and* (ρ) *hold. If* $\rho \in L^1(+\infty)$ *and* $\omega \leqslant 2$ *then* $(\rho\mathrm{KO})$ *is equivalent to* (KO)*.*

Proof. Observe first of all that since $0 \leqslant \rho \in L^1((0,+\infty))$ $(\rho\mathrm{KO})$ is equivalent to

$$\frac{1}{K^{-1}(\widehat{F}(t))} \in L^1(+\infty). \tag{6.1}$$

Since $\omega \leqslant 2$ we also have

$$1 \leqslant e^{(2-\omega)\int_0^s \rho(z)\,dz} \leqslant \Lambda,$$

and therefore

$$F(t) = \int_0^t f(s)\,ds \leqslant \widehat{F}(t) = \int_0^t f(s) e^{(2-\omega)\int_0^s \rho(z)\,dz} \leqslant \Lambda F(t). \tag{6.2}$$

Recalling that K^{-1} is increasing, the left-hand side inequality in (6.2) shows that

$$\int^{+\infty} \frac{dt}{K^{-1}(\widehat{F}(t))} \leqslant \int^{+\infty} \frac{dt}{K^{-1}(F(t))}$$

and, by (6.1), (KO) implies (ρKO).

On the other hand, since, by (F_1), f is C-increasing with constant $C \geqslant 1$, so is also the integrand in the definition of $\widehat{F}$, and therefore the right-hand side inequality in (6.2) and the argument in the proof of Lemma 3.1, with $\sigma = \Lambda^{-1}$ and F replaced by $\widehat{F}$, show that

$$\int^{+\infty} \frac{ds}{K^{-1}(F(s))} \leqslant \int^{+\infty} \frac{ds}{K^{-1}(\Lambda^{-1}\widehat{F}(s))} \leqslant C\Lambda \int^{+\infty} \frac{dt}{K^{-1}(\widehat{F}(t))}, \tag{6.3}$$

and, again by (6.1), (ρKO) implies (KO). □

Remark 6.1. The above proposition generalizes Proposition 6.1 in [12].

Proposition 6.3. *Assume that* (Φ_0), (F_1), (L_1), (L_2), $(\varphi\ell)_1$, (θ), (b), (ρ) *and* (ρKO) *with* $\omega = \theta$ *hold. Let* $A > 0$, $\beta \geqslant -2$, *and, if* $\lambda > 0$ *and* θ *are the constants in* (b) *and* (θ), *suppose that* $\theta \leqslant 1$ *and*

$$\begin{cases} \lambda \geqslant 1, \quad t^{\beta/2}\tilde{b}(t)^{-1} \displaystyle\int_1^t \tilde{b}(s)^{\lambda}\, ds \leqslant C \quad \forall t \geqslant 1 \qquad \text{if } \theta = 1, \\ \lambda(2-\theta) \geqslant 1, \quad t^{\beta/2}\tilde{b}(t)^{\lambda(1-\theta)-1} \leqslant C \quad \forall t \geqslant 1 \quad \text{if } \theta < 1, \end{cases} \tag{6.4}$$

for come constant $C > 0$. *The there exists* $T > 0$ *sufficiently large such that, for every* $T \leqslant t_0 < t_1$ *and* $0 < \epsilon < \eta$, *there exist* $\bar{T} > t_1$ *and a* C^2 *function* $\alpha : [t_0, \bar{T}) \to [\epsilon, +\infty)$ *which is a solution of the problem*

$$\begin{cases} \varphi'(\alpha')\alpha'' + At^{\beta/2}\varphi(\alpha') \leqslant \tilde{b}(t) f(\alpha)\ell(\alpha) - \rho(\alpha)\varphi'(\alpha')(\alpha')^2 \quad \text{on } [t_0, \bar{T}), \\ \alpha' > 0 \quad \text{on } [t_0, \bar{T}), \quad \alpha(t_0) = \epsilon, \quad \alpha(t) \to +\infty \quad \text{as } t \to \bar{T}^- \end{cases} \tag{6.5}$$

and satisfies

$$\epsilon \leqslant \alpha \leqslant \eta \quad \text{on } [t_0, t_1]. \tag{6.6}$$

Proof. The proof is a modification of that of Proposition 3.4 so we only sketch it.

Note that since $(\theta)_1$ holds with $\theta \leqslant 1$, so does $(\varphi\ell)_2$. Thus K defines a C^1-diffeomorphism of $\mathbb{R}_0^+$ and condition (ρKO) is meaningful.

As in the proof of Proposition 3.4, we may assume that $\tilde{b} \leqslant 1$ for t large. Choose $T > 0$ large enough that $\tilde{b}'(t) \leqslant 0$ and $0 < \tilde{b}(t) \leqslant 1$ in $[T, +\infty)$, let t_0, t_1, ϵ, η be as in the statement, use Lemma 6.1, (b) and condition (ρKO), to define T_σ by means of the formula

$$\int_{t_0}^{T_\sigma} \tilde{b}(s)^\lambda\,ds = \int_{\epsilon}^{+\infty} \frac{e^{\int_0^s \rho}}{K^{-1}(\sigma \widehat{F}(s))},$$

and choose $\sigma \in (0,1]$ small enough to guarantee that $T_\sigma > t_1$.

Next let $\alpha : [t_0, T_\sigma) \to [\epsilon, +\infty)$ be defined by the formula

$$\int_{t}^{T_\sigma} \tilde{b}(s)^\lambda\,ds = \int_{\alpha(t)}^{+\infty} \frac{e^{\int_0^s \rho}}{K^{-1}(\sigma \widehat{F}(s))},$$

so that

$$\alpha(t_0) = \epsilon \quad \text{and} \quad \alpha\bigl(T_\sigma^-\bigr) = +\infty.$$

Differentiating we obtain

$$\alpha' = \tilde{b}^\lambda K^{-1}(\sigma \widehat{F}) e^{-\int_0^\alpha \rho},$$

so that $\alpha' > 0$, and rearranging, differentiating once again, and simplifying we obtain

$$\sigma f(\alpha) e^{(2-\theta)\int_0^\alpha \rho} = \left(\frac{e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right) \frac{\varphi'(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})}{\ell(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})} \left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right)', \tag{6.7}$$

so that, in particular, $(\alpha' e^{\int_0^\alpha \rho}/\tilde{b}^\lambda)' > 0$.

We use the fact that $e^{\int_0^\alpha \rho}/\tilde{b} \geqslant 1$ to apply $(\theta)_1$, we expand the derivative of $(\alpha' e^{\int_0^\alpha \rho}/\tilde{b}^\lambda)$, use $\tilde{b}' \leqslant 0$, and rearrange to obtain

$$\varphi'\bigl(\alpha'\bigr)\alpha'' \leqslant C\sigma f(\alpha)\ell\bigl(\alpha'\bigr)\tilde{b}^{\lambda(2-\theta)} - \rho(\alpha)\varphi'\bigl(\alpha'\bigr)^2. \tag{6.8}$$

On the other hand, we rewrite (6.7) in the form

$$\varphi'\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right)\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right)' = \sigma \tilde{b}^\lambda f(\alpha)\ell\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right) e^{(1-\theta)\int_0^\alpha \rho},$$

integrate between t_0 and t, and use the C-monotonicity of f and ℓ and $(\theta)_2$ to obtain

$$\varphi\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right) - \varphi\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right)(t_0) \leqslant C\sigma f(\alpha) e^{(1-\theta)\int_0^\alpha \rho} \ell\left(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right) \int_0^t \tilde{b}^\lambda,$$

whence, rearranging and using the C-monotonicity of $t^{\theta-1}\varphi(t)/\ell(t)$, f and ℓ, and the $\theta \leqslant 1$ shows that (see the argument that led to (3.15) in the proof of Proposition 3.4)

$$\frac{\varphi(\alpha')}{\ell(\alpha')} \leqslant C\left(\frac{e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda}\right)^{\theta-1} \frac{\varphi(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})}{\ell(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})}$$

$$\leqslant C\tilde{b} f(\alpha)\left\{\sigma \tilde{b}^{\lambda(1-\theta)-1}\int_0^t \tilde{b}^\lambda + \frac{\varphi(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})(t_0)}{f(\epsilon)\ell(\frac{\alpha' e^{\int_0^\alpha \rho}}{\tilde{b}^\lambda})}(t_0)\right\}. \tag{6.9}$$

Thus, combining (6.8) and (6.9) and arguing as in Proposition 3.4 we deduce that

$$\varphi'(\alpha')\alpha'' + At^\beta/2\varphi(\alpha') \leqslant N(\sigma)\tilde{b} f(\alpha)\ell(\alpha') - \rho(\alpha)\varphi'(\alpha')(\alpha')^2$$

with

$$N_\sigma(t) = C\sigma\tilde{b}^{\lambda(2-\theta)-1} + ACt^{\beta/2}\tilde{b}^{\lambda(1-\theta)-1}\frac{\varphi(K^{-1}(\sigma F(\epsilon)))}{\ell(K^{-1}(\sigma F(\epsilon)))f(\epsilon)}$$
$$+ AC\sigma t^{\beta/2}\tilde{b}^{\lambda(1-\theta)-1}\int_{t_0}^t \tilde{b}(s)^\lambda.$$

The proof now proceeds exactly as in Proposition 3.4. □

The next result is the analogue of Theorem 3.5 and Theorem C in the Introduction follows from it using Remark 3.3.

Theorem 6.4. *Let $(M, \langle\,,\rangle)$ be a complete manifold satisfying*

$$\mathrm{Ricc}_{n,m}(L_D) \geqslant H^2(1+r^2)^{\beta/2}, \tag{6.10}$$

for some $n > m$, $H > 0$ and $\beta \geqslant -2$, and assume that (h), (g), (ρ), (Φ_0), (F_1), (L_1) (L_2), $(\varphi\ell)_1$ and (θ) hold. Let also $b(x) \in C^0(M)$ be strictly positive on M and such that

$$b(x) \geqslant \tilde{b}(r(x)) \quad \textit{for } r(x) \gg 1, \tag{6.11}$$

with $\tilde{b}$ satisfying (b), *and* (6.4). *Finally, suppose that* $(\rho\mathrm{KO})$ *holds with $\omega = \theta$ in the definition of $\widehat{F}$. Then any entire classical weak solution of the differential inequality*

$$L_{D,\varphi}u \geqslant b(x) f(u)\ell(|\nabla u|) - g(u)h(|\nabla u|) \tag{1.19}$$

is either non-positive or constant. Moreover, if $u \geqslant 0$ and $\ell(0) > 0$, then $u \equiv 0$.

Proof. The proof is modeled on that of Theorem 3.5. However, in the case where u is bounded above, in order to prove that, if u takes on positive values and is non-constant then

$$u_o^* = \sup_{B_{r_o}} u < \sup u = u^*,$$

we argue as follows. Assume that u attains its supremum $u^* > 0$ and let $\Gamma = \{x\colon\ u(x) = u^*\}$. Clearly Γ is closed and nonempty. We are going to show that it is also open so, by connectedness, $\Gamma = M$ and u is constant. To this end, let $x_o \in \Gamma$. We have $b(x)f(u) \geqslant \frac{1}{2}b(x_o)f(u^*) > 0$ and $g(u) \leqslant 2C\rho(u^*)$ in a suitable neighborhood U of x_o. Moreover, by $(\theta)_1$ and (h), we may estimate

$$h(s) \leqslant Cs^2\varphi'(s) \leqslant C\frac{\varphi'(1)}{\ell(1)}s^{2-\theta}\ell(s) = Cs^{2-\theta}\ell(s) \quad \forall s \leqslant 1,$$

so that, in U,

$$b(x)f(u)\ell\big(|\nabla u|\big) - g(u)h\big(|\nabla u|\big) \geqslant \ell\big(|\nabla u|\big)\left(\frac{b(x_o)}{2}f\big(u^*\big) - C\rho\big(u^*\big)|\nabla u|^{2-\theta}\right).$$

Since $\nabla u(x_o) = 0$ it is now clear that there exists a neighborhood $U' \subset U$ of x_o where the right-hand side the above inequality is non-negative. Thus,

$$L_{D,\varphi}u \geqslant 0 \quad \text{in } U'$$

and $u = u^*$ in U' by the strong maximum principle.

We note in passing that if $\ell(0) > 0$ the required conclusion may be obtained without having to appeal to condition $(\theta)_1$.

The rest of the proof proceeds as in Theorem 3.5 using Proposition 6.3 instead of Proposition 3.4. □

As we did for Theorem 3.5 in Section 3, even in this case we can provide a version of the above result valid for a class of operators which include the mean curvature operator. In order to do this we need to introduce the appropriate Keller–Osserman condition. Given $\omega \in \mathbb{R}$, let ρ satisfy (ρ) and let $\widehat{F}$ be defined in (1.20). We assume $(\varphi\ell)_3$ holds and let $\widehat{K}$ be defined in (4.1). The version of Keller–Osserman condition we consider is then

$$\frac{e^{\int_0^t \rho}}{\widehat{K}^{-1}(\widehat{F}(t))} \in L^1(+\infty). \qquad (\rho\widehat{\mathrm{K}}\mathrm{O})$$

Modifications of the arguments of Section 4 allow to obtain the following

Theorem 6.5. *Let $(M, \langle\, ,\rangle)$ be a complete manifold satisfying* (6.10) *for some $n > m$, $H > 0$ and $\beta \geqslant -2$, and assume that (h), (g), (ρ), (Φ_0), (F_1), (L_1) (L_2), $(\varphi\ell)_1$ and (θ) hold. Let also $b(x) \in C^0(M)$ be strictly positive on M and satisfying* (6.11) *with $\tilde{b}$ satisfying* (b)*, and* (6.4). *Finally, suppose that $(\rho\widehat{\mathrm{K}}\mathrm{O})$ holds with $\omega = \theta$ in the definition of $\widehat{F}$. Then any entire classical weak solution of the differential inequality*

$$L_{D,\varphi}u \geqslant b(x)f(u)\ell\big(|\nabla u|\big) - g(u)h\big(|\nabla u|\big) \qquad (1.19)$$

is either non-positive or constant. Moreover, if $u \geqslant 0$ and $\ell(0) > 0$, then $u \equiv 0$.

We leave the details to the interested reader, and merely point out that, according to what remarked in the proof of Theorem 6.4, if $\ell(0) > 0$ then it suffices to assume $(\theta)_2$ in the statement of Theorem 6.5.

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