Cyclicity 1 and 2 Conditions for a 2-Polycycle of Integrable Systems on the Plane*

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We obtain generic conditions for a 2-polycycle of an integrable system to have cyclicity one and two respectively under multiple parameter perturbations.

Key Words: polycycle; cyclicity; limit cycle; bifurcation.

1. INTRODUCTION AND MAIN RESULTS

Consider the $C^\infty$ planar system

$$
\begin{align*}
\dot{x} &= P(x, y) + \varepsilon P(x, y, \delta), \\
\dot{y} &= Q(x, y) + \varepsilon Q(x, y, \delta),
\end{align*}
$$

(1.1)

where $P$, $Q$, $P$, $Q$ are $C^\infty$ functions, $\varepsilon > 0$ is small, and $\delta \in U \subset \mathbb{R}^n$ with $U$ bounded. Suppose for $\varepsilon = 0$ (1.1) has a 2-polycycle $L = L_1 \cup L_2 \cup S_{10} \cup S_{20}$, where $L_1$ and $L_2$ are heteroclinic orbits connecting hyperbolic saddle points $S_{10}$ and $S_{20}$. Without loss of generality, we suppose that $L$ is counterclockwise oriented and that

$$
\begin{align*}
z_1(t) &\to S_{10} \quad (\text{or} \quad S_{20}), \\
z_2(t) &\to S_{20} \quad (\text{or} \quad S_{10}) \quad \text{as} \quad t \to -\infty \quad (\text{or} \quad +\infty),
\end{align*}
$$

(1.2)

where $z_i(t)$ is a time-parameter representation of $L_i$, $i = 1, 2$. Let $S_i$ be the saddle point of (1.1) near $S_{10}$ for $\varepsilon$ small. The hyperbolic ratio of $S_i$ is given

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by \( r_i(\varepsilon, \delta) = -\lambda_{i1}/\lambda_{i2} \) where \( \lambda_{i1} < 0 < \lambda_{i2} \) are the two eigenvalues of \( S_i \), \( i = 1, 2 \). For small \( \varepsilon \) we have

\[
r_i(\varepsilon, \delta) = r_{i0} + \varepsilon r^*_i(\delta) + O(\varepsilon^2), \quad i = 1, 2.
\]

From [1, 2] we know that if \( L \) is non-trivial (i.e., it is isolated) and \( r_{10}r_{20} \neq 1 \) then (1.1) has at most two limit cycles near \( L \) for \( \varepsilon > 0 \) small and \( \delta \in U \). In other words, \( L \) has cyclicity 1 or 2 if \( r_{10}r_{20} \neq 1 \). From [2, 3] we know that if \( r_{10}r_{20} = 1 \), (1.1) may have more than 2 limit cycles.

Now we suppose for \( \varepsilon = 0 \) (1.1) has a first integral \( I(x, y) \) of class \( C^1 \) such that \( L \) is given by \( I(x, y) = 0 \) and the periodic orbits near \( L \) are given by \( L_h: I(x, y) = \varepsilon h \), \( 0 < h << 1 \). In this case, \( L \) is nonisolated and \( r_{10}r_{20} = 1 \). Hence, from (1.3) we have

\[
r_1(\varepsilon, \delta) r_2(\varepsilon, \delta) = 1 + \varepsilon r^*(\delta) + O(\varepsilon^2), \quad r^*(\delta) = r_{10}r^*_1(\delta) + r_{20}r^*_2(\delta).
\]

From [4] if

\[
M^*(\delta) \equiv \lim_{h \to 0} \frac{1}{L_h} \int_{L_h} (I_x P_0 + I_y Q_0) \, dt
\]

is finite and not identically zero and one of the connections \( L_1 \) and \( L_2 \) is fixed under perturbations then a necessary condition for \( L \) to generate a limit cycle is \( M^*(\delta_0) = 0 \) for some \( \delta = \delta_0 \) and at most one limit cycle can be generated for \( \varepsilon + |\delta - \delta_0| \) small as \( r^*(\delta_0) \neq 0 \). It was proved in [5] that if the unperturbed system is Hamiltonian, then a necessary condition for \( L \) to generate a limit cycle is \( M^*(\delta_0) = 0 \) for some \( \delta = \delta_0 \), and at most two limit cycles can be generated for \( \varepsilon + |\delta - \delta_0| \) small as \( r_i^*(\delta_0) \neq 0 \), \( i = 1, 2 \), and \( r^*(\delta_0) \neq 0 \). When the vector parameter \( \delta \) in (1.1) does not appear, (1.4) becomes

\[
r_1(\varepsilon) r_2(\varepsilon) = 1 + \varepsilon r^* + O(\varepsilon^2).
\]

In this case, it was proved in [6] that if (1.1) is an analytic system, and \( r^* \neq 0 \) at most two limit cycles can appear near \( L \) for \( \varepsilon > 0 \) small.

In this paper we consider the multiple parameter perturbed system (1.1) under the assumption that for \( \varepsilon = 0 \) (1.1) is an integrable system. Our main results can be stated as follows.

**Theorem 1.1.** Suppose that for \( \varepsilon = 0 \) (1.1) has a \( C^4 \) first integral \( I(x, y) \) which is of class \( C^\infty \) when \( r_{10} \) is rational. Let (1.2) hold. Then

(i) A necessary condition for \( L \) to generate a limit cycle is \( M(\delta_0) = 0 \) for some \( \delta = \delta_0 \in U \), where
$$M_1(\delta) = \begin{cases} M_1(\delta) + M_2(\delta), & \text{if } r_{10} = 1, \\ M_1(\delta), & \text{if } r_{10} > 1, \\ M_2(\delta), & \text{if } r_{10} < 1, \end{cases}$$

$$M_i(\delta) = \int_{-\infty}^{\infty} e^{-\int_{0}^{t}(P_0 + Q_0 z_i(t)) dt} (P_0 Q - Q_0 P)(z_i(t), 0, \delta) dt, \quad i = 1, 2.$$  

(ii) If $M(\delta_0) = 0$, $r^*(\delta_0) \neq 0$, then (1.1) has at most two limit cycles near $L$ for $\varepsilon + |\delta - \delta_0|$ small.

**Theorem 1.2.** Let the assumption of Theorem 1.1 hold. If the connection $L_1$ (resp., $L_2$) is fixed for all $\varepsilon > 0$ small and $\delta \in U$, then a necessary condition for $L$ to generate a limit cycle is $M_2(\delta_0) = 0$ (resp., $M_1(\delta_0) = 0$) for some $\delta = \delta_0 \in U$, and if further, $r^*(\delta_0) \neq 0$, at most one limit cycle can appear near $L$ for $\varepsilon + |\delta - \delta_0|$ small.

**Theorem 1.3.** Let the assumption of Theorem 1.1 hold. If one of the following conditions holds

(i) $r_{10} = 1$, $M_1(\delta_0) = 0$, $r_1^*(\delta_0) r_2^*(\delta_0) > 0$;
(ii) $r_{10} = 1$, $M_1(\delta_0) = -M_2(\delta_0) < 0$, $r_2^*(\delta_0)[r_2^*(\delta_0) + r_2^*(\delta_0)] > 0$;
(iii) $r_{10} = 1$, $M_1(\delta_0) = -M_2(\delta_0) > 0$, $r_2^*(\delta_0)[r_2^*(\delta_0) + r_2^*(\delta_0)] > 0$;
(iv) $r_{10} > 1$, $M_1(\delta_0) = 0$, $M_2(\delta_0) \neq 0$;
(v) $r_{10} < 1$, $M_2(\delta_0) = 0$, $M_1(\delta_0) \neq 0$,

then at most one limit cycle can exist near $L$ for $\varepsilon + |\delta - \delta_0|$ small.

We have immediately from Theorems 1.1 and 1.3.

**Corollary.** Under the condition of Theorem 1.1 the 2-polycycle $L$ has cyclicity 1 (resp., 2) if one of the conditions (i)--(v) of Theorem 1.3 holds (resp., $M(\delta_0) = 0$, $r^*(\delta_0) \neq 0$, and none of the conditions (i)--(v) of Theorem 1.3 holds) for $\varepsilon + |\delta - \delta_0|$ small.

The paper is organized as follows. In Section 2, a complete proof of Theorems 1.1--1.3 is presented. In Section 3 bifurcation diagrams near $L$ are given for $\varepsilon + |\delta - \delta_0|$ small with $\delta = (\delta_1, \delta_2) \in R^2$.

We remark that the proof here is very different from that of [6], and the technique used in [6] is not valid for the multiple parameter perturbed system (1.1).
2. PROOF OF THE MAIN RESULTS

Let $\sigma_i$ and $\tau_i$ be segments normal to $L$ near the points $S_{i0}$, $i = 1, 2$. Using positive or negative orbits of (1.1) we can define Dulac maps $D_i: \sigma_i \to \tau_i$, $i = 1, 2$ and regular maps $R_1: \tau_1 \to \tau_2$, and $R_2: \sigma_1 \to \sigma_2$ (see Fig. 2.1).

Let

$$F_i(u, \varepsilon, \delta) = F_1(u, \varepsilon, \delta) - F_2(u, \varepsilon, \delta), \quad F_1 = R_1 \circ D_1, \quad F_2 = D_2 \circ R_2.$$  

(2.1)

The function $F$ is called a bifurcation function of (1.1). Let $D_1$ and $D_2$ be defined for $0 < u < 1$ with $D_i(0) = 0$. Then from [7, 8] the functions $R_i$ are of the form

$$R_i(u) = (-1)^i \alpha_i(\varepsilon, \delta) + \alpha_i(\varepsilon, \delta) u + O(u^2), \quad i = 1, 2, \quad (2.2)$$

where $\alpha_i(0, \delta) = \alpha_i > 0$, and

$$d_i(\varepsilon, \delta) = \varepsilon N_i M_i(\delta) + O(\varepsilon^2), \quad i = 1, 2,$$

$$N_i = [P_0^i(A_i) + Q_0^i(A_i)]^{1/2}, \quad A_1 = \tau_2 \cap L_1, \quad A_2 = \sigma_2 \cap L_2.$$  

(2.3)

Denote by $X_\varepsilon$ the vector field defined by (1.1). Let $r_0 = r_{10}$. From [9], if $r_0$ is irrational for any fixed natural number $k$, $X_\varepsilon$ is $C^k$-equivalent to

$$x' = x, \quad y' = -r_1(\varepsilon, \delta) y$$  

(2.4)

in a neighborhood of $S_{10}$, and $-X_\varepsilon$ $C^k$-equivalent to

$$x' = -s(\varepsilon, \delta) x, \quad y' = y$$  

(2.5)

in a neighborhood of $S_{20}$, where $s(\varepsilon, \delta) = r_2^{-1}(\varepsilon, \delta)$. If $r_0 = p/q$ is rational, for any fixed natural number $k$ there exists an integer $N(k)$ such that $X_\varepsilon$ is $C^k$-equivalent to

$$x' = x, \quad y' = y \left[ -r_0 + \sum_{i=0}^{N(k)} \alpha_i(\varepsilon, \delta) (x^py^q)^i \right]$$  

(2.6)

in a neighborhood of $S_{10}$, and $-X_\varepsilon$ $C^k$-equivalent to

$$x' = x \left[ -r_0 + \sum_{i=0}^{N(k)} \beta_i(\varepsilon, \delta) (x^py^q)^i \right], \quad y' = y$$  

(2.7)

in a neighborhood of $S_{20}$. In particular,

$$\alpha_1(\varepsilon, \delta) = r_0 - r_1(\varepsilon, \delta), \quad \beta_1(\varepsilon, \delta) = r_0 - s(\varepsilon, \delta).$$  

(2.8)
We can suppose that the formulae (2.4)–(2.7) are valid in a ball of radius 2. This implies that we can take the segments $\sigma_i$, $\tau_i$ as
\[
\sigma_1 = \tau_2 = \{(x, y) \mid y = 1, 0 \leq x \leq 1\}, \quad \tau_1 = \sigma_2 = \{(x, y) \mid x = 1, 0 \leq y \leq 1\}.
\]

By (2.3)–(2.7), this choice leads to
\[
N_1 = N_2 = 1. \quad (2.9)
\]

From [9, 10], the Dulac maps $D_i$ have the following well ordered asymptotic expansions,
\[
D_1(u) = u^{\omega_1(u, r_0)}, \quad D_2(u) = u^{\omega_2(u, r_0)} \quad \text{for } r_0 \text{ irrational}, \quad (2.10)
\]

and
\[
D_1(u) = u^{\omega_0} + \alpha_1 u^{\omega_1} + \sum_{1 \leq j \leq k + 1, 1 \leq i} \alpha_i u^{(q+1)\omega_1 + \psi_1}, \quad (2.11)
\]
\[
D_2(u) = u^{\omega_0} + \beta_1 u^{\omega_2} + \sum_{1 \leq j \leq k + 1, 1 \leq i} \beta_i u^{(q+1)\omega_2 + \psi_2},
\]

for $r_0 = p/q$ rational, where $\alpha_i$ (resp., $\beta_i$) are polynomials in $\omega_1, \ldots, \omega_{N(k)+1}$ (resp., $\beta_1, \ldots, \beta_{N(k)+1}$). $\psi_1, \psi_2$ are $C^k$ functions, $k$-flat with respect to $u$ at $u=0$ and satisfy
\[
\frac{\partial^n}{\partial u^n} (u^{-k}\psi_i) \to 0 \quad \text{uniformly in } \varepsilon \text{ as } u \to 0
\]

for all natural numbers $n$, and $K$ is an integer and
\[
\omega_1 = \omega(u, \alpha_1), \quad \omega_2 = \omega(u, \beta_1) \text{ with } \omega(u, \gamma) = \int_u^1 \gamma dt.
\]
Note that
\[ u^0 + \alpha_i u^0 \omega_i = u^i, \quad u^a + \beta_i u^a \omega_2 = u^i. \] (2.12)

From (2.6) and (2.7), we have \( \alpha_i = \beta_{k+1} = 0 \) (resp., \( \beta_i = \beta_{k+2} = 0 \)) if \( i = 0 \) (resp., \( \beta_i = 0 \)) for \( i = 2, \ldots, N(k+1) \). Since (1.11) \( \varepsilon = 0 \) has a \( C^\infty \) first integral in the case \( r_0 = p/q \), it follows from [12] that \( \alpha_i(0, \delta) = \beta_i(0, \delta) = 0, i = 2, \ldots, N(k+1) \). Hence, we have that
\[ \alpha_i = \beta_i = \psi_{k+1} = \psi_{k+2} = 0 \quad \text{for} \quad \varepsilon = 0. \]

Therefore, by (2.8), (2.10)–(2.12) we can write
\[ D_1(u) = u^0[1 + \epsilon f_1(u, \varepsilon, \delta)], \quad D_2(u) = u^0[1 + \epsilon f_2(u, \varepsilon, \delta)], \] (2.13)
where \( f_1 \) and \( f_2 \) satisfy for any \( 0 < l < p \) and \( n = 0, 1, 2 \)
\[ u^n - \frac{\partial^n f_i}{\partial u^n} \to 0 \quad \text{as} \quad u \to 0. \] (2.14)

Let \( D_{i0} = D_i|_{x=0} \), \( R_{i0} = R_i|_{x=0} \), \( i = 1, 2 \). The mean value theorem gives that
\[ R_1(D_1) - R_1(D_{10}) = R_{10} - (D_1 - D_{10}), \]
\[ R_1(D_{10}) - R_{10}D_{10} = R_{10}e, \] (2.15)
\[ D_2(R_2) - D_2(R_{20}) = D_{20} - R_{20} = D_{20} - R_{20}, \]
\[ D_{10} = M_1(\delta) + O(e + D_{10}), \]
\[ R_{10} = -M_2(\delta) + O(e + u), \]
\[ R_{10} = a_1 + o(1), \]
\[ D_{20} = \int_0^1 [R_{20} + t(R_2 - R_{20})]^{t-1} (1 + o(e)) dt. \]

Notice that from (1.3), (2.8), (2.12), and (2.13)
\[ D_1(u) = u^0 - (1) \int a u^a \omega_2 \cdot (r_2^*(\delta) + o(1)), \quad R_{20} = a_{20} u + O(u^2) \]
and that \( F(u, 0, \delta) = R_{10} + D_{10} - D_{20} - R_{20} = 0 \). We have from (2.1) and (2.15)

\[
F(u, e, \delta) = e[M_1(\delta) + sM_2(\delta)R(u, e, \delta)(1 + o(e)) + f_0(u, e, \delta)]
\]

\[
= eF_0(u, e, \delta),
\]

where

\[
R(u, e, \delta) = \int_0^1 \left[ R_{20} + t(R_2 - R_{20}) \right]^{\tau - 1} dt,
\]

\[
f_0(u, e, \delta) = -u^s[a_1\omega_1(r_1^*(\delta) + o(1)) + a_2\omega_2(r_2^*(\delta) + o(1))] + O(e + u).
\]

Now it is clear that from (2.16) and (2.17)

\[
F_0(u, 0, \delta) = \begin{cases} 
M_1(\delta) + M_2(\delta) + u(1 + o(1)), & \text{if } r_0 = 1, \\
M_1(\delta) + O(u^{r_0 - 1}), & \text{if } r_0 > 1, \\
M_1(\delta) + r_0 M_2(\delta)(a_20 u + O(u^2))^{r_0 - 1} + O(|u^0 \ln u|), & \text{if } r_0 < 1.
\end{cases}
\]

The first conclusion of Theorem 1.1 follows from (2.18) easily. Further, from (2.2), (2.13), and (2.14) we have

\[
R_1(u) = a_i(1 + g_i(u, e, \delta)), \quad g_i = O(u) \in C^{\delta - 1}, \quad i = 1, 2,
\]

\[
D_1(u) = r_1 u^{r_1 - 1}(1 + e h_1(u, e, \delta)), \quad D_2(u) = su^{s - 1}(1 + e h_2(u, e, \delta)),
\]

\[
u^{-\delta_1}, u^{1 - \delta_2} \to 0, \quad i = 1, 2, \quad \text{as } u \to 0.
\]

Hence, we have

\[
F'_u = R_1(D_1) - D'_1(D_2) - R'_2 = sa_i(1 + g_2)(A(1 + e h_1)
\]

\[
\times (1 + g_2(D_1, e, \delta))(1 + g_2)^{-1} u^{r_2 - 1} - R_2^{r_2 - 1}(1 + e h_2(R_2, e, \delta))),
\]

where \( A = A(e, \delta) = a_i r_i / (a_2 s) \). Now let \( M(\delta_0) = 0, r^*(\delta_0) \neq 0 \). We want to prove that \( F \) has at most two roots in \( u > 0 \) for \( e + |\delta - \delta_0| \) small. We first consider the case of \( r_0 = 1 \). Then \( r_0^*(\delta_0) \neq 0 \) becomes \( r^*_1(\delta_0) \neq 0 \). In particular, \( r^*_1(\delta_0) \neq 0 \). Without loss of generality, we assume \( r^*_1(\delta_0) \neq 0 \). Let

\[
x - 1 = e\lambda(\varepsilon, \delta), \quad r_1 - 1 = e\mu(\varepsilon, \delta), \quad \gamma(\varepsilon, \delta) = (r_1 - 1)/(s - 1) = \mu/\lambda.
\]

(2.21)
Using (1.2) and \( r_0 = 1 \), we have

\[
\lambda(0, \delta_0) = -r^*_2(\delta_0) \neq 0, \quad \gamma_0 = \gamma(0, \delta_0) = -r^*_2(\delta_0) \neq 1. \tag{2.22}
\]

Since for \( u > 0 \), \( F'_u |_{\varepsilon = 0} = 0 \), we have from (2.20), \( A(0, \varepsilon) = a_{20}/a_{20} = 1 \), \( g_1(D_{10}, 0, \delta) = g_1(u, 0, \delta) = g_2(u, 0, \delta) \equiv g_{20} \). It follows from (2.19) that \( g_1(D_1, v, \varepsilon, \delta) = g_{20} + \varepsilon g^*_1 \), \( g_2(u, v, \delta) = g_{20} + \varepsilon g^*_2 \) with \( g^*_1 = O(u\omega_1) \), \( g^*_2 = O(u) \). Hence

\[
A(1 + \varepsilon h_1)(1 + g_1(D_1, v, \varepsilon, \delta))(1 + g_2) - 1 = 1 + \varepsilon g_0(u, v, \delta),
\]

where from (2.19) \( g_0 \) satisfies

\[
\frac{1}{u} g_0, \frac{1}{u} g'_0 u \to 0, \quad i = 1, 2, \quad \text{as} \quad u \to 0. \tag{2.23}
\]

From (2.20) and (2.21), \( F'_u = 0 \) if and only if

\[
G(u, v, \delta) \equiv R_2(1 + \varepsilon h_2(1 + g_0))^{1/2} - u'(1 + \varepsilon g_0)^{1/2} = 0. \tag{2.24}
\]

It suffices to prove that \( G \) has at most one root in \( u > 0 \) for \( \varepsilon + |\delta - \delta_0| \) small. From (2.22) and using \( \ln(1 + u) = u + O(u^2) \) we have

\[
(1 + \varepsilon h_2)^{1/2} = 1 + O(h_2), \quad (1 + \varepsilon g_0)^{1/2} = 1 + O(g_0). \tag{2.25}
\]

It is direct that

\[
[u(1 + \varepsilon h_2(u, v, \delta))^{1/2}]_u = (1 + \varepsilon h_2)^{1/2} \left( 1 + \frac{uh'_2u}{\lambda(1 + \varepsilon h_2)} \right),
\]

\[
[u'(1 + \varepsilon g_0)^{1/2}]_u = u'(1 + \varepsilon g_0)^{1/2} \left( 1 + \frac{ug'_0u}{\lambda(1 + \varepsilon g_0)} \right) u.
\]

Therefore, from (2.24) and (2.25) we obtain that

\[
G'_u = a_{20}(1 + o(1)) - u^{-1}(1 + o(1)) [\gamma + O(\varepsilon g'_0u)] = 0. \tag{2.26}
\]

Noting (2.22) we have that if \( \gamma_0 \neq 0 \), then \( G'_u \neq 0 \) for small \( u > 0 \) and \( \varepsilon + |\delta - \delta_0| \). If \( \gamma_0 = 0 \), and if \( G \) has a root \( u > 0 \) for \( \varepsilon + |\delta - \delta_0| \) small, then from (2.24) and (2.25) the root satisfies \( u' = R_2(1 + O(g_0 + h_2)) \to 0 \) as \( \varepsilon + u \to 0 \). This leads to \( \gamma > 0 \) and \( 0 < u < \exp(-1/\gamma) \). Hence

\[
\frac{u^\gamma}{\gamma} < \gamma \exp \left( -\frac{\gamma - 1}{\gamma} \right) \to +\infty, \quad \frac{u^\gamma}{\gamma} < (\gamma \exp(1/\gamma))^{-1} \to 0 \tag{2.27}
\]

as \( \varepsilon + |\delta - \delta_0| \to 0 \). Using (2.27) and (2.23), it implies from (2.26) that

\[
G'_u |_{\varepsilon = 0} \quad \text{for} \quad \varepsilon + |\delta - \delta_0| \quad \text{small}. \tag{2.28}
\]
Thus, we see that no matter whether $\gamma_0 = 0$, $G$ has at most one root in $u > 0$ for $\varepsilon + |\delta - \delta_0|$ small. Hence by (2.24), $F$ has at most two roots in $u > 0$ for $\varepsilon + |\delta - \delta_0|$ small. Theorem 1.1 is proved in the case $r_0 = 1$.

Next we consider the case $r_0 \neq 1$. From (2.20), $F'_u = 0$ if and only if

$$G \equiv R_0(1 + \varepsilon h_0(R_2, \varepsilon, \delta)) \prod_{i=1}^{n}(1 + \varepsilon h_i(D_1, \varepsilon, \delta)) \prod_{i=1}^{n}(1 + g_i) \prod_{i=1}^{n}(1 + g_2) - A(1 - (s - 1)u^i (r - 1) = 0.$$  

Note that $s = r_0 - \varepsilon \alpha^2 \beta_1^2 \delta + O(\delta^2), \ (r_1 - 1)/(s - 1) = 1 + [(r_1 - s)/(s - 1)] = 1 + \varepsilon \alpha \beta \delta$, $r_0 \neq 1$, $\lambda(0, \delta_0) = r_0 r^*(\delta)/(r_0 - 1) \neq 0$. We can write

$$G = R_0(1 + \varepsilon h_2(R_2, \varepsilon, \delta)) - (A_0 + \varepsilon A_1)(1 + \varepsilon h_2) H_1 H_2 u^{\varepsilon + i_2},$$

where from (2.19)

$$H_1 = H_1(u, \varepsilon, \delta) = 1 + g_1(D_1, \varepsilon, \delta), \ H_2 = H_2(u, \varepsilon, \delta) = 1 + g_2(D_1, \varepsilon, \delta), \ g_i = O(u) \in C^k, \ u^{-1} g_i \to 0, \ i = 1, 2, \ as \ u \to 0.$$

The equality $G|_{u=0} = 0$ gives that $R_{10} \equiv A_0 H_{10} H_{20} u, H_{i0} = H_i(u, 0, \delta), i = 1, 2.$ The mean value theorem implies that

$$H_1 - H_{10} = H_1(D_1 - D_{10}) + H_{12} e = H_{11} H_3 e + H_{12} e, \ H_2 - H_{20} = H_{21} e, \ R_2 - R_{20} = R_{21} e,$$

where

$$H_{11} = \int_0^1 g_1(D_{10} + t(D_1 - D_{10}), \varepsilon, \delta) \ dt, \ H_{12} = \int_0^1 g_2(D_{10}, \varepsilon, \delta) \ dt, \ H_{21} = O(u), \ R_{21} = M_2(\delta) + O(\varepsilon + u) \in C^k, \ u^{-1} H_3, u^{-1} H_{i0} \to 0, \ as \ u \to 0.$$

Then from (2.29) we have

$$G = \varepsilon R_0 h_2(R_2, \varepsilon, \delta) + R_2 - R_{20} - [\varepsilon A_1(1 + \varepsilon h_1) H_1 H_2 u^{\varepsilon + i_2} + A_0 h_1 H_1 H_2 u^{\varepsilon + i_2} + A_0 h_2 H_1 H_2 u^{\varepsilon + i_2} - A_0 h_1 H_1 H_2 H_{10} u^{\varepsilon + i_2} + H_{21} H_{10} u^{\varepsilon + i_2} + H_{20} H_{10} u(u^{i_2} - 1)/\varepsilon) ].$$
Note that \( u^\delta - 1 = -\varepsilon \lambda \vartheta(u, -\varepsilon \lambda) \). We obtain
\[
G = \varepsilon R_h \lambda \vartheta(\varphi \lambda, \varepsilon \lambda, \delta) + R_{\varphi \lambda} - uH_4 + \left( A_0 H_{20} H_{10} + H_4 \right) \lambda \vartheta(u, -\varepsilon \lambda)
\]
\[
= \varepsilon H(u, \varepsilon, \delta),
\]
(2.32)
where \( H_4 = A_1 (1 + \varepsilon \delta_1) H_1 H_2 + A_0 H_1 H_2 + A_0 H_1 H_3 + H_{12} H_2 + H_{21} H_{10} \). By (2.30) and (2.31) we can prove easily that
\[
H_4 = A_1 (1 + \varepsilon \delta_1) H_1 H_2 + A_0 H_1 H_2 + A_0 H_1 H_3 + H_{12} H_2 + H_{21} H_{10}.
\]
Hence
\[
H_4 = A_1 (1 + \varepsilon \delta_1) H_1 H_2 + A_0 H_1 H_2 + A_0 H_1 H_3 + H_{12} H_2 + H_{21} H_{10}.
\]
where \( \varphi_1 \to 0 \) as \( \varepsilon + u \to 0 \) and \( \varphi_2 \) is bounded for \( \varepsilon + u \) small. Hence \( \lambda H_4 > 0 \) for \( \varepsilon + u \) small, which implies that \( G = \varepsilon H \) has at most one root in \( u > 0 \) for \( \varepsilon + |\delta - \delta_0| \) small. By (2.24), the proof of Theorem 1.1 is completed.

To prove Theorem 1.2, let the connection \( L_2 \) is fixed. This is equivalent to \( d_2(\varepsilon, \delta) \equiv 0 \). Then from (2.16) and (2.17), a necessary condition for \( L \) to generate a limit cycle is \( M_1(\delta_0) = 0 \) for some \( \delta = \delta_0 \). For the uniqueness of limit cycles, in the case \( r_0 = 1 \) we have from (2.24) and (2.25)
\[
G = u[\alpha_2 + O(u)][1 + O(h_2)] - u^{-1}[1 + O(g_0)] \equiv uG^*.
\]
Obviously, \( G^* > 0 \) (<0) for \( u + \varepsilon + |\delta - \delta_0| \) small if \( \gamma > 1 \) (<1). This implies that \( G \) has no roots in \( u > 0 \) for \( \varepsilon + |\delta - \delta_0| \) small. In the case \( r_0 \neq 1 \), noting that \( R_{21} = O(u) \) since \( d_2(\varepsilon, \delta) \equiv 0 \) we have from (2.29) and (2.32) \( H(0, \varepsilon, \delta) = 0 \). Hence, the fact that \( \lambda H_4 > 0 \) leads to \( G = \varepsilon H \neq 0 \) for all small \( u > 0 \). Theorem 1.2 is proved in the case that \( L_2 \) is fixed. If \( L_1 \) is fixed, instead of (2.1), we consider the function
\[
\bar{F} = R_1^{-1} - D_1^{-1} - D_2^{-1} R_2^{-1}.
\]
(2.33)
The conclusion can be proved in the same procedure as in the first case. This ends the proof of Theorem 1.2.

Finally let us turn to the proof of Theorem 1.3. Suppose that condition (i) holds. Then \( \gamma_0 < 0 \) and it follows from (2.24) and (2.25) that \( G < 0 \). Under condition (ii) we have \( M_2(\delta_0) > 0 \), \( r_2^*(\delta_0) \neq 0 \) and \( \gamma_0 < 1 \). We need only to prove that \( G \neq 0 \) for \( u + \varepsilon + |\delta - \delta_0| \) small. Let us suppose \( G \) vanishes at some \( u > 0 \). As before we have \( \gamma > 0 \). Hence by (2.2), (2.3), (2.5), and from (2.24) we have
\[
G(0, \varepsilon, \delta) = -d_2(\varepsilon, \delta)(1 + O(h_2)) < 0.
\]
(2.34)
Using \( 0 \leq \gamma_0 < 1 \) and (2.34), from (2.26) and (2.28) we can prove that \( G < 0 \) for all \( u + \varepsilon + |\delta - \delta_0| \) small, a contradiction. Therefore, \( G \neq 0 \), and hence \( F \) has at most one root in \( u > 0 \).
If condition (iii) holds, then \( r_1^*(\delta_0) \neq 0 \), and instead of (2.24) we can use the function
\[
\bar{G}(v, \varepsilon, \delta) = v^{1/\mu}(1 + \varepsilon b_2(v, \varepsilon, \delta))^{1/\mu} - R_2^{-1}(1 + \varepsilon g_0(R_2^{-1}, \varepsilon, \delta))^{1/\mu}
\]
to prove the conclusion in the same way, where \( R_2^{-1} \) denotes the inverse function of \( v = R_2(u) \).

Let condition (iv) holds. By (2.31) and (2.32), we have \( H(0, 0, \delta) = M_1(\delta_0) \), which follows that \( G \neq 0 \) for \( u + \varepsilon + |\delta - \delta_0| \) small. Hence \( F \) has at most one root in \( u > 0 \) for \( \varepsilon + |\delta - \delta_0| \) small. If the condition (v) holds, the conclusion can be proved by using the function \( \bar{F} \) given by (2.33).

### 3. BIFURCATION DIAGRAMS

In this section, we give bifurcation diagrams for \( \delta = (\delta_1, \delta_2) \in \mathbb{R}^2 \) under certain nondegenerate conditions. Let us consider the case \( r_0 = 1 \) first. Suppose there exists \( \delta_0 = (\delta_{10}, \delta_{20}) \) such that
\[
M(\delta_0) = M_1(\delta_0) + M_2(\delta_0) = 0, \quad \mu_0 = \frac{\partial M}{\partial \delta_2}(\delta_0) \neq 0. \quad (3.1)
\]
If \( M_1(\delta_0) \neq 0 \), without loss of generality, we can assume
\[
M_1(\delta_0) = -M_2(\delta_0) > 0. \quad (3.2)
\]

From (2.16) and (2.17), we have
\[
F_0(u, \varepsilon, \delta) = M(\delta) + (s - 1) M_2(\delta) \\
+ sM_2(\delta)[R(u, \varepsilon, \delta)(1 + o(\varepsilon)) - 1] + f_0(u, \varepsilon, \delta), \\
F_0(0, \varepsilon, \delta) = M(\delta) + (s - 1) M_2(\delta) \\
+ sM_2(\delta)[( - d_2)^s - 1 + o(\varepsilon)) - 1] + O(\varepsilon).
\]

From (3.1) and (3.2) we have obviously
\[
\lim_{\varepsilon \to 0} (-d_2)^s - 1 = 1, \quad \lim_{\varepsilon \to 0} \frac{\partial}{\partial \delta_2} F_0(0, \varepsilon, \delta) = \mu_0 + O(|\delta - \delta_0|).
\]
The implicit function theorem implies that a continuous function \( \delta_2 = \delta_2^*(\varepsilon, \delta_1) = \delta_2^0(\delta_1) + O(\varepsilon |\ln \varepsilon|) \) exists such that

\[
M(\delta_1, \delta_2^0) = 0, \quad F_\varepsilon(0, \varepsilon, \delta_1, \delta_2^0) = 0, \quad \mu_0(\delta_2 - \delta_2^0) F_\varepsilon(0, \varepsilon, \delta) \geq 0 \tag{3.3}
\]

for \( \varepsilon + |\delta - \delta_0| \) small. By (2.2), (2.3), and (3.2) it is easy to see that for \((\varepsilon, \delta)\) satisfying \( \delta_2 = \delta_2^*(\varepsilon, \delta_1) \) (1.1) has a homoclinic orbit \( I_{01} \) homoclinic to \( S_1 \). The graph of the function \( \delta_2^* \) on the \( \delta^- \) plane for fixed \( \varepsilon > 0 \) is called a homoclinic bifurcation curve, denoted by \( HoB_1 \). From (2.6) we can suppose the divergence of (1.1) takes the value

\[
1 + \lambda_1 - r_0 = 1 - r_1 = - \varepsilon \left( r_1^* + O(\varepsilon) \right) \tag{3.4}
\]

at \( S_1 \). If \( r_1^*(\delta_0)[r_1^*(\delta_0) + r_2^*(\delta_0)] > 0 \), from Theorem 1.3, (1.1) has at most one limit cycle near \( L \). In this case the bifurcation diagram is simple and easy to give. Let \( r_1^*(\delta_0)[r_1^*(\delta_0) + r_2^*(\delta_0)] \leq 0 \). We have equivalently either \( r_1^*(\delta_0) = 0 \) or

\[
0 < \gamma_0 < 1. \tag{3.5}
\]

For the former case, we add

\[
r_1^*(\delta_0) = 0, \quad \mu_1 = \frac{\partial}{\partial \delta_1} r_1^*(\delta_1, \delta_2^0(\delta_1)) \bigg|_{\delta_1 = \delta_0} \neq 0. \tag{3.6}
\]

We consider (3.5) and (3.6), respectively. Let (3.5) hold first. Then using (2.25) and (3.2), we have from (2.24) and (2.26), \( G(0, \varepsilon, \delta) > 0 \) for \( \varepsilon > 0 \), \( G(0, 0, \delta) = 0 \), \( G(0, \varepsilon, \delta) < 0 \) for \( \varepsilon \neq u_0 \). Noting (2.21) and (2.22), from (2.20) and (2.24) we deduce that

\[
r_2^*(\delta_0)[u - u_0] F_\varepsilon(u) < 0 \quad \text{for} \quad u \neq u_0, \quad \varepsilon > 0. \tag{3.7}
\]

Let \( f(u, \varepsilon, \delta_1) = F_\varepsilon(u, \varepsilon, \delta_1, \delta_2^*(\varepsilon, \delta_1)). \) From (3.3), \( f(0, \varepsilon, \delta_1) = 0. \) From (2.18) and (3.5) we have

\[
r_2^*(\delta_0) f(u, 0, \delta_0) = - |u \ln u| a_10 r_2^*(\delta_0) [r_1^*(\delta_0) + r_2^*(\delta_0)](1 + o(\varepsilon)) < 0 \quad \text{for} \quad 0 < u \ll 1.
\]

Hence, from (3.7), there exists \( u_1 = u_1(\varepsilon) > u_0(\varepsilon) \) such that \( f(u_1, \varepsilon, \delta_1) = 0 \) for \( \varepsilon + |\delta_1 - \delta_0| \) small. In other words, when \( \delta_2 = \delta_2^*(\varepsilon, \delta_1) \) besides the homoclinic orbit \( I_{01} \) (1.1) has a limit cycle. From (3.4), \( I_{01} \) is stable (unstable) if \( r_1^*(\delta_0) > 0 \) (\(<0\)). Hence, from (3.3) we know that (1.1) has two limit cycles for \( r_1^*(\delta_0) \mu_0(\delta_2 - \delta_2^*) > 0, |\delta_2 - \delta_2^*| \ll 1. \) Therefore, from
(3.1) there exists a function \( \delta_2 = \delta_1^*(0, \delta_1) \) satisfying \( r_1^*(\delta_0) \mu_s(\delta_1^*-\delta_1^*) > 0 \) such that (1.1) has a unique 2-multiple limit cycle for \( \delta_2 = \delta_1^* \). The corresponding bifurcation curve is denoted by 2MB. Now it is easy to give the bifurcation diagram for (1.1). For instance, if
\[
\mu_0 > 0, \quad r_1^*(\delta_0) > 0
\]
we have Fig. 3.1.

Now suppose that (3.6) is satisfied. Noting (3.4), the implicit function theorem yields that a function \( \delta_1 = \delta_1^*(\varepsilon) = \delta_{10} + O(|\varepsilon|) \) exists such that
\[
r_1(\varepsilon, \delta_1^*, \delta_1^*(\varepsilon, \delta_1^*)) = 1,
\]
\[
\mu_s(\delta_1 - \delta_1^*)[r_1(\varepsilon, \delta_1, \delta_1^*) - 1] \geq 0 \quad \text{for } \varepsilon + |\delta_1 - \delta_{10}| \text{ small. (3.9)}
\]
The divergence of (1.1) at \( S_2 \) has the same sign as the value \( r_0 + \beta_0 - 1 = s - 1 = -\varepsilon[r_1^*(\delta) + O(\varepsilon)] \). From (1.3) and (3.6), \( r_1^*(\delta_0) = r_1^*(\delta_0) \neq 0 \). Hence,
\[
- r_1^*(\delta_0) \int_{r_0}^{r_1} \text{div}(1.1) \, dt \bigg|_{\delta_1 = (\delta_1^*, \delta_1^*)} > 0 \quad \text{for } \varepsilon > 0 \text{ small.}
\]
This shows that \( \Gamma_{01} \) is stable (unstable) if \( r_1^*(\delta_0) > 0 \) (\(< 0 \)). Now let \( (\varepsilon, \delta) \) satisfy
\[
\delta_2 = \delta_1^*, \quad 0 < |\delta_1 - \delta_1^*| << 1, \quad r_1^*(\delta_0) \mu_s(\delta_1 - \delta_1^*) < 0. \quad (3.10)
\]
From (3.9) and (3.4) we see that the stability of the homoclinic loop \( \Gamma_{01} \) has got changed, and a limit cycle \( \Gamma_1 \) is born out inside \( \Gamma_{01} \). When

![FIG. 3.1. Bifurcation diagram under (3.1), (3.2), (3.5), and (3.8).](image-url)
The point is the only intersection point of the curves $H_{eB}$: $\delta_2 = \delta_2^*(e, \delta_1)$ satisfying $\mu_0 \mu_1 (|\delta_1 - \delta_1^*| |\delta_2 - \delta_2^*|) > 0$, $0 < |\delta_2 - \delta_2^*(e, \delta_1)| \ll |\delta_1 - \delta_1^*|$. $I_{01}$ has broken, and a second limit cycle $I_2$ is created. From (3.1), there exists a function $\delta_2 = \delta_2^*(e, \delta_1)$ satisfying $\mu_0 \mu_1 (|\delta_1 - \delta_1^*| |\delta_2^* - \delta_2^*|) > 0$ such that for $\delta$ on the $2MB$ curve: $\delta_2 = \delta_2^* \ast I_1$ and $I_2$ become a 2-multiple limit cycle $I^*$ and then $I^*$ disappears when $\delta$ is in a side of the curve. Consequently, we have $\delta_2^*(e, \delta_1(e, \delta_1^*)) = \delta_2^*(e, \delta_1^*)$ for $e > 0$ small. Now we can give the bifurcation diagram. For example, if

$$\mu_0 > 0, \quad \mu_1 > 0, \quad r^*(\delta_0) > 0$$

we have Fig. 3.2.

Still let $r_0 = 1$. We suppose now $M_i(\delta_0) = 0$, $i = 1, 2$, and $\gamma_1^*(\delta_0), \gamma_2^*(\delta_0) \leq 0$. Assume that

$$\lambda_0 = \frac{\partial M}{\partial \delta_2}(\delta_0) \neq 0, \quad \mu_0 = \frac{\partial M}{\partial \delta_1}(\delta_1, \delta_2^*) \mid_{\delta_1^* - \delta_2^*} \neq 0,$$

where $M_2(\delta_1, \delta_2^*) = 0$. There exist functions: $H_{eB_2}: \delta_2 = \delta_2^*(e, \delta_1) = \delta_2^* + O(e)$ and $\delta_1 = \delta_1^*(e, \delta_1^*)$ such that

$$d_2(e, \delta_1^*, \delta_2^*), \quad d_1(e, \delta_1^*, \delta_2^*), \quad d_3(e, \delta_1^*, \delta_2^*) = 0,$$

$$\lambda_0 (\delta_2 - \delta_2^*), \quad d_1(e, \delta_1^*, \delta_2^*) \geq 0.$$

The point $M = (\delta_1^*, \delta_2^*(e, \delta_1^*))$ on the $\delta$-plane corresponds to a heteroclinic loop. The loop is stable (unstable) if $r^*(\delta_0) > 0$ ($< 0$). Then for $(e, \delta)$ satisfying

$$\delta_2 = \delta_2^*(e, \delta_1), \quad 0 < |\delta_1 - \delta_1^*| \ll 1, \quad r^*(\delta_0) \mu_0(\delta_1 - \delta_1^*) > 0,$$

a limit cycle $I_2$ appears. Let $x_i = (\partial M_i / \partial \delta_2)(\delta_0)$. From (3.12) we have $|x_1| + |x_2| \neq 0$. For definiteness, we can assume $x_2 \neq 0$. It follows that there exists a function $H_{eB_2}: \delta_2 = \delta_2^*(e, \delta_1)$ such that $d_1(e, \delta_1, \delta_2^*) = 0$, $x_2(\delta_2 - \delta_2^*) d_1 \geq 0$. Obviously, $\delta_2^*(e, \delta_1^*) = \delta_2^*(e, \delta_1^*)$. That is, the point $M$ is the only intersection point of the curves $H_{eB_1}$ and $H_{eB_2}$. Let further $\sum_{i=1}^2 |(\partial M_i / \partial \delta_2)(M_1 + M_2)(\delta_0)| \neq 0$. For definiteness, we suppose

$$\beta = \frac{\partial}{\partial \delta_2} (M_1 + M_2)(\delta_0) \neq 0.$$

As before, there exists a function $\delta_2 = \delta_2^*(e, \delta_1)$ such that

$$\beta(\delta_2 - \delta_2^*(e, \delta_1)) F_0(0, e, \delta) \geq 0.$$
We claim that
\[
\begin{align*}
    r^*(\delta_0) \lambda_0 \left[ \delta_0^X(e, \delta_1) - \delta_0^Y(e, \delta_1) \right] < (>) 0 & \quad \text{if} \quad r^*(\delta_0) \mu_0 (\delta_1 - \delta_1^*) > (>) 0,
    \\
    & \quad \text{(3.17)}
\end{align*}
\]

We can assume \( r^*(\delta_0) > 0 \). Fix \( \delta_1 \) satisfying \( \mu_0(\delta_1 - \delta_1^*) > 0 \). Then noting (2.2), (1.1) has no separatix loops near \( L \) if \( d_2 > 0 \). Hence, \( d_2 < 0 \) when \( F_0(0, e, \delta) = 0 \). By (3.13) and (3.16), this implies that \( \lambda_0(\delta_0^* - \delta_0^*) < 0 \) when \( \delta_2 = \delta_0^X(e, \delta_1) \). Therefore, \( \lambda_0(\delta_0^* - \delta_0^*) < 0 \) if \( \mu_0(\delta_1 - \delta_1^*) > 0 \). In the same way we can prove \( \lambda_0(\delta_0^* - \delta_0^*) > 0 \) if \( \mu_0(\delta_1 - \delta_1^*) < 0 \). Then (3.17) follows. It is obvious from (3.17) that \( \delta_0^X(e, \delta_1^*) = \delta_0^X(e, \delta_1^*) \). Denote by HoBr the half curve \( \delta_2 = \delta_0^X(e, \delta_1^*) \) for \( (-1)^{i+1} r^*(\delta_0) \mu_0(\delta_1 - \delta_1^*) > 0 \), \( i = 1, 2 \). Along HoBr, (1.1) has a homoclinic loop \( I_0 \) through \( S_i \), \( i = 1, 2 \). Notice that \( I_0 \) is stable (unstable) as \( r^*(\delta_0) > 0 \) \( (>) 0 \) or \( r^*(\delta_0) = 0 \), \( r^*(\delta_0) > 0 \) \( (,) 0 \). We can change \( \delta_2 \) suitably such that a limit cycle \( I_1 \) appears by breaking \( I_0 \). From (3.14) in the case of \( r^*(\delta_0) \mu_0(\delta_1 - \delta_1^*) > 0 \), there exists a 2MB curve \( \delta_2 = \delta_0^X(e, \delta_1) \) satisfying \( r^*(\delta_0) \lambda_0(\delta_0^* - \delta_0^*) < 0 \) such that along the curve (1.1) has a 2-multiple limit cycle \( I_1^{(2)} \). The limit cycle approaches the heteroclinic loop of (1.1) as \( r^*(\delta_0) \mu_0(\delta_1 - \delta_1^*) \rightarrow 0^+ \). This implies that the 2MB curve ends at \( M \) as \( \delta_1 \rightarrow \delta_1^* \). Therefore, the four curves HeB1, HeB2, HoB1, \( \cup \) HoB2 and 2MB intersect at \( M \). Now the bifurcation diagram can follow. For instance, if
\[
\lambda_0 > 0, \quad \mu_0 > 0, \quad r^*(\delta_0) > 0, \quad r_1^*(\delta_0) < 0, \quad r_2^*(\delta_0) > 0,
\]
the bifurcation diagram is given by Fig. 3.3.
For the case $r_0 \neq 1$, if $M_1(\delta_0) \neq 0$ or $M_2(\delta_0) \neq 0$, from Theorem 1.3, (1.1) has at most one limit cycle. The bifurcation diagram is easy in this case. If $M_i(\delta_0) = 0$, $i = 1, 2$, we can further suppose
\[ \det \left( \frac{\partial (M_1, M_2)}{\partial \delta} (\delta_0) \right) \neq 0 \]
and give bifurcation diagrams in the analogous procedure as above. We omit the details here.

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