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On the probability that integrated random walks stay positive

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Abstract

Let S_n be a centered random walk with a finite variance, and consider the sequence $A_n := \sum_{i=1}^{n} S_i$, which we call an *integrated random walk*. We are interested in the asymptotics of

$$p_N := \mathbb{P}\left\{ \min_{1 \le k \le N} A_k \ge 0 \right\}$$

as $N \to \infty$. Sinai (1992) [15] proved that $p_N \asymp N^{-1/4}$ if S_n is a simple random walk. We show that $p_N \asymp N^{-1/4}$ for some other kinds of random walks that include double-sided exponential and double-sided geometric walks, both not necessarily symmetric. We also prove that $p_N \le cN^{-1/4}$ for integer-valued walks and upper exponential walks, which are the walks such that $\text{Law}(S_1 | S_1 > 0)$ is an exponential distribution.

MSC: 60G50; 60F99

Keywords: Integrated random walk; Area of random walk; Unilateral small deviations; One-sided exit probability; Excursion; Area of excursion

1. Introduction

Let S_n be a centered random walk with a finite variance, and consider the sequence of r.v.'s $A_n := \sum_{i=1}^{n} S_i$, which we call an *integrated random walk*. We are interested in the asymptotical

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behavior of the probabilities

$$p_N := \mathbb{P}\left\{\min_{1\le k\le N} A_k \ge 0\right\}$$

as $N \to \infty$. We came to this problem while studying properties of so-called sticky particle systems; see [19]. One may consider this question as a particular case of the general problem on finding one-sided small deviation probabilities of a random sequence.

The only known sharp result on p_N is due to Sinai [15], who showed that $p_N \asymp N^{-1/4}$ for a simple random walk. Sinai studied this problem in connection with solutions of the Burgers equation with random initial data. Caravenna and Deuschel [2] considered such probabilities in relation to random polymers, and they obtained a rough non-polynomial upper bound for p_N for general random walks. A rough lower bound is given by the trivial $p_N \ge \mathbb{P}\{\min_{1\le k\le N} S_k \ge 0\} \sim cN^{-1/2}$.

For the continuous version of the problem,

$$\mathbb{P}\left\{\min_{0\le s\le N} \int_0^s W(u)\mathrm{d}u \ge -1\right\} \sim cN^{-1/4}, \tag{1}$$

where $W(u)$ is a Wiener process and c is a positive constant that could be found explicitly. This result of Isozaki and Watanabe [9] refines a weaker version of (1) obtained by Sinai [15], who had $\asymp$ instead of $\sim$ in the right-hand side. Isozaki and Watanabe actually conclude (1) from the results of McKean [12].

These asymptotical results of [9,15] prompted the author to conjecture in [19] that $p_N \asymp N^{-1/4}$ for any centered random walk with a finite variance. In this paper we obtain several results that partially prove the conjecture. Note that it seems impossible to get the relation $p_N \asymp N^{-1/4}$ directly from (1) because even if $S_n = W(n)$ is a standard Gaussian random walk, $\int_0^n W(u)\mathrm{d}u - \sum_{i=1}^n W(i)$ has order $n^{1/2}$.

Let us first state a result on the upper bound for p_N. We say that a r.v. X is *upper exponential* if $\mathrm{Law}(X|X > 0)$ is an exponential distribution. A typical example is an exponential r.v. centered by its expectation. An integer-valued r.v. X is called *upper geometric* if $\mathrm{Law}(X|X > 0)$ is a geometric distribution. In what follows, we refer to random walks by the type of common distribution of their increments.

Theorem 1. *Let S_n be a centered random walk with a finite variance that is either integer-valued or upper exponential. Then $p_N \le cN^{-1/4}$ for some constant $c > 0$.*

Our proof is based on the fact that any integer-valued random walk S_n with $\mathbb{E}S_1 = 0$ and $\mathrm{Var}(S_1) < \infty$ returns to zero almost surely. This of course does not hold for the "continuous" case, and we need to impose the condition of upper exponentiality. It is unclear whether it is possible to remove this additional assumption using discretization and the result for integer-valued walks: the discretized centered walk should be also centered. On the other hand, it worth citing the comment from the book of Feller [8, p. 404]: "At first sight the distribution F [an upper exponential distribution] ... appears artificial, but the type turns up frequently in connection with Poisson processes, queuing theory, ruin problems, etc.". Moreover, Theorem 1 is important for the results of [19], where the primary interest was in exponential walks centered by expectation.

We prove lower bounds for p_N under more restrictive conditions, which are imposed on $\mathrm{Law}(S_1|S_1 < 0)$. A r.v. X is called *two-sided exponential* if both X and $-X$ are upper exponential. A typical example is the Laplace distribution but two-sided exponential distributions are not necessarily symmetric. Further, we follow Spitzer [16] and say that a r.v. X is *right-continuous* if $\mathbb{P}\{X \in \{\dots, -1, 0, 1\}\} = 1$. Finally, define a *slackened simple random walk* as a nondegenerate symmetric right-continuous walk. Informally speaking, these are simple random walks allowed to stay immobile.

Note that upper exponential, upper geometric, and right-continuous random walks have the same common property, which plays the key role in our proofs: the overshoot over any fixed level is independent of the moment when its occurs and also of the trajectory of the walk up to this moment.

Theorem 2. 1. *Let S_n be a centered random walk such that both S_n and $-S_n$ are either upper geometric or right-continuous. Then $N^{-1/4} l(N) \le p_N$ for some function $l(n)$ that is slowly varying at infinity.*
2. *Let S_n be a centered random walk that is either double-sided exponential or satisfies conditions of Part* 1 *and is symmetric. Then $cN^{-1/4} \le p_N$ for some constant $c > 0$.*

Note that Part 1 covers walks that are lower geometric and right-continuous or vice versa, and both Parts 1 and 2 cover walks with $\mathbb{P}\{S_1 = 0\} > 0$. From Theorems 1 and 2, we conclude the following.

Corollary. *Let S_n be a centered random walk that is two-sided exponential, slackened simple, or symmetric two-sided geometric. Then $p_N \asymp N^{-1/4}$.*

We prove the upper bound following the main idea of the proof of Sinai [15], although we make significant simplifications. For the lower bound, only a sketch of the proof was given in [15] but all interesting details were omitted. We failed to conclude these missing arguments, and therefore we prove the lower bounds in an entirely different way. In fact, [15] implicitly uses a local limit theorem for bivariate walks whose first component is conditioned to stay positive and, as the main difficulty, has increments from the domain of attraction of an α-stable law (with $\alpha = 1/3$). It was only recently that Vatutin and Wachtel [17] proved a weaker result, a local limit theorem for such heavy-tailed (univariate) walks conditioned to stay positive. Thus, the other contribution of our paper is the first complete proof of the lower bound for p_N.

The paper is organized as follows. In Section 2 we give a heuristic explanation of why $p_N \asymp N^{-1/4}$ for a simple random walk, and then develop and generalize the basic idea of this heuristic approach making it applicable to the random walks considered here. In Section 3 we prove preparatory results on durations and areas of "cycles" of random walks; a *cycle* is a positive excursion together with the consecutive negative excursion. In particular, in Proposition 1 we find the asymptotics of the "tail" of the joint distribution of these variables. This simplifies and generalizes the analogous result of Sinai [15] obtained by sophisticated but tedious arguments which work only for simple random walks. In Sections 4 and 5 we prove upper and lower bounds for p_N, respectively. Finally, in Section 6 we make concluding remarks and discuss possible ways to prove the lower bound under less restrictive conditions.

2. From heuristics to proofs

2.1. *Heuristics for the asymptotics of p_N*

Let us give a heuristic explanation of why $p_N \asymp N^{-1/4}$ for a simple random walk. We took the following arguments from the survey paper [18], which provides a simple informal explanation of the complicated proofs of Sinai [15]. The approach itself was introduced in [15] although p_N was estimated there in a different way.

The main idea of Sinai's method is to decompose the trajectory of the random walk S_k into independent excursions. Define the moments of hitting zero as $\tau_0^0 := 0$ and $\tau_{n+1}^0 := \min\{k > \tau_n^0 : S_k = 0\}$ for $n \geq 0$. Let $\theta_n^0 := \tau_n^0 - \tau_{n-1}^0$ be durations of excursions, let $\xi_n^0 := \sum_{i=\tau_{n-1}^0+1}^{\tau_n^0} S_i$ be their areas, and let $\eta^0(N)$ be the number of complete excursions by the time N, namely, $\eta^0(N) := \max\{k \geq 0 : \tau_k^0 \leq N\} = \max\{k \geq 0 : \sum_{i=1}^k \theta_i^0 \leq N\}$. Since for each n it holds that

$$\left\{ \min_{1 \leq k \leq \tau_n^0} \sum_{i=1}^k S_i \geq 0 \right\} = \left\{ \min_{1 \leq k \leq n} \sum_{i=1}^k \xi_i^0 \geq 0 \right\},$$

as $\tau_{\eta^0(N)}^0 \leq N < \tau_{\eta^0(N)+1}^0$, we have

$$\mathbb{P}\left\{ \min_{1 \leq k \leq \eta^0(N)+1} \sum_{i=1}^k \xi_i^0 \geq 0 \right\} \leq \mathbb{P}\left\{ \min_{1 \leq k \leq N} \sum_{i=1}^k S_i \geq 0 \right\} \leq \mathbb{P}\left\{ \min_{1 \leq k \leq \eta^0(N)} \sum_{i=1}^k \xi_i^0 \geq 0 \right\}. \quad (2)$$

Note that the r.v.'s ξ_n^0 are i.i.d. and symmetric; hence $\sum_{i=1}^k \xi_i^0$ is a symmetric random walk. It is well-known that for such random walks

$$\mathbb{P}\left\{ \min_{1 \leq k \leq n} \sum_{i=1}^k \xi_i^0 \geq 0 \right\} \sim \frac{c}{\sqrt{n}}$$

as $n \to \infty$ for a certain constant $c > 0$. On the other hand, $\eta^0(N) \asymp N^{1/2}$ in probability as $N \to \infty$ because of another well-known fact that θ_1^0 belongs to the domain of normal attraction of an α-stable law with exponent 1/2. Were $\eta^0(N)$ independent with the walk $\sum_{i=1}^k \xi_i^0$, these asymptotical estimates and (2) would immediately imply $p_N \asymp N^{-1/4}$.

Unfortunately, $\eta^0(N) = \max\{k \geq 0 : \sum_{i=1}^k \theta_i^0 \leq N\}$ and $\sum_{i=1}^k \xi_i^0$ are dependent, and a careful study of the joint distributions of (ξ_1^0, θ_1^0) is required. Sinai [15] gives a tedious analysis of the generating function of (ξ_1^0, θ_1^0) using the theory of continuous fractions. However, these arguments cannot be generalized since the crucial recursive relation for the generating function of (ξ_1^0, θ_1^0) was obtained in [15] using binary structure of increments of simple random walks.

2.2. *Preparatory definitions*

In our proofs, we use a generalization of the described approach of decomposing the trajectory of the walk into independent excursions. In this section we introduce appropriate definitions.

Suppose, at first, that S_n is an integer-valued random walk. We keep the previous notation but define τ_n^0 as the moments of *returning* to zero: $\tau_0^0 := 0$ and $\tau_{n+1}^0 := \min\{k > \tau_n^0 + 1 : S_k = 0, S_{k-1} \neq 0\}$ for $n \geq 0$, which coincide with the moments of *hitting* zero if S_n is a simple random walk. The variables τ_{n+1}^0 are finite with probability 1 because the walk is integer-valued, centered, and has a finite variance. Only the upper bound in (2) remains valid because the walk can jump over the zero level without hitting it.

Clearly, the described approach does not work for general walks. We shall consider different stopping times.

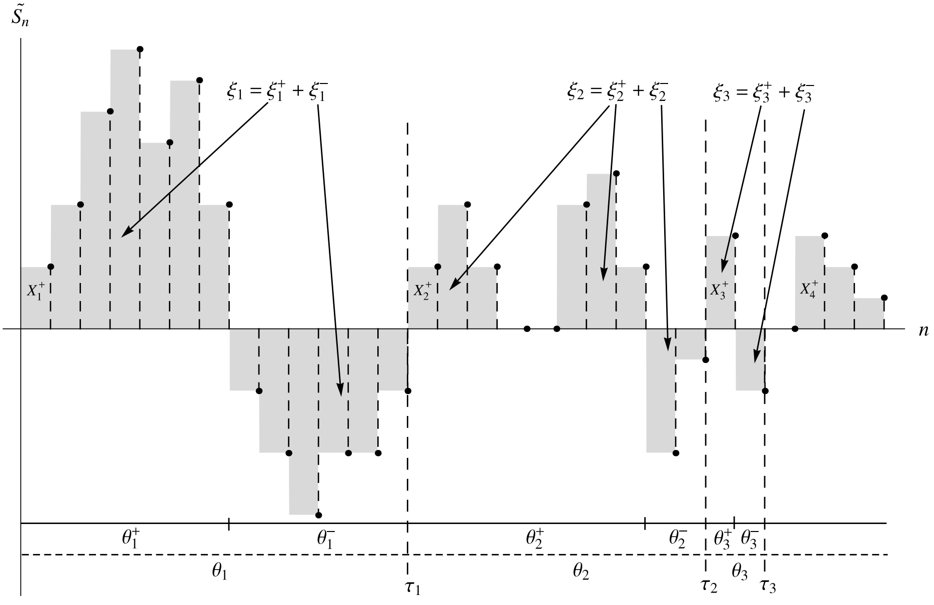

Fig. 1. Decomposition of the trajectory of $\tilde{S}_n$ into "cycles".

Define conditional probability $\widetilde{\mathbb{P}}\{\cdot\} := \mathbb{P}\{\cdot|S_1 > 0\}$ and define $\tilde{p}_N$ as p_N but with $\mathbb{P}$ replaced by $\widetilde{\mathbb{P}}$. Note that it suffices to prove Theorems 1 and 2 for $\tilde{p}_N$ instead of p_N. Indeed,

$$p_N = \mathbb{P}\left\{\min_{1\le k\le N}\sum_{i=1}^{k} S_i \ge 0\right\} = a_+ \sum_{n=0}^{N} a_0^n\, \mathbb{P}\left\{\min_{1\le k\le N-n}\sum_{i=1}^{k} S_i \ge 0 | S_1 > 0\right\}$$
$$= a_+ \sum_{n=0}^{N} a_0^n\, \tilde{p}_{N-n},$$

where

$$a_+ := \mathbb{P}\{S_1 > 0\}, \qquad a_0 := \mathbb{P}\{S_1 = 0\}, \qquad a_- := \mathbb{P}\{S_1 < 0\}.$$

Hence

$$p_N \asymp \tilde{p}_N \tag{3}$$

if $\tilde{p}_N$ decays polynomially.

Now, let X_1^+ be a r.v. with the distribution $\mathrm{Law}(S_1|S_1 > 0)$ and independent with the walk S_n, and put $\widetilde{S}_n := X_1^+ + S_n - S_1$ for $n \ge 1$. Clearly,

$$\mathrm{Law}(\widetilde{S}_1, \widetilde{S}_2, \ldots) = \mathrm{Law}(S_1, S_2, \ldots | S_1 > 0).$$

For the convenience of the reader, the following definitions are represented in comprehensive Fig. 1. Define the moments τ_n when $\widetilde{S}_k$ overshoots the zero level *from below*: $\tau_0 := 0$ and $\tau_{n+1} := \max\left\{k > \tau_n : \widetilde{S}_k \le 0\right\}$ for $n \ge 0$. It is readily seen that $\tau_n + 1$ are stopping times. Define $\theta_n := \tau_n - \tau_{n-1}$ and $\xi_n := \sum_{i=\tau_{n-1}+1}^{\tau_n} \widetilde{S}_i$, and let $\eta(N)$ be the number of overshoots of the zero level from below by the time N, namely,

$$\eta(N) := \max\{k : \tau_k \le N\} = \max\left\{k : \sum_{i=1}^{k} \theta_i \le N\right\}.$$

Now, by analogy with (2), we write

$$\mathbb{P}\left\{\min_{1\le k\le \eta(N)+1} \sum_{i=1}^{k} \xi_i \ge 0\right\} \le \widetilde{\mathbb{P}}\left\{\min_{1\le k\le N} \sum_{i=1}^{k} S_i \ge 0\right\} \le \mathbb{P}\left\{\min_{1\le k\le \eta(N)} \sum_{i=1}^{k} \xi_i \ge 0\right\}. \quad (4)$$

It is clear that the moments of overshoots τ_n partition the trajectory of $\widetilde{S}_k$ into "cycles" that consist of one weak positive and the consequent weak negative excursion (that is, nonnegative and nonpositive, respectively, but we will omit "weak" in what follows). Let $\theta_n^+ := \max\left\{k > 0 : \widetilde{S}_{\tau_{n-1}+k} \ge 0\right\}$ and $\theta_n^- := \max\{k > 0 : \widetilde{S}_{\tau_{n-1}+\theta_n^+ + k} \le 0\}$ be the lengths and let $\xi_n^+ := \sum_{i=\tau_{n-1}+1}^{\tau_{n-1}+\theta_n^+} \widetilde{S}_i$ and $\xi_n^- := \sum_{i=\tau_{n-1}+\theta_n^+ +1}^{\tau_n} \widetilde{S}_i$ be the areas of these excursions, respectively; obviously, $\xi_n = \xi_n^+ + \xi_n^-$ and $\theta_n = \theta_n^+ + \theta_n^-$. The following observation plays the key role in our paper.

Lemma 1. *Let S_n be a centered random walk with a finite variance.*
(a) *If S_n is integer-valued, then random vectors $(\xi_n^0, \theta_n^0)_{n\ge1}$ are i.i.d.*
(b) *If S_n is upper exponential, upper geometric, or right-continuous, then the random vectors $(\xi_n, \theta_n)_{n\ge1}$ are i.i.d., $(\xi_n^+, \theta_n^+)_{n\ge1}$ are i.i.d., and $(\xi_n^-, \theta_n^-)_{n\ge1}$ are i.i.d. If, in addition, S_n satisfies the assumptions of Theorem 2, then $(\xi_n^+, \theta_n^+)_{n\ge1}$ and $(\xi_n^-, \theta_n^-)_{n\ge1}$ are mutually independent.*

Note: from this point on, S_n satisfies the assumptions of Theorem 2 means that it satisfies the assumptions of Part 1 or Part 2 of the theorem. The lemma, basically, shows that under the assumptions made, the cycles of the walk are i.i.d.

Proof. Part (a) is trivial. For Part (b), note that the overshoots over the zero level $X_n^+ := \widetilde{S}_{\tau_{n-1}+1}$ are i.i.d. and their common distribution is $\mathrm{Law}(S_1 | S_1 > 0)$, which is exponential, geometric, or δ_1. This naturally follows from the memoryless property of these distributions; a proof can be found in Example XII.4(a) from [8]. In the same way, we show that X_n^+ are independent from the "past" $\widetilde{S}_1, \dots, \widetilde{S}_{\tau_{n-1}}$. Now from $\xi_n = \sum_{i=\tau_{n-1}+1}^{\tau_n} \widetilde{S}_i = \sum_{i=\tau_{n-1}+1}^{\tau_n} (X_n^+ + \widetilde{S}_i - \widetilde{S}_{\tau_{n-1}+1})$ and $\theta_n = \max\left\{k > 0 : X_n^+ + \widetilde{S}_{\tau_{n-1}+k} - \widetilde{S}_{\tau_{n-1}+1} \le 0\right\}$ we see that (ξ_n, θ_n) are i.i.d. as $\tau_n + 1$ are stopping times. The proof of the other statements is analogous. □

3. Areas and durations of excursions and cycles

We already explained in Section 2.1 why it is important to study properties of the joint distribution of ξ_1 and θ_1. Here we prove several crucial results on (ξ_1, θ_1), (ξ_1^+, θ_1^+), (ξ_1^-, θ_1^-), and (ξ_1^0, θ_1^0), which are used in the proofs of Theorems 1 and 2.

We start with a surprising lemma which allows us, in certain cases, to reduce a complicated study of the joint distribution of (ξ_1, θ_1) to a much simpler consideration of its marginal distributions.

Lemma 2. *Let S_n be a centered random walk with a finite variance. If S_n is upper exponential, then the distribution of ξ_1 is symmetric, and moreover, $(\xi_1, \theta_1) \stackrel{\mathcal{D}}{=} (-\xi_1, \theta_1)$ and*

$(\xi_1^+, \theta_1^+, \xi_1^-, \theta_1^-) \stackrel{\mathcal{D}}{=} (-\xi_1^-, \theta_1^-, -\xi_1^+, \theta_1^+)$. *If* S_n *is integer-valued, then the distribution of* ξ_1^0 *is symmetric, and moreover,* $(\xi_1^0, \theta_1^0) \stackrel{\mathcal{D}}{=} (-\xi_1^0, \theta_1^0)$.

Proof. Let us start with the upper exponential case assuming, without loss of generality, that $\mathrm{Law}(X|X > 0)$ is a standard exponential distribution. Since $\xi_1 = \widetilde{S}_1 + \cdots + \widetilde{S}_{\theta_1}$, it suffices to show that for each $i, j \geq 1$, the measures $\mathbb{P}\left\{(\widetilde{S}_1, \ldots, \widetilde{S}_{\theta_1}) \in \cdot, \theta_1^+ = i, \theta_1^- = j\right\}$ and $\mathbb{P}\left\{(-\widetilde{S}_{\theta_1}, \ldots, -\widetilde{S}_1) \in \cdot, \theta_1^+ = j, \theta_1^- = i\right\}$ coincide. This statement follows from the observation that for any $x_1, \ldots, x_i > 0$ and $x_{i+1}, \ldots, x_{i+j} < 0$,

$$\begin{aligned} &\mathbb{P}\left\{\widetilde{S}_1 \in \mathrm{d}x_1, \ldots, \widetilde{S}_{i+j} \in \mathrm{d}x_{i+j}, \theta_1^+ = i, \theta_1^- = j\right\} \\ &\quad = a_+ \mathrm{e}^{x_{i+j} - x_1} \mathbb{E}\left\{S_2 \in \mathrm{d}x_2, \ldots, S_{i+j-1} \in \mathrm{d}x_{i+j-1} | S_1 = x_1, S_{i+j} = x_{i+j}\right\} \mathrm{d}x_1 \mathrm{d}x_{i+j} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\left\{\widetilde{S}_{i+j} \in -\mathrm{d}x_1, \widetilde{S}_{i+j-1} \in -\mathrm{d}x_2, \ldots, \widetilde{S}_1 \in -\mathrm{d}x_{i+j}, \theta_1^+ = j, \theta_1^- = i\right\} \\ &\quad = a_+ \mathrm{e}^{x_{i+j} - x_1} \mathbb{E}\left\{S_2 \in -\mathrm{d}x_{i+j-1}, \ldots, S_{k-1} \in -\mathrm{d}x_2 | S_1 = -x_{i+j}, S_{i+j} = -x_1\right\} \\ &\qquad \times \mathrm{d}x_1 \mathrm{d}x_{i+j}. \end{aligned}$$

Indeed, the conditional expectations in the right-hand sides coincide for any random walk: this is, essentially, the well-known property of duality of random walks.

The proof for the lattice case is analogous: since $\xi_1^0 = S_1 + \cdots + S_{\theta_1^0}$, use that for any $i \geq 0$, $j \geq 1$, and any integer $x_{i+1}, \ldots, x_{i+j} \neq 0$, it holds that

$$\begin{aligned} &\mathbb{P}\left\{S_1 = \cdots = S_i = 0, S_{i+1} = x_{i+1}, \ldots, S_{i+j} = x_{i+j}, S_{i+j+1} = 0\right\} \\ &\quad = \mathbb{P}\left\{S_1 = \cdots = S_i = 0, S_{i+1} = -x_{i+j}, \ldots, S_{i+j} = -x_{i+1}, S_{i+j+1} = 0\right\} \end{aligned}$$

for any random walk. □

Note that the distribution of ξ_1 is not symmetric even for two-sided geometric random walks unless $a_- = a_+$. The proof presented above for the upper exponential case does not work here because two-sided geometric walks can return to zero.

In order to state the next result, recall that r.v.'s $Y_1, \ldots, Y_k$ are *associated* if

$$\mathrm{cov}\,(f(Y_1, \ldots, Y_k),\ g(Y_1, \ldots, Y_k)) \geq 0$$

for any coordinatewise nondecreasing functions $f, g : \mathbb{R}^k \to \mathbb{R}$ such that the covariance is well-defined. An infinite set of r.v.'s is associated if any finite subset of its variables is associated. The following sufficient conditions of association are well-known; see [7]:

(a) A set consisting of a single r.v. is associated.
(b) Independent r.v.'s are associated.
(c) Coordinatewise nondecreasing functions (of a finite number of variables) of associated r.v.'s are associated.
(d) If $Y_{1,u}, \ldots, Y_{k,u}$ are associated for every u and $(Y_{1,u}, \ldots, Y_{k,u}) \xrightarrow{\mathcal{D}} (Y_1, \ldots, Y_k)$ as $u \to \infty$, then $Y_1, \ldots, Y_k$ are associated.
(e) If two sets of associated variables are independent, then the union of these sets is also associated.

We now state the other result that allows us, in some cases, to proceed from study of the joint distribution of (ξ_1, θ_1) to a consideration of the distributions of ξ_1 and θ_1.

Lemma 3. *Under the assumptions of Theorem 2, the random variables $\{\xi_n, \theta_n^+\}_{n\geq 1}$ are associated.*

Proof. We first show that ξ_1^+ and θ_1^+ are associated. Indeed, by (b) and (c), the r.v.'s $\sum_{i=1}^{\min\{k,\theta_1^+\}} \widetilde{S}_i$ and $\min\{k, \theta_1^+\}$ are associated for each k as coordinatewise nondecreasing functions of the first k independent increments of the walk. Since $(\sum_{i=1}^{\min\{k,\theta_1^+\}} \widetilde{S}_i, \min\{k, \theta_1^+\}) \to (\xi_1^+, \theta_1^+)$ with probability 1 as $k \to \infty$, ξ_1^+ and θ_1^+ are associated by (d).

Now $\xi_1^+, \xi_1^-, \theta_1^+$ are associated by (a) and (e) because ξ_1^- is independent of ξ_1^+ and θ_1^+, and then $\xi_1 = \xi_1^+ + \xi_1^-$ and θ_1^+ are also associated by (c). This concludes the proof of the lemma since $(\xi_n, \theta_n)_{n\geq 1}$ are i.i.d. □

The following Proposition 1 describes the "tails" of ξ_1 and θ_1. The proposition consists of two sections, Parts (a) and (b). We stress that only Part (a), whose proof is straightforward, is used to prove Theorem 1 and Part 2 of Theorem 2. The proof of Part 1 of Theorem 2 requires more complicated Corollary 1 of Part (b). Although Part (b) itself is not used directly in the proofs of our main results, it is interesting because of its Corollary 2 and because it generalizes the crucial Theorem 1 of Sinai [15].

Let $\xi_{ex} := \int_0^1 W_{ex}(u)\mathrm{d}u$ be the area of a standard Brownian excursion. The latter is defined as $W_{ex}(u) := (\overline{\nu} - \underline{\nu})^{-1/2} \left|W(\underline{\nu} + u(\overline{\nu} - \underline{\nu}))\right|$, where $W(u)$ is a standard Brownian motion, $\underline{\nu}$ is the last zero of $W(u)$ before 1 and $\overline{\nu}$ is the first zero after 1. For $x \geq 0$, put

$$F(x) := \mathbb{E} \min\left\{x^{-1/3}\xi_{ex}^{1/3}, 1\right\}.$$

Clearly, $F(x)$ is decreasing, $F(0) = 1$, and $F(\infty) = 0$. By Janson [10], ξ_{ex} is continuous and has finite moments of any order, so $F(x)$ is continuous and, by $F(x) = x^{-1/3}\mathbb{E}\min\left\{\xi_{ex}^{1/3}, x^{1/3}\right\}$, we have $\lim_{x\to\infty} x^{1/3}F(x) = \mathbb{E}\xi_{ex}^{1/3} < \infty$.

Proposition 1. *Let S_n be a centered random walk with a finite variance.*
(a) *θ_1^+ belongs to the domain of normal attraction of a spectrally positive α-stable law with exponent $1/2$, and the same holds for θ_1^0 if S_n is integer-valued.*
(b) *If S_n satisfies the assumptions of Theorem 2 or S_n is upper exponential, then for any $s, t \geq 0$ such that $s + t > 0$ it holds that*

$$\begin{aligned}\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1^+ > sn^{3/2}, \theta_1^+ > tn\right\} &= \lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1^- < -sn^{3/2}, \theta_1^- > tn\right\} \\ &= \lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1 > sn^{3/2}, \theta_1 > tn\right\} \\ &= \lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1 < -sn^{3/2}, \theta_1 > tn\right\} \\ &= C_{\mathrm{Law}(S_1)}\, t^{-1/2}F(\sigma st^{-3/2}), \qquad (5)\end{aligned}$$

where $C_{\mathrm{Law}(S_1)} = \frac{(1-a_0)\mathbb{E}|S_1|}{\sqrt{2\pi}a_+a_-\sigma}$ or $C_{\mathrm{Law}(S_1)} = \sqrt{\frac{2}{\pi}}\frac{\sigma}{\mathbb{E}|S_1|}$, respectively. The right-hand side of (5) at $t = 0$ is defined by continuity. If S_n is integer-valued and the lattice span of S_1 is 1, then

$$\begin{aligned}\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1^0 > sn^{3/2}, \theta_1^0 > tn\right\} &= \lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1^0 < -sn^{3/2}, \theta_1^0 > tn\right\} \\ &= \frac{\sigma}{\sqrt{2\pi t}}F(\sigma st^{-3/2}). \qquad (6)\end{aligned}$$

Corollary 1. *Suppose S_n satisfies the assumptions of Theorem 2. Then ξ_1 belongs to the domain of normal attraction of a symmetric α-stable law with exponent $1/3$.*

As an immediate consequence of de Haan et al. [3], we have the following.

Corollary 2. *Under conditions of Part* (b) *of Proposition 1,*

$$\left(\frac{\xi_1 + \cdots + \xi_n}{n^3}, \frac{\theta_1 + \cdots + \theta_n}{n^2}\right) \xrightarrow{\mathcal{D}} (\xi, \theta),$$

where $\mathrm{Law}(\theta)$ *is spectrally positive α-stable with exponent* $1/2$ *and* $\mathrm{Law}(\xi)$ *is symmetric α-stable with exponent* $1/3$. *The same holds for sums of ξ_i^0 and θ_i^0.*

Before we get to the proofs, recall some important facts on ladder variables of random walks from Feller [8]. For any random walk U_n, define the first descending and ascending ladder epochs as $\tau_+ := \min\{k > 0 : U_k < 0\}$ and $\tau_- := \min\{k > 0 : U_k > 0\}$, respectively, where by definition $\min_\varnothing := \infty$. We introduce such notation considering τ_+ as the duration of the first *positive excursion* of U_k (increased by 1 of course) rather than the first moment when U_k becomes negative. It is readily seen that

$$\mathbb{P}\{\tau_+ > n\} = \mathbb{P}\left\{\min_{1 \le i \le n} U_i \ge 0\right\}. \tag{7}$$

Define

$$c_+ := \sum_{n=1}^{\infty} \frac{1}{n}\left(\mathbb{P}\{U_n > 0\} - 1/2\right), \qquad c_0 := \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}\{U_n = 0\},$$

$$c_- := \sum_{n=1}^{\infty} \frac{1}{n}\left(\mathbb{P}\{U_n < 0\} - 1/2\right)$$

if the sums are well-defined. If c_+ and c_- are finite, then

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\tau_+ > n\} = \frac{\mathrm{e}^{c_+ + c_0}}{\sqrt{\pi}}, \qquad \lim_{n\to\infty} n^{1/2}\mathbb{P}\{\tau_- > n\} = \frac{\mathrm{e}^{c_- + c_0}}{\sqrt{\pi}}. \tag{8}$$

It is known that c_0 is *always* finite while c_+ and c_- are finite if $\mathbb{E}U_1 = 0$ and $0 < \mathbb{D}U_1 =: \sigma^2 < \infty$. Under the latter conditions, we also have

$$\mathbb{E}U_{\tau_+} = -\frac{\sigma}{\sqrt{2}}\mathrm{e}^{c_+ + c_0}, \qquad \mathbb{E}U_{\tau_-} = \frac{\sigma}{\sqrt{2}}\mathrm{e}^{c_- + c_0} \tag{9}$$

for the ladder heights U_{τ_+} and U_{τ_-}. Finally, if $\mathbb{P}\{U_n > 0\} \to 1/2$, then

$$\mathbb{P}\{\tau_+ > n\} \sim n^{-1/2}L(n), \tag{10}$$

for some function $L(n)$ that is slowly varying at infinity; see [13].

Proof of Proposition 1. I. The statements on ξ_1^+ and θ_1^+.
Case 1: $s = 0$ and $t > 0$. Without loss of generality, put $t = 1$. We have

$$\begin{aligned}\mathbb{P}\{\theta_1^+ > n\} &= \widetilde{\mathbb{P}}\{\tau_+ > n+1\} = a_+^{-1}\mathbb{P}\{\tau_+ > n+1, S_1 > 0\} \\ &= a_+^{-1}\left(\mathbb{P}\{\tau_+ > n+1\} - a_0\mathbb{P}\{\tau_+ > n\}\right),\end{aligned} \tag{11}$$

and by (8), since S_k is centered and has a finite variance,

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1^+ > n\} = \frac{1-a_0}{a_+}\cdot\frac{\mathrm{e}^{c_++c_0}}{\pi^{1/2}}. \tag{12}$$

This relation proves Part (a) of the proposition.

To simplify the right-hand side of (12), write $\mathbb{E}|S_1| = 2a_+\mathbb{E}(S_1|S_1 > 0)$, which follows from $\mathbb{E}S_1 = 0$. Under the assumptions of Part (b), S_k is upper exponential, right-continuous, or upper geometric, so $\mathrm{Law}(S_1|S_1 > 0) = \mathrm{Law}(S_{\tau_-})$, and recalling (9), $\mathbb{E}|S_1| = 2a_+\mathbb{E}S_{\tau_-} = \sqrt{2}a_+\sigma\mathrm{e}^{c_-+c_0}$. Then $\mathrm{e}^{c_-+c_0} = \frac{\mathbb{E}|S_1|}{\sqrt{2}a_+\sigma}$, and by $\mathrm{e}^{c_++c_0+c_-} = 1$, we get $\mathrm{e}^{c_+} = \frac{\sqrt{2}a_+\sigma}{\mathbb{E}|S_1|}$. If S_k is upper exponential, then clearly $\mathbb{P}\{S_k = 0\} = a_0^k$; hence $\mathrm{e}^{c_0} = \frac{1}{1-a_0}$, and from (12) we have $C_{\mathrm{Law}(S_1)} = \sqrt{\frac{2}{\pi}}\frac{\sigma}{\mathbb{E}|S_1|}$ for the constant in (5). If S_k satisfies the assumptions of Theorem 2, by the same arguments as above, $\mathrm{e}^{c_-} = \frac{\sqrt{2}a_-\sigma}{\mathbb{E}|S_1|}$. Now $\mathrm{e}^{c_++c_0+c_-} = 1$ implies $\mathrm{e}^{c_0} = \frac{(\mathbb{E}|S_1|)^2}{2a_+a_-\sigma^2}$, and from (12), $C_{\mathrm{Law}(S_1)} = \frac{(1-a_0)\mathbb{E}|S_1|}{\sqrt{2\pi}a_+a_-\sigma}$.

Case 2: $s \geq 0$ and $t > 0$. We state one important particular case of the result of Shimura [14] on convergence of discrete excursions. Let $W(t)$ be a standard Brownian motion, and let $\bar{W}(t) := W(t) - \inf_{0\leq s\leq t} W(s)$ be a reflecting Brownian motion. Then for any random walk U_n such that $\mathbb{E}U_1 = 0$ and $0 < \mathbb{D}U_1 =: \sigma^2 < \infty$, for any $\varepsilon > 0$,

$$\mathrm{Law}\left(\left(\frac{\tau_+}{n}, \frac{U_{\min\{\tau_+,[n\cdot]\}}}{\sigma n^{1/2}}\right)\middle|\, \tau_+ > \varepsilon n\right) \xrightarrow{\mathcal{D}} \mathrm{Law}\left(\left(\nu''_\varepsilon - \nu'_\varepsilon, \bar{W}(\nu'_\varepsilon + \min\{\cdot, \nu''_\varepsilon - \nu'_\varepsilon\})\right)\right) \tag{13}$$

in $\mathbb{R}\times\mathcal{D}[0,\infty)$ as $n\to\infty$, where $\mathcal{D}$ stands for Skorokhod space and $(\nu'_\varepsilon, \nu''_\varepsilon)$ is the first pair of successive zeros of $\bar{W}$ such that $\nu''_\varepsilon - \nu'_\varepsilon > \varepsilon$.

Since the r.v.'s $\nu''_\varepsilon - \nu'_\varepsilon$ and $\int_{\nu'_\varepsilon}^{\nu''_\varepsilon} \bar{W}(u)\mathrm{d}u$ are continuous, from (8) and (13) we find that for any $s \geq 0$ and $t \geq \varepsilon$,

$$\begin{aligned}\mathbb{P}\left\{\xi_+ > sn^{3/2}, \tau_+ > tn\right\} &\sim \mathbb{P}\{\tau_+ > \varepsilon n\}\mathbb{P}\left\{\nu''_\varepsilon - \nu'_\varepsilon > t, \int_{\nu'_\varepsilon}^{\nu''_\varepsilon} \bar{W}(u)\mathrm{d}u > \sigma s\right\}\\ &\sim \frac{\mathrm{e}^{c_++c_0}}{(\pi\varepsilon n)^{1/2}}\mathbb{P}\left\{\nu''_\varepsilon - \nu'_\varepsilon > t, (\nu''_\varepsilon - \nu'_\varepsilon)\int_0^1 \bar{W}(\nu'_\varepsilon + u(\nu''_\varepsilon - \nu'_\varepsilon))\mathrm{d}u > \sigma s\right\}\end{aligned} \tag{14}$$

as $n\to\infty$, where $\xi_+ := \sum_{k=1}^{\tau_+-1} S_k$ and by definition, $\Sigma_\varnothing := 0$.

We claim that, first, the process $W_{ex}^{(\varepsilon)}(\cdot) := (\nu''_\varepsilon - \nu'_\varepsilon)^{-1/2}\bar{W}(\nu'_\varepsilon + \cdot(\nu''_\varepsilon - \nu'_\varepsilon))$ is a standard Brownian excursion $W_{ex}(\cdot)$ on $[0,1]$ and, second, $W_{ex}^{(\varepsilon)}(\cdot)$ is independent of $\nu''_\varepsilon - \nu'_\varepsilon$. Recall the definition $W_{ex}(\cdot) := (\nu'' - \nu')^{-1/2}\bar{W}(\nu' + \cdot(\nu'' - \nu'))$, where ν' is the last zero of $\bar{W}(\cdot)$ before 1 and ν'' is the first zero after 1. $W_{ex}(\cdot)$ is usually defined in terms of $|W(\cdot)|$ but we used that $\bar{W}(\cdot) \overset{\mathcal{D}}{=} |W(\cdot)|$.

Indeed, it is known (for instance, see [4]) that if U_n is a simple random walk, then

$$\mathrm{Law}\left(\frac{U_{[\tau_+\cdot]}}{\sigma\tau_+^{1/2}}\middle|\,\tau_+ = n\right) = \mathrm{Law}\left(\frac{U_{[n\cdot]}}{\sigma n^{1/2}}\middle|\,\tau_+ = n\right) \xrightarrow{\mathcal{D}} \mathrm{Law}\left(W_{ex}(\cdot)\right)$$

in $\mathcal{D}[0,1]$. Hence for any $a > 0$ and any cylindrical set $\mathcal{A} \subset \mathcal{D}[0,1]$ that is generated by the product of intervals (the latter ensures $\mathbb{P}\{W_{ex}(\cdot) \in \partial\mathcal{A}\} = \mathbb{P}\{W_{ex}^{(\varepsilon)}(\cdot) \in \partial\mathcal{A}\} = 0$),

$$\mathbb{P}\left\{\frac{U_{[\tau_+ \cdot]}}{\sigma \tau_+^{1/2}} \in \mathcal{A}, \tau_+ > an\right\} = \mathbb{P}\{\tau_+ > an\}\left(\mathbb{P}\{W_{ex}(\cdot) \in \mathcal{A}\} + o(1)\right). \tag{15}$$

On the other hand, (13) yields

$$\text{Law}\left(\left(\frac{\tau_+}{n}, \frac{U_{[\tau_+ \cdot]}}{\sigma \tau_+^{1/2}}\right) \Bigg| \tau_+ > \varepsilon n\right) \xrightarrow{\mathcal{D}} \text{Law}\left(\left(\nu_\varepsilon'' - \nu_\varepsilon', W_{ex}^{(\varepsilon)}(\cdot)\right)\right)$$

in $\mathbb{R} \times \mathcal{D}[0,1]$. Hence if $a \geq \varepsilon$, then

$$\mathbb{P}\left\{\frac{U_{[\tau_+ \cdot]}}{\sigma \tau_+^{1/2}} \in \mathcal{A}, \tau_+ > an\right\} = \mathbb{P}\{\tau_+ > \varepsilon n\}\left(\mathbb{P}\left\{\nu_\varepsilon'' - \nu_\varepsilon' > a, W_{ex}^{(\varepsilon)}(\cdot) \in \mathcal{A}\right\} + o(1)\right). \tag{16}$$

Finally, comparing (15) and (16) and using (8), we obtain

$$\left(\frac{\varepsilon}{a}\right)^{1/2} \mathbb{P}\{W_{ex}(\cdot) \in \mathcal{A}\} = \mathbb{P}\left\{\nu_\varepsilon'' - \nu_\varepsilon' > a, W_{ex}^{(\varepsilon)}(\cdot) \in \mathcal{A}\right\},$$

which implies $W_{ex}^{(\varepsilon)}(\cdot) \stackrel{\mathcal{D}}{=} W_{ex}(\cdot)$ and independence of $\nu_\varepsilon'' - \nu_\varepsilon'$ and $W_{ex}^{(\varepsilon)}(\cdot)$.

Now, since $\mathbb{P}\left\{\nu_\varepsilon'' - \nu_\varepsilon' > t\right\} = (\frac{\varepsilon}{t})^{1/2}$ for $t \geq \varepsilon$, we rewrite (14) as

$$\begin{aligned}
&\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_+ > sn^{3/2}, \tau_+ > tn\right\} \\
&= \frac{e^{c_+ + c_0}}{2\pi^{1/2}} \int_t^\infty z^{-3/2}\mathbb{P}\left\{\int_0^1 W_{ex}(u)\mathrm{d}u > \sigma s z^{-3/2}\right\} \mathrm{d}z \\
&= \frac{e^{c_+ + c_0}}{3(\sigma s)^{1/3}\pi^{1/2}} \int_0^{\sigma s t^{-3/2}} v^{-2/3}\mathbb{P}\{\xi_{ex} > v\}\,\mathrm{d}v,
\end{aligned}$$

where we changed variables and put $\xi_{ex} := \int_0^1 W_{ex}(u)\mathrm{d}u$. For any $x > 0$, write

$$\begin{aligned}
\frac{1}{3x^{1/3}} \int_0^x v^{-2/3}\mathbb{P}\{\xi_{ex} > v\}\,\mathrm{d}v &= \mathbb{P}\{\xi_{ex} > x\} - \frac{1}{x^{1/3}} \int_0^x v^{1/3}\mathrm{d}\mathbb{P}\{\xi_{ex} \leq v\} \\
&= x^{-1/3}\mathbb{E}\min\left\{\xi_{ex}^{1/3}, x^{1/3}\right\} \\
&= \mathbb{E}\min\left\{x^{-1/3}\xi_{ex}^{1/3}, 1\right\} =: F(x).
\end{aligned}$$

Then

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_+ > sn^{3/2}, \tau_+ > tn\right\} = \frac{e^{c_+ + c_0}}{(\pi t)^{1/2}} F(\sigma s t^{-3/2}),$$

and arguing as in (11),

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\xi_1^+ > sn^{3/2}, \theta_1^+ > tn\right\} = \frac{1 - a_0}{a_+} \cdot \frac{e^{c_+ + c_0}}{(\pi t)^{1/2}} F(\sigma s t^{-3/2}).$$

We already explained above why the constant in the right-hand side has the required form.

Case 3: $s > 0$ and $t = 0$. Since the right-hand side of (5) at $t = 0$ is defined by continuity and (5) is already proved for $s, t > 0$, we should check that

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\xi_1^+ > sn^{3/2}\} = \lim_{t\to 0}\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\xi_1^+ > sn^{3/2}, \theta_1^+ > tn\}.$$

By the law of total probability, it suffices to show

$$\lim_{t\to 0}\limsup_{n\to\infty} n^{1/2}\mathbb{P}\{\xi_1^+ > sn^{3/2}, \theta_1^+ \le tn\} = 0.$$

But

$$\begin{aligned}\mathbb{P}\{\xi_1^+ > sn^{3/2}, \theta_1^+ < tn\} &\le \widetilde{\mathbb{P}}\left\{\max_{1\le k\le \tau_+ - 1} S_k > t^{-1}sn^{1/2}, \tau_+ < tn\right\}\\ &\le a_+^{-1}\mathbb{P}\left\{\max_{1\le k\le \tau_+ - 1} S_k > t^{-1}sn^{1/2}\right\},\end{aligned}$$

where the second estimate was obtained as in (11), and by definition, $\max_{\varnothing} := -\infty$. Now the required estimate follows from Theorem 2 of Simura [14].

II. The statements on ξ_1^- and θ_1^-.

If S_n is upper exponential, simply use $(\xi_1^-, \theta_1^-) \stackrel{\mathcal{D}}{=} (-\xi_1^+, \theta_1^+)$ from Lemma 2 and the part of (5) on ξ_1^+ and θ_1^+ proven above. If S_n satisfies the assumptions of Theorem 2, (θ_1^-, ξ_1^-) has the same distribution as $(\bar{\theta}_1^+, -\bar{\xi}_1^+)$, where the bar means that the walk $\bar{S}_n := -S_n$ is considered. Since $\bar{S}_n$ satisfies the assumptions of Theorem 2 if S_n does, we use the part of (5) on ξ_1^+ and θ_1^+ proven above and $C_{\mathrm{Law}(S_1)} = C_{\mathrm{Law}(-S_1)}$.

III. The statements on ξ_1 and θ_1.

We only consider the case $s = 0$ letting, without loss of generality, $t = 1$. The proof of the other cases is absolutely similar. Let us check that for $\theta_1 = \theta_1^+ + \theta_1^-$,

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1 > n\} = \lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1^+ > n\} + \lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1^- > n\}.$$

By standard arguments, it suffices to show that

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\left\{\theta_1^+ > n, \theta_1^- > n\right\} = 0. \tag{17}$$

Under the assumptions of Theorem 2, θ_1^+ and θ_1^- are independent, and the statement is trivial.

Otherwise, consider an independent copy S_n' of the walk S_n. For any $x \ge 0$, put $\tau_-'(x) := \min\{k \ge 1 : S_k' > x\}$. Since $\theta_1^- = \max\{k \ge 1 : \widetilde{S}_{\theta_1^+ + k} - \widetilde{S}_{\theta_1^+ + 1} \le -\widetilde{S}_{\theta_1^+ + 1}\}$, we have $\theta_1^- \stackrel{\mathcal{D}}{=} \tau_-'\left(-\widetilde{S}_{\theta_1^+ + 1}\right)$, and for any $M > 0$,

$$\mathbb{P}\left\{\theta_1^+ > n, \theta_1^- > n\right\} \le \mathbb{P}\left\{\theta_1^+ > n, \widetilde{S}_{\theta_1^+ + 1} < -M\right\} + \mathbb{P}\left\{\theta_1^+ > n\right\}\mathbb{P}\left\{\tau_-(M) > n\right\}.$$

Arguing as in (11), we get (17) from (8) and

$$\lim_{M\to\infty}\limsup_{n\to\infty} n^{1/2}\mathbb{P}\left\{\tau_+ > n, S_{\tau_+} < -M\right\} = 0,$$

which follows from Lemma 4 in [6].

IV. The statements on ξ_1^0 and θ_1^0.

It is well-known [16, Section 32] that

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1^0 > n\} = \sqrt{\frac{2}{\pi}}\sigma \tag{18}$$

for any integer-valued random walk with a finite variance. Then we find the asymptotics of the "tail" of (θ_1^0, ξ_1^0) exactly as that of (θ_1^+, ξ_1^+), up to the following differences. First, we use (18) instead of (8). Second, instead of referring to (13), we use the result of Kaigh [11] that $\frac{U_{[n\cdot]}}{\sigma n^{1/2}}$ conditioned on $\{\theta_1^0 = n\}$ weakly converges to a signed Brownian excursion $\varrho W_{ex}(\cdot)$, where $\mathbb{P}\{\varrho = 1\} = \mathbb{P}\{\varrho = -1\} = 1/2$ and ϱ is independent of $W_{ex}(\cdot)$. The additional assumption that S_1 has span 1 is required for using the result of Kaigh [11]. □

4. The upper bound

Case 1. S_n is an upper exponential random walk.

Define $\nu := \min\{k > 0 : \xi_1 + \cdots + \xi_k < 0\}$. Then

$$\xi_1 + \cdots + \xi_\nu = \sum_{i=1}^{\tau_1} \widetilde{S}_i + \cdots + \sum_{i=\tau_{\nu-1}+1}^{\tau_\nu} \widetilde{S}_i = \sum_{i=1}^{\tau_\nu} \widetilde{S}_i < 0$$

implying $\mathbb{P}\{\tau_\nu \le N\} \le \mathbb{P}\{\min_{1\le k\le N} \sum_{i=1}^{k} \widetilde{S}_i < 0\} = 1 - \tilde{p}_N$, and hence

$$\tilde{p}_N \le \mathbb{P}\{\tau_\nu > N\}. \tag{19}$$

We stress that (19) is true for every random walk, but the r.v.'s ξ_i are i.i.d. if S_n is upper exponential (or, of course, if S_n is integer-valued and either upper geometric or right-continuous).

By a Tauberian theorem (see [8, Ch. XIII]), the asymptotics of $\mathbb{P}\{\tau_\nu > N\}$ as $N \to \infty$ can be found if we know the behavior of the generating function $\chi(t)$ of τ_ν as $t \nearrow 1$: for any $p \in (0, 1)$ and $c > 0$,

$$\mathbb{P}\{\tau_\nu > N\} \sim \frac{c}{\Gamma(p)N^{1-p}} \Longleftrightarrow 1 - \chi(t) \sim c(1-t)^{1-p}. \tag{20}$$

Let us first find the generating function of the joint distribution of ν and τ_ν. For any positive integer k and l,

$$\begin{aligned}\mathbb{P}\{\nu = k, \tau_\nu = l\} = \mathbb{P}\{&\xi_1 \ge 0, \ldots, \xi_1 + \cdots + \xi_{k-1} \ge 0,\\ &\xi_1 + \cdots + \xi_k < 0, \theta_1 + \cdots + \theta_k = l\}.\end{aligned}$$

The r.v. ν is the first descending ladder epoch of the walk $\xi_1+\cdots+\xi_n$, and its generating function is described by the Sparre–Andersen theorem; see [8, Ch. XII]. Sinai [15, Lemma 3] gives the following straightening of this result: the generating function

$$\chi(s,t) := \sum_{k,l\ge 1} \mathbb{P}\{\nu = k, \tau_\nu = l\} s^k t^l$$

of the random vector (ν, τ_ν) satisfies

$$\ln \frac{1}{1-\chi(s,t)} = \sum_{k,l\ge 1} \frac{s^k t^l}{k} \mathbb{P}\{\xi_1 + \cdots + \xi_k < 0, \theta_1 + \cdots + \theta_k = l\}.$$

By Lemma 2, for the generating function $\chi(t) := \chi(1,t)$ of τ_ν it holds that

$$\ln \frac{1}{1-\chi(t)} = \sum_{k,l\geq 1} \frac{t^l}{k} \mathbb{P}\{\xi_1 + \cdots + \xi_k < 0,\ \theta_1 + \cdots + \theta_k = l\}$$
$$= \frac{1}{2} \sum_{k,l\geq 1} \frac{t^l}{k} \mathbb{P}\{\theta_1 + \cdots + \theta_k = l\}. \tag{21}$$

Since θ_k are i.i.d.,

$$\sum_{k,l\geq 1} \frac{t^l}{k} \mathbb{P}\{\theta_1 + \cdots + \theta_k = l\} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} t^l \mathbb{P}\{\theta_1 + \cdots + \theta_k = l\}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} \zeta^k(t) = \ln \frac{1}{1-\zeta(t)},$$

where $\zeta(t)$ is the generating function of θ_1. Then

$$1 - \chi(t) = \sqrt{1-\zeta(t)}, \tag{22}$$

and using Part (a) of Proposition 1 and the Tauberian theorem (20) twice, we get $\mathbb{P}\{\tau_\nu > N\} \sim cN^{-1/4}$. By (3) and (19), the upper bound follows.

Case 2. S_n is an integer-valued random walk.

We argue exactly as in the proof of the first part. Replacing everywhere ξ_n and θ_n by ξ_n^0 and θ_n^0, respectively, we get $p_N \leq \mathbb{P}\{\tau_{\nu^0}^0 > N\}$ instead of (19) and

$$1 - \chi^0(t) = \sqrt{1-\zeta^0(t)} \mathrm{e}^{H(t)}$$

instead of (22), where

$$H(t) := \frac{1}{2} \sum_{k,l\geq 1} \frac{t^l}{k} \mathbb{P}\left\{\xi_1^0 + \cdots + \xi_k^0 = 0,\ \theta_1^0 + \cdots + \theta_k^0 = l\right\}$$

emerges in the analogue of (21). The limit $\lim_{t\to 1} H(t)$ exists and is finite because $H(t)$ is increasing and the series

$$H(1) = \sum_{k,l\geq 1} \frac{1}{k} \mathbb{P}\left\{\xi_1^0 + \cdots + \xi_k^0 = 0, \theta_1^0 + \cdots + \theta_k^0 = l\right\}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}\left\{\xi_1^0 + \cdots + \xi_k^0 = 0\right\} = c_0$$

is convergent for any random walk. Hence the upper bound follows from Part (a) of Proposition 1 and the Tauberian theorem (20) as above.

5. The lower bound

By (4), we estimate

$$\widetilde{\mathbb{P}}\left\{\min_{1\leq k\leq N} \sum_{i=1}^{k} S_i \geq 0\right\} \geq \mathbb{P}\left\{\min_{1\leq k\leq \sqrt{N}} \sum_{i=1}^{k} \xi_i \geq 0, \eta(N) + 1 \leq \sqrt{N}\right\}$$

$$= \mathbb{P}\left\{\min_{1\le k\le\sqrt{N}}\sum_{i=1}^{k}\xi_i \ge 0, \theta_1+\cdots+\theta_{\sqrt{N}} > N\right\}$$
$$\ge \mathbb{P}\left\{\min_{1\le k\le\sqrt{N}}\sum_{i=1}^{k}\xi_i \ge 0, \theta_1^+ +\cdots+\theta_{\sqrt{N}}^+ > N\right\}.$$

By Lemma 3 and the sufficient condition of association (c),

$$\widetilde{\mathbb{P}}\left\{\min_{1\le k\le N}\sum_{i=1}^{k}S_i \ge 0\right\} \ge \mathbb{P}\left\{\min_{1\le k\le\sqrt{N}}\sum_{i=1}^{k}\xi_i \ge 0\right\}\cdot\mathbb{P}\left\{\theta_1^+ +\cdots+\theta_{\sqrt{N}}^+ > N\right\}$$
$$\ge c\mathbb{P}\left\{\min_{1\le k\le\sqrt{N}}\sum_{i=1}^{k}\xi_i \ge 0\right\}$$

for some $c > 0$ and all N, where we used Part (a) of Proposition 1 for the last line.

Under the assumptions of Part 2 of Theorem 2, the distribution of ξ_1 is symmetric; see Lemma 2 for the case of two-sided exponential walks. Hence for the random walk $\sum_{i=1}^{k}\xi_i$ we have $c_+ = -c_0/2$, which is always finite, and Part 2 of Theorem 2 follows from (3), (7) and (8).

The proof of Part 1 of Theorem 2, actually, takes much more effort because it requires the use of Corollary 1 of Proposition 1. The latter implies that $\mathbb{P}\{\xi_1+\cdots+\xi_n > 0\} \to 1/2$. Unfortunately, we cannot verify that the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\mathbb{P}\{\xi_1+\cdots+\xi_n > 0\} - 1/2\right) \tag{23}$$

converges, and we should use (10) instead of (8).

Convergence of series of the type (23) was studied by Egorov [5], who considered rates of convergence in stable limit theorems and stated his results exactly in the form of (23). It is, however, unclear how to check his conditions for our case. A proof of the convergence would eliminate the slowly varying factor $l(N)$ in Theorem 2.

6. Open questions and concluding remarks

1. Obtaining the lower bound under less restrictive conditions.

The most restrictive assumptions of Theorem 2 are the ones imposed on $\mathrm{Law}(S_1|S_1 < 0)$. We used these assumptions *only* in the proof of association of ξ_1 and θ_1^+. It seems that these variables are associated under much less restrictive conditions and, possibly, under no assumptions at all. Simulations show that association holds in many cases. Note that the direct use of the sufficient condition of association (c) is impossible because ξ_1 is *not* a coordinatewise increasing function of associated r.v.'s $\widetilde{S}_1, \widetilde{S}_2, \ldots$.

2. Elimination of the slowly varying term in Theorem 2.

As we explained above, the slowly varying factor could be eliminated if we show that the series (23) is convergent. The rate of convergence in stable limit theorems is usually estimated under the existence of so-called pseudomoments of ξ_1. The pseudomoment of ξ_1 of order 1/3 exists if the functions $x^{1/3}\mathbb{P}\{\xi_1 > x\}$ and $x^{1/3}\mathbb{P}\{\xi_1 < -x\}$ have a regular behavior as $x \to \infty$. It seems that the "tails" of ξ_1 could be controlled if we had an appropriate rate of convergence of discrete excursions to a Brownian excursion. We know only one result on this question: Drmota and Marckert [4] gives the rate of convergence of positive excursions of left-continuous random walks. Since we need rates for both positive and negative excursions, the only slackened random walks would be covered, giving no refinement to Theorem 2.

3. When the first draft of this paper was already written, the author became aware that Frank Aurzada and Steffen Dereich were also working on one-sided small deviation probabilities of integrated random processes, and they considered p_N as a particular case. The methods of their paper [1] are entirely different from the ones presented here.

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References

[1] F. Aurzada, S. Dereich, Universality of the asymptotics of the one-sided exit problem for integrated processes, Preprint. www.math.tu-berlin.de/~aurzada/publications/onesidedint.pdf.

[2] F. Caravenna, J.-D. Deuschel, Pinning and wetting transition for (1 + 1)-dimensional fields with Laplacian interaction, Ann. Probab. 36 (2008) 2388–2433.

[3] L. de Haan, E. Omey, S. Resnick, Domains of attraction and regular variation in $\mathbb{R}^d$, J. Multivariate Anal. 14 (1984) 17–33.

[4] M. Drmota, J.-F. Marckert, Reinforced weak convergence of stochastic processes, Statist. Probab. Lett. 71 (2005) 283–294.

[5] V.A. Egorov, On the rate of convergence to a stable law, Theory Probab. Appl. 25 (1980) 180–187.

[6] M.S. Eppel, A local limit theorem for the first overshoot, Sib. Math. J. 20 (1979) 130–138.

[7] J.D. Esary, F. Proschan, D.W. Walkup, Association of random variables, with applications, Ann. Math. Statist. 38 (1967) 1466–1474.

[8] W. Feller, An Introduction to Probability Theory and its Applications, vol. 2, Wiley, New York, 1966.

[9] Y. Isozaki, S. Watanabe, An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai's estimates for the integral of Brownian motion, Proc. Japan Acad. Ser. A 70 (1994) 271–276.

[10] S. Janson, Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas, Probab. Surv. 4 (2007) 80–145.

[11] W.D. Kaigh, An invariance principle for random walk conditioned by a late return to zero, Ann. Probab. 4 (1976) 115–121.

[12] H.P. McKean, A winding problem for a resonator driven by a white noise, J. Math. Kyoto Univ. 2 (1963) 227–235.

[13] B.A. Rogozin, The distribution of the first ladder moment and height and fluctuation of a random walk, Theory Probab. Appl. 16 (1971) 575–595.

[14] M. Shimura, A class of conditional limit theorems related to ruin problem, Ann. Probab. 11 (1983) 40–45.

[15] Ya.G. Sinai, Distribution of some functionals of the integral of a random walk, Theoret. Math. Phys. 90 (1992) 219–241.

[16] F. Spitzer, Principles of Random Walk, Springer, New York, 1964.

[17] V.A. Vatutin, V. Wachtel, Local probabilities for random walks conditioned to stay positive, Probab. Theory Related Fields 143 (2009) 177–217.

[18] M. Vergassola, B. Dubrulle, U. Frisch, A. Noullez, Burgers' equation, Devil's staircases and the mass distribution function for large-scale structures, Astron. Astrophys. 289 (1994) 325–356.

[19] V. Vysotsky, Clustering in a stochastic model of one-dimensional gas, Ann. Appl. Probab. 18 (2008) 1026–1058.