



Conservation of particle multiplicities between chemical and thermal freeze-out

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Abstract

The evolution of a hadronic system after its chemical decomposition is described through a model that conserves the hadronic multiplicities to their values at chemical freeze-out. In the partition function describing the model all known hadronic resonances with masses up to 2400 MeV have been included. The state of the system is found as function of temperature and the corresponding baryon density is evaluated. The baryon density at thermal decoupling is also computed.

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1. Introduction

Thermal approaches have extensively been used to describe the particle multiplicities which emerge from high energy collisions [1–16]. The results of such approaches are satisfactory since they are able to predict quite accurately a large number of different experimentally measured hadronic multiplicities as function of a few thermodynamic variables, such as temperature, volume and chemical potentials.

The extracted parameters from such approaches define the “chemical freeze-out”, i.e., the point where the chemical composition of the system that produces the particle multiplicities is fixed. Another point called “thermal freeze-out” can also be defined. This second point is associated with the particle momentum distribution which is measured experimentally. After this point this distribution remains fixed and the particles no longer interact among themselves.

There is evidence that in a lot of circumstances the two points corresponding to the same system do not occur at the same temperature. Generally the freeze-out temperature is lower than the chemical freeze-out one. Since

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the particles are measured once they have reached the experimental apparatus, after any kind of interaction among themselves has ceased and since the thermodynamic parameters of the chemical freeze-out point predict quite well all those particles, one has to infer that all these abundances will have to remain fixed through the whole process between chemical and thermal freeze-out.

In this Letter the main focus will be to construct a model of relativistic hadronic particles formulated in the grand canonical ensemble that will be able to conserve the particle multiplicities between points which correspond to different temperatures. Of course, the two points with greatest interest are the chemical and the freeze-out point. The question that arises then is why there is need for another model and why not use one of the existing thermal models. In these models the particles are described as thermally equilibrated entities. In some of these models the hadrons are non-interacting particles [2–7] and in others a kind of interaction among them has been included [1,8–16]. But usually the free thermodynamic parameters are the volume V , the temperature T and a set of a few chemical potentials each of which is associated with the conservation of quantum numbers like baryon number B , charge Q , strangeness S , etc. During the evolution of the system among states with different temperatures its content can alter. This is done by adjusting the chemical potentials appropriately so as to keep the relevant quantum numbers fixed. The only parameter that remains unfixed after the conservation of quantum numbers is the volume. If there is need to fix the particle numbers as well it is clear that this cannot be accomplished with the existing parameters. Even if it is assumed that conservation of all the particle numbers automatically conserves the quantum numbers, meaning that only the conservation of the particles is enough, this limits the application of the above models to only situations where the number of the particle entities is equal to the number of the chemical potentials plus one.¹ But generally the hadrons which have to be considered are more numerous than the quantum numbers potentials.

As it is evident these thermal models cannot evolve the system from chemical to thermal freeze-out. In this work the necessity to have fixed particle numbers will be used to construct a model which will determine the evolution of the hadronic system after its chemical freeze-out. This newly constructed model will coincide at chemical freeze-out with “Ideal Hadron Gas” model (IHG) [2–5], one of the aforementioned thermal models, which is formulated in the grand canonical ensemble and describes hadrons as relativistic non-interacting particles.

2. Formulation of the model

Before discussing the new model, the IHG model will be presented. In the context of IHG the grand canonical partition function, formulated in the Boltzmann approximation, has the form

$$\ln Z(V, T, \{\lambda\})_{\text{IHG}} = V \sum_i \lambda_{QN_i} \sum_j Z_{H_{ij}}(T) \equiv V \sum_i \lambda_{QN_i} \sum_j \frac{T}{2\pi^2} g_{ij} m_{ij}^2 K_2\left(\frac{m_{ij}}{T}\right), \quad (1)$$

where i runs over all hadronic families such as mesons, N Baryons, Λ Baryons, etc. and j represents the specific member of the family with degeneracy factor g_{ij} and mass m_{ij} . λ_{QN_i} stands for the product of all the fugacities associated with the particular family. These fugacities can either be quantum numbers fugacities related to Baryon number, Strangeness, etc. or to quark flavour.²

One can evaluate particle abundances if the above partition function is extended by the introduction of a fugacity λ_{ij} for every particle. After calculating the particle number one has to set in IHG $\lambda_{ij} = 1$ [17], so again the particle number is only expressed as function of the quantum numbers fugacities.

¹ In such a case the quantum number potentials and the volume would have to be adjusted to keep the particles fixed between points with different temperatures.

² For example, for Ξ^- baryons, λ_{QN} would read $\lambda_B \lambda_Q^{-1} \lambda_S^{-2} \gamma_S^2$ or $\lambda_d \lambda_s^2 \gamma_s^2$. One can look in [16, Eq. (14)], to find out how the two sets of fugacities are related.

Now, if someone wishes to keep the particle numbers fixed it is only natural to extend (1) by the use of fugacities λ_{ij} (corresponding to particle numbers), but with the difference that the constraint $\lambda_{ij} = 1$ will not be imposed. This model will be called Fixed Particle Numbers (FPN) model and accordingly its partition function will depend on λ_{ij} 's

$$\ln Z(V, T, \{\lambda\})_{\text{FPN}} = V \sum_{ij} \lambda_{H_{ij}} Z_{H_{ij}}(T) \equiv V \sum_{ij} \lambda_{H_{ij}} \frac{T}{2\pi^2} g_{ij} m_{ij}^2 K_2\left(\frac{m_{ij}}{T}\right), \quad (2)$$

where $\lambda_{H_{ij}}$ is product of quantum numbers as well as particle number fugacities.³ Since the experimentally measured multiplicities usually contain feeding from the decay of resonances, all known hadrons with masses up to 2400 MeV have been included in the FPN partition function. The same hadrons have also been included to IHG partition function (1). The mean particle number can be evaluated through the relation

$$\langle N_{ij} \rangle_{\text{FPN}} = \lambda_{ij} \left. \frac{\partial \ln Z(V, T, \{\lambda\})_{\text{FPN}}}{\partial \lambda_{ij}} \right|_{\{\lambda\} \neq \lambda_{ij}}, \quad (3)$$

where $\{\lambda\} \neq \lambda_{ij}$ means that for the evaluation of the partial derivative all fugacities except λ_{ij} are considered as constants.

The next point that has to be elucidated is at what values the particle numbers will stay fixed. Since after chemical freeze-out these values do not alter it is useful to formulate FPN so as to keep the particle numbers fixed at their chemical freeze-out values. One has to remember that the chemical freeze-out values are extracted from a thermal model, like IHG, by a successful fit to the experimentally measured values. The particle numbers can then be fixed at the values calculated through IHG for the chemical freeze-out thermodynamic variables. Thus the results of IHG and FPN should coincide at chemical freeze-out point. So it has to be required⁴

$$\langle N_{ij} \rangle_{\text{IHG}} = \langle N'_{ij} \rangle_{\text{FPN}} \iff V \lambda_{QN_i} Z_{H_{ij}}(T) = V' \lambda_{H_{ij}} Z_{H_{ij}}(T') \iff \lambda_{H_{ij}} = \frac{V \lambda_{QN_i} Z_{H_{ij}}(T)}{V' Z_{H_{ij}}(T')}. \quad (4)$$

The above equation can be used to calculate the total product of fugacities $\lambda_{H_{ij}}$ consisting of quantum numbers fugacities and of hadron fugacities at temperature T' . It has to be pointed out that it is not possible to evaluate each quantum number fugacity separately, but this is irrelevant since the full product of fugacities can be calculated. Another focal point is that all quantum numbers are automatically conserved as linear combination of the particle numbers.

In the right-hand side of Eq. (4) the only term which is left undetermined after the imposition of the conservation of particle numbers is the multiplicand factor V/V' . So an additional constraint has to be applied. For example, conservation of entropy can be assumed.⁵

The entropy of the system can in general be calculated from⁶

$$\tilde{S} = - \left(\frac{\partial [-T \ln Z(V, T, \{\mu\})]}{\partial T} \right)_{V, \{\mu\}}, \quad (5)$$

³ For example, for $\Xi(1530)^-$ baryon, λ_H would read $\lambda_B \lambda_Q^{-1} \lambda_S^{-2} \gamma_S^2 \lambda_{\Xi(1530)^-}$.

⁴ The primed variables in this Letter will generally be related to subsequent points of the chemical freeze-out point.

⁵ A lot of authors assume isentropic evolution of the system, e.g., see [18].

⁶ The symbol of entropy is tilded in order not to be confused with the symbol of Strangeness. K can be set equal to one.

where μ represents the chemical potential associated with fugacity $\lambda = \exp(\mu/T)$. Applying (5) to the partition function (2)⁷ the constraint of fixed entropy will read

$$\begin{aligned}\tilde{S} = \tilde{S}' &\iff \ln Z_{\text{IHG}}(V, T, \{\mu\}) + VT \sum_{ij} \lambda_{QN_i} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - VT \sum_{ij} \lambda_{QN_i} \frac{\mu_{QN_i}}{T^2} Z_{H_{ij}}(T) \\ &= \ln Z_{\text{FPN}}(V', T', \{\mu'\}) + V'T' \sum_{ij} \lambda_{H_{ij}} \frac{\partial Z_{H_{ij}}(T')}{\partial T'} - V'T' \sum_{ij} \lambda_{H_{ij}} \frac{\mu_{H_{ij}}}{T'^2} Z_{H_{ij}}(T').\end{aligned}$$

With the use of (4) the last equation becomes

$$\begin{aligned}VT \sum_{ij} \lambda_{QN_i} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - V \sum_{ij} \lambda_{QN_i} \ln(\lambda_{QN_i}) Z_{H_{ij}}(T) \\ = V'T' \sum_{ij} \frac{V \lambda_{QN_i} Z_{H_{ij}}(T)}{V' Z_{H_{ij}}(T')} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - V' \sum_{ij} Z_{H_{ij}}(T') \frac{V \lambda_{QN_i} Z_{H_{ij}}(T)}{V' Z_{H_{ij}}(T')} \ln\left(\frac{V \lambda_{QN_i} Z_{H_{ij}}(T)}{V' Z_{H_{ij}}(T')}\right) \\ \iff T \sum_{ij} \lambda_{QN_i} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - \sum_{ij} \lambda_{QN_i} \ln(\lambda_{QN_i}) Z_{H_{ij}}(T) \\ = T' \sum_{ij} \lambda_{QN_i} \frac{Z_{H_{ij}}(T)}{Z_{H_{ij}}(T')} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - \sum_{ij} \lambda_{QN_i} Z_{H_{ij}}(T) \ln\left(\frac{V \lambda_{QN_i} Z_{H_{ij}}(T)}{V' Z_{H_{ij}}(T')}\right).\end{aligned}\quad (6)$$

Setting $x \equiv V'/V$, (6) can be solved for x to give

$$x_{\text{FPN}} = \exp\left[\frac{\sum_{ij} \lambda_{QN_i} Z_{H_{ij}}(T) \ln\left(\frac{Z_{H_{ij}}(T)}{Z_{H_{ij}}(T')}\right) + T \sum_{ij} \lambda_{QN_i} \frac{\partial Z_{H_{ij}}(T)}{\partial T} - T' \sum_{ij} \lambda_{QN_i} \frac{Z_{H_{ij}}(T)}{Z_{H_{ij}}(T')} \frac{\partial Z_{H_{ij}}(T)}{\partial T}}{\sum_{ij} \lambda_{QN_i} Z_{H_{ij}}(T)}\right].\quad (7)$$

Eq. (7) can be used to evaluate the volume expansion ratio as the system has cooled to a temperature T' less than the chemical freeze-out temperature T . With the use of the same equation, quantities like the baryon density of the system can be calculated at T' . The baryon chemical potential at which the system is found at temperature T' cannot be calculated separately from the rest of chemical potentials in the context of FPN. But the baryon density has no problem to be evaluated. One has to remember that baryon number is also fixed with the imposition of the constraints (4). So

$$n_{B \text{ FPN}} = \frac{\langle B' \rangle}{V'} = \frac{\langle B \rangle^{\text{ch}}}{V'} = \frac{V}{V'} \cdot \frac{\langle B \rangle^{\text{ch}}}{V} = \frac{n_B^{\text{ch}}}{x_{\text{FPN}}},\quad (8)$$

where n_B^{ch} is the baryon density calculated at chemical freeze-out.

3. Application

The newly constructed model, FPN, can describe thermally equilibrated hadronic systems with fixed particle numbers when their temperature is known. As, an example, the systems formed at different interactions at SPS will be considered. For this reason the chemical freeze-out parameters obtained for these systems through fits to their

⁷ Since the IHG and FPN partition functions coincide at chemical freeze-out point, the IHG partition function (1), where $\lambda_{ij} = 1$, can be used for the evaluation of the entropy at this point.

Table 1

Chemical freeze-out parameters calculated for different interactions at SPS and the corresponding references

Experiment	T^{ch} (MeV)	μ_B^{ch} (MeV)	γ_s^{ch}	Reference
S + S 200 A GeV	180.5 ± 10.9	220.2 ± 18.0	0.747 ± 0.048	[7,18]
S + Ag 200 A GeV	178.9 ± 8.1	241.5 ± 14.5	0.711 ± 0.063	[7,18]
Pb + Pb 158 A GeV	174.7 ± 6.7	240 ± 14	0.900 ± 0.049	[19]

Table 2

Thermal freeze-out temperature calculated in two different references for the Pb + Pb interaction and the corresponding computation of baryon density through FPN. The upper errors of baryon density correspond to the upper errors of temperature. The same is true for the lower errors

Experiment	T^{ther} (MeV)	Reference	n_B^{ther} (fm^{-3})
Pb + Pb 158 A GeV	120 ± 12	[20]	$0.099^{+0.022}_{-0.019}$
Pb + Pb 158 A GeV	95.8 ± 3.5	[21]	$0.0627^{+0.0047}_{-0.0045}$

experimentally measured values will be used. These parameters are listed in Table 1 along with the references where they can be found. From a variety of thermal analyses performed by different authors the particular ones have been chosen because they allow for partial strangeness equilibrium ($\gamma_s \neq 1$) and they use most recent available values for the experimentally measured hadronic multiplicities. The values of Table 1 are then taken, for each interaction separately, as input to the equations $\langle S \rangle = 0$ and $\frac{\langle B \rangle}{2\langle Q \rangle} = \beta$,⁸ to determine the rest of the fugacities. Thus the whole set of chemical freeze-out parameters (T , μ_B , μ_Q , μ_S , γ_s) are calculated and also the products of fugacities λ_{QN_i} in (1) are also set.

Giving different values to temperature T , Eq. (8) can be used to calculate the corresponding baryon density. The resulting paths for FPN for S + S, S + Ag and Pb + Pb interactions are shown in Fig. 1 with solid curves. For the Pb + Pb interaction the thermal freeze-out temperature is calculated in Refs. [20] and [21]. For these values baryon density at thermal freeze-out n_B^{ther} can be evaluated. The results are listed in the last column of Table 2. The path for Pb + Pb is followed until the lower temperature (of the two given in Refs. [20,21]) is reached. The points which correspond to the thermal freeze-out temperatures of these references are depicted with squares on the FPN curve.

FPN has the unique attribute to conserve each particle species separately. So it is not possible to compare its results directly with another thermal model. In order to have a general view we shall depict on the graphs with the FPN results IHG states for different temperatures. The IHG model to be used only conserves the quantum numbers $\langle B \rangle$, $\langle Q \rangle$ and $\langle S \rangle$ and the entropy $\langle \tilde{S} \rangle$. The IHG states are represented by a dotted curve in Fig. 1. It has to be pointed out that IHG does not take the system from chemical to thermal freeze-out. The system remains in the context of IHG all the time at chemical equilibrium where the particles can transform into one another.

The usefulness of Fig. 1 is that it shows that a ‘‘soup’’ of fixed particles cools more efficiently than a chemically equilibrated IHG state. This means that at the same baryon density (which through the conservation of the baryon number is equivalent to equal volumes) smaller temperature corresponds to the FPN state than to the IHG state. The reason is that IHG only conserves a few quantum numbers. As temperature drops the number of particles diminishes but without affecting the preservation of quantum numbers. For example, an equal reduction to the number of protons and antiprotons will not affect the conservation of baryon number. In the FPN case, on the contrary, no particle number is allowed to diminish. So in the IHG state the available energy has to be distributed among less particles than FPN, and their mean kinetic energy has to be greater, leading to greater temperature.

In Fig. 2 the FPN ratios $x = V'/V$, where V is the chemical freeze-out volume, are plotted as function of temperature for the three SPS interactions. For comparison the ratios $x = V'_{\text{IHG}}/V_{\text{IHG}}$ for the particular IHG model

⁸ β is fixed from the baryon number and charge of the participant nucleons, e.g., see [16].

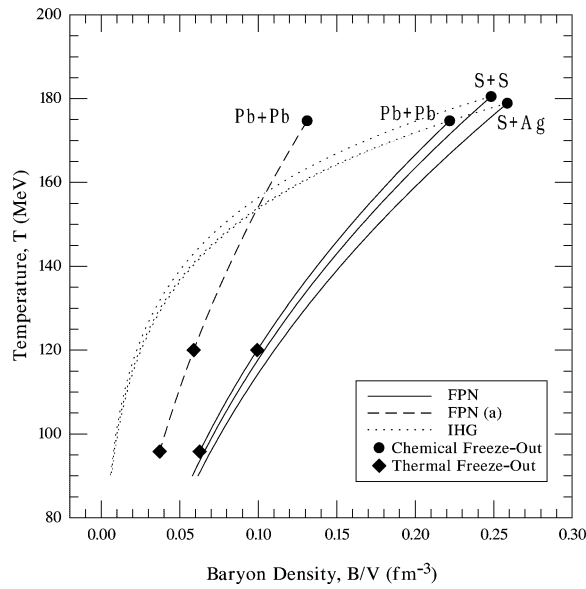


Fig. 1. Contours (solid lines) that follow hadronic systems after chemical freeze-out on (T, n_B) plane for 3 interactions at SPS, calculated through FPN (model of Fixed Particle Numbers). On the same graph points at chemical equilibrium (dotted lines) calculated through an IHG model that conserves baryon number, charge, strangeness and entropy are depicted. The slashed curve represents calculations with FPN including hadrons with masses only up to the Delta mass (FPN (a)) and using as chemical freeze-out parameters the ones that correspond to Pb + Pb interaction of Table 1.

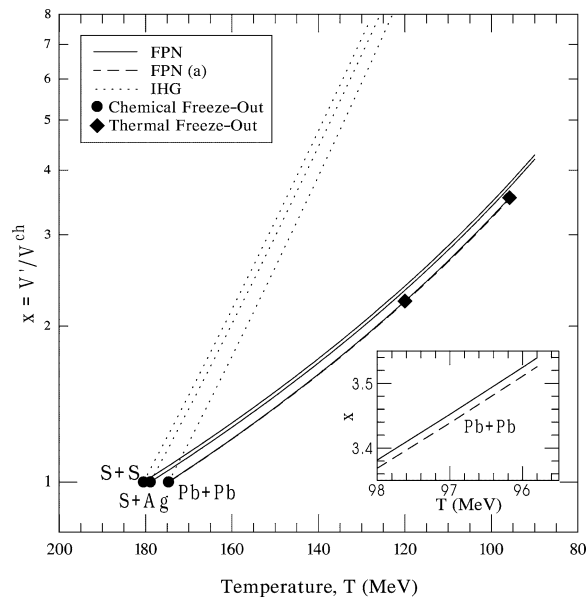


Fig. 2. The ratios of the volumes V' of the SPS hadronic systems at a certain temperature to their volumes V^{ch} at chemical freeze-out calculated for FPN (solid lines). The dotted lines correspond to the ratios of volumes of chemically equilibrated states of IHG model of Fig. 1 calculated at the same temperatures as the volumes of the FPN ratios. The slashed curve represents calculations with model FPN (a) of Fig. 1.

discussed above are plotted with dotted curves. The two volumes in the IHG ratios correspond to chemically equilibrated states at different temperatures. V'_{IHG} and V_{IHG} are calculated for the same temperatures as V' and V , respectively.

Also, in order to show the effect of hadrons with large masses, FPN calculations with hadrons only up to the Delta mass have been included in Figs. 1 and 2. The chemical freeze-out parameters for Pb + Pb of Table 1 have been used for these calculations. From Fig. 1 it is evident that the baryon density calculated through the truncated version of FPN is considerably less than the baryon density calculated at the same temperature with the FPN model using all the hadrons. The expansion ratio x for the truncated FPN, though, is close to the calculated ratio with FPN using all the hadrons, as it can be seen from Fig. 2.

4. Conclusion

After chemical freeze-out the collisions among hadrons that compose the hadronic gas can no longer change its chemical composition. Following this requirement a non-interacting hadron gas model (FPN) has been presented that keeps the multiplicity of every particle fixed to the value dictated by the chemical freeze-out conditions. In the context of FPN the constraints of conservation of quantum numbers are broken up to a larger number of constraints, these of conservation of particle numbers. The chemical potentials of quantum numbers are no longer “good” variables to describe the evolution of the system. Of course the fugacities of particle numbers used as variables in FPN are not “free” parameters. Their values are fixed from the given set of the quantum numbers fugacities at chemical freeze-out. So the evolution of a hadronic system is described as function of temperature (after imposing conservation of entropy). This is done for three SPS interactions.

Following this evolution and using values of thermal freeze-out temperature extracted for the Pb + Pb interaction the baryon density at freeze-out is evaluated. As the temperature at thermal decoupling for various interactions can be calculated using transverse mass spectra or HBT analysis [22] the same procedure can be applied to evaluate the corresponding baryon density before free streaming for these interactions.

In this Letter the IHG has been used as the thermal model which would coincide with FPN at chemical freeze-out point. Any other thermal model, interacting or non-interacting, can also be used in place of IHG and, with the use of particle fugacities, a model that conserves particle multiplicities can also be formed. Recently in model II of [23]⁹ the particle fugacities have been used to conserve particle multiplicities. This a three-dimensional hydrodynamic model describing radial and elliptic flow but it includes hadrons with masses only up to Delta mass and it is restricted to zero baryon chemical potential.

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